Group Actions

on Aspherical Manifolds

Dubrovnik 2011

Joint with Cappell + Yan
and Block
and Ferry etc...
and Bartels + Lueck

Conjectures.

<u>Borel</u>: If M is aspherical and $f: M' \to M$ is a homotopy equivalence, then f is homotopic to a homeo.

<u>Wall</u>: If $B\pi$ satisfies Poincaré duality, then $\exists$ a closed aspherical manifold M with $\pi_1 M \cong \pi$.

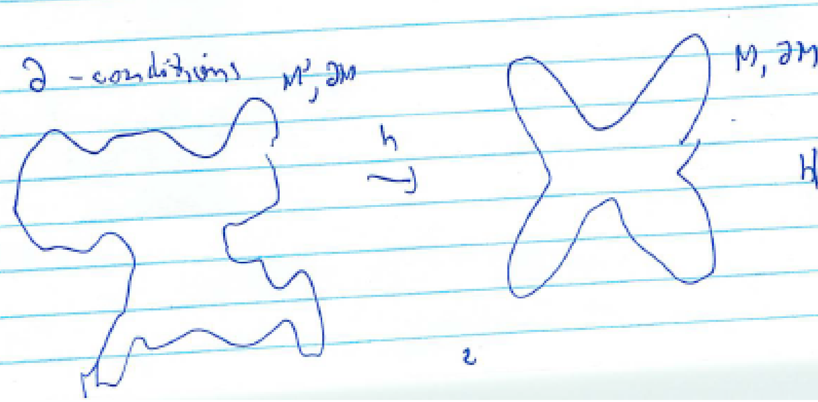
Theorem: (Bryant, Ferry, Mio, W)

If $BC(\pi)$ and $\dim \geq 6$ then $\exists$ a closed aspherical $\mathbb{Z}$-homology mfld like in Wall's conj.

Bartels + Lueck proved BC for hyperbolic groups $\Rightarrow$ $\therefore$ homology manifolds always exist.

Digression: What does BC mean if there is no aspherical mfld?

∂-conditions



$h|_{\partial M}$ a homeo.

$BC(\pi)$ means this for all M w/ ∂.

If a closed M exists $BC(\pi) \Leftrightarrow BC(\pi \times \mathbb{Z}^i)$

$\Leftrightarrow BC(M \times T^i) \Leftrightarrow$ old version for all $M \times T^i$

(It's a slight stabilization; like replacing $\tilde{K}_*(M) = 0$ by $\tilde{H}_*(M) = 0$.)

So e.g. Wall's Conj is true if π is hyperbolic with $\partial\pi \approx S^{n-1}$.

<u>Conj</u>: Wall's Conj is False

Why? Every homology manifold X

has an invariant (sort of $p_0(X) \in H^0(X;\mathbb{Z})$)

$I(X) \in \mathbb{Z}$ so that $I(X) = 1 \iff$

X is resolvable.

Borel Conj $\Longrightarrow$ I is homotopy invariant

$\therefore$ If one had an example with $I \neq 1$

Wall Conj $\Rightarrow$ $\neg$ Borel Conj.

However, only evidence for Wall Conj.

is Borel Conj.

But maybe

(Modified ~~Borel~~ Wall Conj) If $B\pi$ is P.D.

then $\exists$ $\mathbb{Z}$-hlgy mfld X (aspherical)

s.t. $\pi_1 X = \pi$.

<u>M. Davis Conj</u> If $B\pi$ is $PD/_R$

R another ring, then $\exists$ R-hlgy mfld X

("aspherical") s.t. $\pi_1 X = \pi$.

<u>Fowler</u> disproved this for $R = \mathbb{Q}$.

Later I will explain some of Fowler's work.

Fowler's work + rest of the talk is based on thinking about group actions on aspherical manifolds.

Theorem (Borel) If X is aspherical and $\pi_1 X$ is centerless then if $G \curvearrowright X$, $G \longrightarrow \mathrm{Out}(\pi_1 X)$ is 1-1.

(Proof: Smith theory ...) $\boxed{\text{Cor: } G \text{ is finite.}}$

Remarkable New Result (Avramidi)

Same is true if

$$X = {}_K\backslash G /_\Gamma$$

for any lattice.

Theorem: If G is compact and connected

$G \curvearrowright X$, X aspherical, then

$$\pi_1 G \longrightarrow \pi_1 X$$

is 1-1 and central.

Cor: G is always a torus.

Action is possible $\Rightarrow$ $\pi_1 X$ contains center

Conjecture (Conner - Raymond '69)

If $\pi_1 X$ has center, then

X has an S^1 action.

True in dim ≤ 3...

Kind of a parametrised Borel conjecture.

