1. Prove that every group of order 15 is cyclic.

2. Determine, up to isomorphism, the groups of order 18.

3. Let $\mathbb{F}$ be a finite field with $q$ elements. Determine the number of $2 \times 2$ nilpotent matrices with entries from $\mathbb{F}$.

4. Let $M$ and $N$ be finitely generated modules over a commutative local ring $A$. If $M \otimes_A N = 0$, prove that $M = 0$ or $N = 0$. Is this true if $A$ is not local? Prove or disprove.

5. Let $A$ be a commutative ring and let

$$0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0$$

be a short exact sequence of $A$-modules. If $L$ and $M$ are injective, show that the sequence splits, and that $N$ is also injective.

6. Let $\mathbb{F}$ be a field and let $A$ be the polynomial ring $\mathbb{F}[x_1, x_2, \ldots, x_n]$. Regard $\mathbb{F}$ as an $A$-module via the map $A \rightarrow A/(x_1, x_2, \ldots, x_n)A$. Prove that

$$\text{Tor}_i^A(\mathbb{F}, \mathbb{F}) \cong \text{Ext}_A^i(\mathbb{F}, \mathbb{F}) \cong \wedge^i(\mathbb{F}^n)$$

for each $i \geq 0$.

7. Determine, in each case, whether the given ring is a unique factorization domain:

(a) $\mathbb{Z}[\sqrt{-1}]$,  
(b) $\mathbb{Z}[\sqrt{-5}]$,  
(c) $\mathbb{Z}[2\sqrt{2}]$

8. Compute the Galois group of the polynomial $(x^2 - 2)(x^3 - 2)(x^3 - 3)$ over the field $\mathbb{Q}(\sqrt{-3})$.

9. Given an integer $n$, prove that there exists a monic polynomial with integer coefficients whose Galois group over $\mathbb{Q}$ is $S_n$.

10. Let $\alpha$ be a complex root of $x^6 + x^3 + 1$. Determine all field homomorphisms $\mathbb{Q}(\alpha) \rightarrow \mathbb{C}$. 