PRELIMINARY EXAMINATION IN ALGEBRA

August 18, 2009

Instructions: Answer as many questions or parts of questions as you wish. A passing score consists of four complete answers or a reasonable equivalent.

1. Compute the number of elements of order 4 in the symmetric group $S_7$.

2. Let $G$ be a group of order $p^n$, where $p$ is a prime number and $n > 0$ is an integer. Prove that the center of $G$ is not trivial.

3. Let $E \subseteq F$ be a finite Galois extension of fields. Suppose that there is an element $\alpha$ in $F$ such that $\alpha \notin E$ and such that $\alpha$ is in every proper extension of $E$ contained in $F$. Show that the Galois group of $F$ over $E$ is cyclic of prime power order.

4. Show that every finitely generated subgroup of $\mathbb{Q}/\mathbb{Z}$ is cyclic.

5. Show that there are no simple groups of order 520.

6. Let $M$ be a matrix with real coefficients such that $M^2 + M + I = 0$. Give the possible canonical forms for $M$

   (a.) Over the real numbers $\mathbb{R}$ (rational canonical form).

   (b.) Over the complex numbers $\mathbb{C}$ (Jordan canonical form).

7. Let $R$ be a commutative ring, and let $S$ be a multiplicative subset of $R$ ($1 \in S$, and if $s$ and $t$ are in $S$, then $st$ is in $S$). Show that an ideal $I$ that is maximal with the property that $S \cap I = \emptyset$ is prime.

8. Determine for which integers $q$ the polynomial $X^3 + 1$ has three distinct roots in the field with $q$ elements.

9. Show that there exists a Galois extension of the field $\mathbb{Q}$ of rational numbers with Galois group $\mathbb{Z}/5\mathbb{Z}$.

10. State and prove the Eisenstein Irreducibility Criterion.