<u>Theorem</u> (Cappell - Weinberger - Yan)

This conjecture is false in all sufficiently large dimensions.

Several inputs.

① Conner - Raymond theory of injective S^1 actions.

② Work on Nielson problem for <u>homology manifolds</u>. (Earlier work w/ Block)

③ Ψ-equivalence invariance (Extension of earlier work w/ J. Rosenberg)

④ Methods of constructing aspherical manifolds.

Conceivably the parametrized version is correct, i.e. $\exists\ \varphi : S^1 \to \mathrm{Homeo}(X)$ s.t. $\varphi(\theta)\,p$ defines a nontrivial element of the center.

This probably follows from the "Borel Package" but I haven't checked this.

S^1 actions on aspherical mflds

$\longrightarrow$ Seifert fibrations

<u>Theorem</u> (Conner-Raymond) If $S^1 \curvearrowright X$ (not nec aspherical!!)

and $[S^1 \cdot x] \neq 0 \in H_1(X;\mathbb{Q})$ then $\exists\, Y$ so that

$\mathbb{Z}_n \curvearrowright Y$ and $X \cong \mathbb{R} \times_{\mathbb{Z}_n} Y$

(You know what I mean...)

For smooth free S^1 actions this can be seen by inspection. (Wish I had a blackboard)

We will rig it so that n should be 2

and then the Y will have to be a h'lgy

mfld and involution won't exist as a

result of failure of Nielsen.

<u>Conjecture:</u> Equivariant Resolution theory.

(we won't need it, but it's appealing.)

Now for Nielsen.

Nielsen Conj: If $\pi = \pi_1$(Aspherical mfld M)

and $G \subset \mathrm{Out}(\pi)$ is finite then $\exists\, G \curvearrowright M$.

(doing the right thing.)

- For $\dim M = 2$ true (Kerkhoff + many others)
- For $\dim M = 3$ false for certain nilmanifolds (Raymond - Scott)

The problem is that $\nexists\, \Gamma$ s.t.

$$1 \to \pi_1 M \to \Gamma \twoheadrightarrow G \to 1$$

And this is associated to $Z(\pi_1 M) \neq 0$

and prevents G from acting on any space.

We can't make use of this.

We are interested in the Question

$$G \curvearrowright M \overset{\varphi}{\dashrightarrow} X \curvearrowleft G$$

$$X = B_\pi$$

Two different versions:

① φ is an equivariant h.e.

- STRONGER THAN NIELSEN
- VERY NATURAL for analogues of B.C.
- Not enough for C-R problem.
  where Nielsen is relevant

② φ is equivariant and a h.e.

This is called a φ-equivalence.

$\Longleftrightarrow$

$$M \times EG \xrightarrow{\varphi \times id} X \times EG$$

is a G h.e. So φ-eq is a "reasonable equiv relation" but no known surgery theory for these in any great generality, but

See work of Petrie, Dovermann, Rothenberg... for partial results.

Defn: If $G \curvearrowright M$ (G finite)

then can define $\sigma^*(M) \in L^p(\mathbb{Q}\pi_1(M \times_G EG))$

and can also $\frac{1}{|G|}$ rather than $\mathbb{Q}$.

[Rosenberg-W] $\qquad K(C^*(\pi(M) \rtimes G))$

And then is it is a φ equiv invariant.

Thm If $G \curvearrowright M \xrightarrow{\varphi} K\backslash G/\Gamma \curvearrowleft G$ then (Note $G \neq \mathbb{Q}$)

$$\varphi_*(\Delta^G(M)) \in K^G(K\backslash G/\Gamma)$$

is a φ-eq invariant.

For Γ a discrete subgroup of G.

This suggests Fowler's Thm

If X is a $\mathbb{Q}$-hlgy mfld

with $\tilde{H}_*(X;\mathbb{Q}) = 0$ and $\pi_1 X = \Gamma$

$\Gamma \subset G$ a lattice with torsion of order $\neq 2$.

Then

$$\sigma^*(X) \in L(\mathbb{Q}\Gamma)$$

$$= \sigma^*\left(\left(\Gamma' \backslash G/K\right)^{\Gamma/\Gamma'}\right)$$

and then Novikov conj + Localization theorem in K-theory + calculate

$\Rightarrow \sigma^*(X) \notin \operatorname{Im} K(B\Gamma) \otimes \mathbb{Q}$

and $\therefore$ X does not exist.

However we will need the prime 2 as we will see.

One attempt (Block - W '09)

make sure f.p.s. = pts or circles

and then ψ-eq $\Leftrightarrow$ equiv h.e.

(especially after equiv ~~resolutions~~ taming)

and then can use surgery.

Let me explain this first.

Easier Analogue.

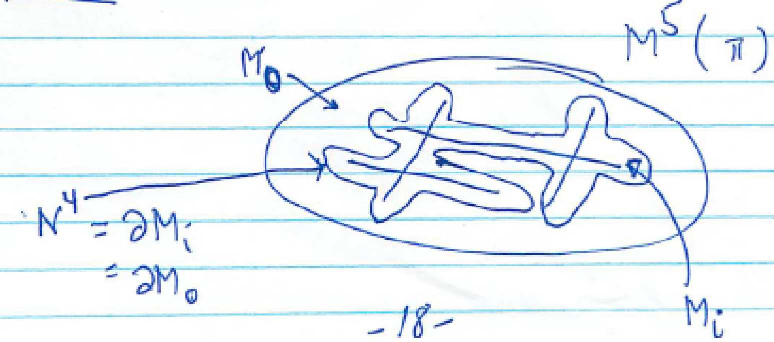
<u>Proposition:</u> If $H_n(B\pi;\mathbb{L}(e))$

$$\longrightarrow L_n(\pi)$$

is not onto then $\exists$ n-dim'l Poincaré complex which is not h.e. to a mfld. with $\pi_1 X = \pi$ $(n \geq 5)$.

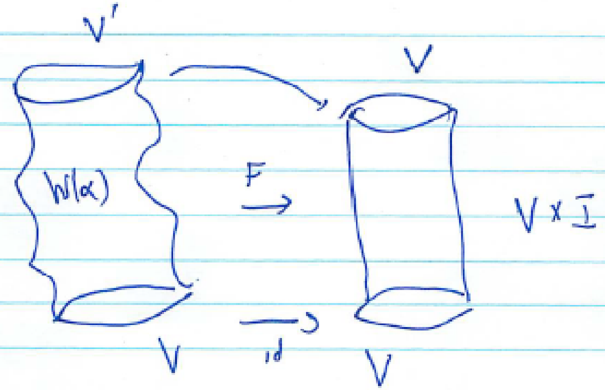
<u>Remark:</u> Easy to improve on this, but we don't care.

<u>Proof:</u> Consider



Recall <u>Wall Realization Thm:</u> Given $\alpha \in L_k(V^{k-1})$

$\exists$

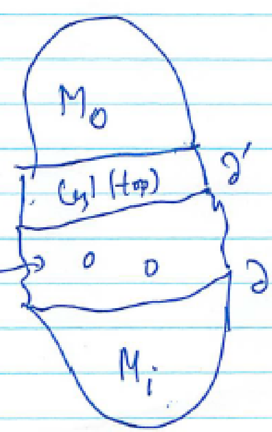


$\Theta(F \text{ rel } \partial) = \alpha$.

(For $k=4$ need also $\#(S^2 \times S^2 \times I)$ due to Cappell-Shaneson)

Now for

$X =$ $W(\alpha) \rightarrow$



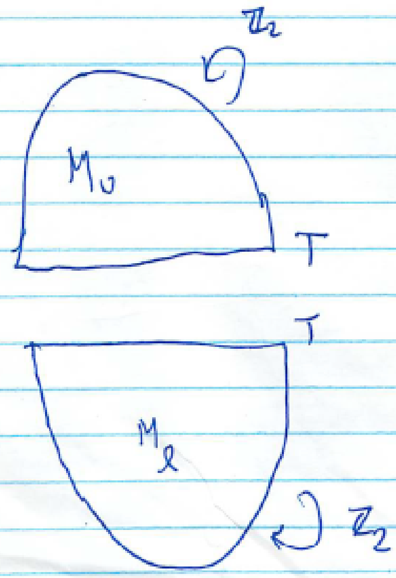
X is closed mfld.

Note: Similarity to both [BFMW] and Gromov-Piatetski-Shapiro non-arithmetic lattices.

We will build a $\mathbb{Z}_2$-equiv Poincaré space as follows.

The T's are tori,

The involutions on T are equivariant h.e. but not equiv. homeo.



The $M_{u,l}$ are aspherical and $\pi_1 T \rightarrowtail \pi_1(M_{u,l})$

$\boxed{+\ \text{No center}}$

This requires several constructions.

① Counterexamples to Equiv Bord Conj.

Ultimately these are push forwards

of $S(S^1 \times (\mathbb{RP}^{4k+1} \# \mathbb{RP}^{4k+1}))$

$\longrightarrow S^{\mathbb{Z}_2}(T)$

of examples of Cappell ('73)

(See recent work of Connolly-Davis-Khan.)

② Constructions of the aspherical manifolds with

involution $M_{v,l}$.

These are based on Gromov's

relative hyperbolization (as developed

by Davis - Januszkiewicz - W).

Note: The Equivariant h.e.

$$\varphi : T' \longrightarrow T$$

is unique homotopic to a homeo Φ.

So $M_u \cup_{\Phi} M_l$ has $\mathbb{Z}_2 \subset \mathrm{Out}(\pi_1)$ but

it's not realized (if we use f.p.s. = pts or circles)

Note: The involution on $Y = M_u \cup_{\Phi} M_l$

is homotopic to a homeomorphism

$$\psi : Y \to Y.$$

Theorem: (CWY) The manifold

$$Z = T(\psi)$$ (the mapping torus of ψ)

has nontrivial center $\subset \pi_1(Z)$ but does not admit a circle action

<u>Scholium</u>: Nor does $Z \times V$ where V is any centerless nonpos. curved closed manifold.

<u>Sketch</u>: The center of $\pi_1(Z)$ is generated by t^2. One checks that the n in Conner-Raymond must be 2 and that a homology manifold h.e. to Y (or $Y \times V$) must admit an involution homotopic to φ. It would be equiv. h.e. (or $\mathbb{Z}/2$-eq) to the action whose total surgery obstruction is nonzero.

Scholia

From Wikipedia, the free encyclopedia

Not to be confused with skolion

Scholia (singular, **scholium** or **scholion**, from Greek σχόλιον "comment", "interpretation"), are grammatical, critical, or explanatory comments, either original or extracted from pre-existing commentaries, which are inserted on the margin of the manuscript of an ancient author, as glosses. One who writes scholia is a **scholiast**. The earliest attested use of the word dates to the 1st century BC.[1]

Suppose we have $G \curvearrowright X$ a Poincaré complex.

Now suppose that there is a G map

$$M \xrightarrow{f} X.$$

- We suppose that $f \times id_{EG}$

$$M \times_G EG \xrightarrow{f \times id} X \times_G EG$$

is covered by bundle data.

- Also suppose that $f|_{\Sigma(M)}$ is a $\mathbb{Z}_{(p)}$ homology equivalence.

(This is the wrong condition, but for us $G = \mathbb{Z}/p$ so it doesn't make a difference)

Then:

<u>Proposition:</u> We can define $\theta(f) \in L^p_X(\mathbb{Z}\pi_1(X \times_G EG))$

<u>Note:</u> 1. It really is L^p not L^h

as one can see in examples.

2. If the element vanishes one cannot guarantee that surgery is possible.

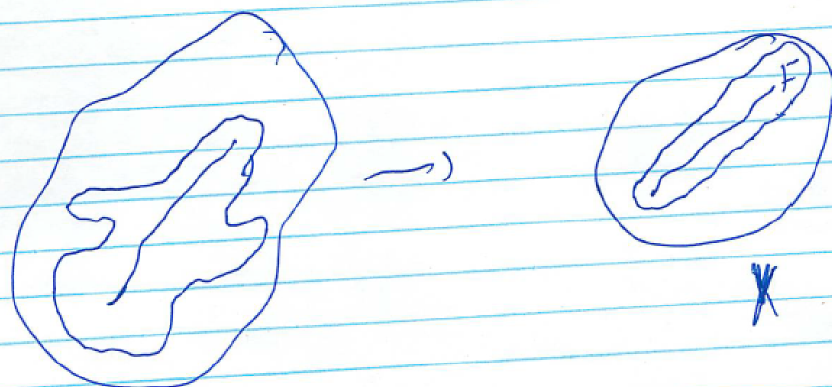
It often is, and that leads to theorems, but theory is still missing

3. The proof uses ideas of Wall finiteness obstruction and Ranicki's quadratic signature.

If $\dim \Sigma(M) < \frac{1}{2} \dim M$ this can be done by hand.

Suppose $M \to Y$ is a ψ eq.

then form the picture:



X

M

$\longrightarrow X - F$

Comp

s.o. s.t. $\partial = 0 \in L(\pi_1 F \times G)$

If ψ.eq s.o. $= 0$.

This s.o. is from a normal map

Cor: If the $\mathbb{Z}/_2$ equiv. P.S. $Y = M_0 \cup M_1$

has Ψ-eq manifold M then

Y - singular set

has vanishing t.s.o. $\in L_n(\pi_1 Y, \pi_1(fps) \times \mathbb{Z}_2)$

(1) Note: $Y = E\pi$.

(2) Now one has to check that Cappell's UNil element

survives this restriction.

This would be easy with a blackboard.

Final Remarks

1. Gp actions on aspherical manifolds are rich and interesting.

2. The Davis Q about acyclic-spherical R-homology manifolds shows how little we know about homology manifolds for $R \neq \mathbb{Z}$

   (We have no good clue about DQ for $R = \mathbb{Q}$ by Fowler's examples).

3. Does anyone have a method for producing aspherical Poincaré complexes? Please tell me!