Poincaré inequalities and rigidity for actions on Banach spaces

Piotr Nowak

Texas A&M University

Dubrovnik VII – June 2011
Property (T) was defined by Kazhdan in late 1960’ies.

We use a characterization of (T) due to Delorme – Guichardet as a definition.

**Definition**

A group $G$ has Kazhdan’s property (T) if every action of $G$ by affine isometries on a Hilbert space has a fixed point. Equivalently,

$$H^1(G, \pi) = 0$$

for every unitary representation $\pi$. 
Generalizing (T) to other Banach spaces

$X$ – Banach space, reflexive ($X^{**} = X$)

Example: $L_p$ are reflexive for $1 < p < \infty$, not reflexive for $p = 1, \infty$.

We are interested in groups $G$ for which the following property holds:

*every affine isometric action of $G$ on $X$ has a fixed point*

or equivalently,

$$H^1(G, \pi) = 0$$

for every isometric representation $\pi$ of $G$ on $X$.

This is much more more difficult than for $L_2$, even when $X = L_p$. 
Generalizing (T) to other Banach spaces

$X$ – Banach space, reflexive ($X^{**} = X$)

Example: $L_p$ are reflexive for $1 < p < \infty$, not reflexive for $p = 1, \infty$.

We are interested in groups $G$ for which the following property holds:

*every affine isometric action of $G$ on $X$ has a fixed point*

or equivalently,

$$H^1(G, \pi) = 0$$

for every isometric representation $\pi$ of $G$ on $X$.

This is much more more difficult than for $L_2$, even when $X = L_p$. 
Previous results

Only a few positive results are known:

- $(T) \iff \text{fixed points on } L_p \text{ and any subspace, } 1 < p \leq 2$

- $(T) \implies \exists \varepsilon = \varepsilon(G) \text{ such that fixed points always exists on } L_p \text{ for } p \in [2, 2 + \varepsilon) \text{ (Fisher – Margulis 2005)}$
  (a general argument, $\varepsilon$ unknown)

- lattices in products of higher rank simple Lie groups for $X = L_p$ for all $p > 1$
  (Bader – Furman – Gelander – Monod, 2007)

- $\text{SL}_n(\mathbb{Z}[x_1, \ldots, x_k])$ for $n \geq 4$; $X = L_p$ for all $p > 1$ (Mimura, 2010)
  [both use a representation-theoretic Howe-Moore property]

- Gromov’s random groups containing expanders for $X = L_p$, $p$-uniformly convex Banach lattices for all $p > 1$ (Naor – Silberman, 2010)
  [Some of these arguments also apply to Shatten $p$-class operators]
Previous results

Only a few positive results are known:

1. \((T) \iff \text{fixed points on } L_p \text{ and any subspace, } 1 < p \leq 2\)

2. \((T) \implies \exists \varepsilon = \varepsilon(G) \text{ such that fixed points always exists on } L_p \text{ for } p \in [2, 2 + \varepsilon) \) (Fisher – Margulis 2005)
   
   (a general argument, \(\varepsilon\) unknown)

3. Lattices in products of higher rank simple Lie groups for \(X = L_p\) for all \(p > 1\)
   (Bader – Furman – Gelander – Monod, 2007)

4. \(\text{SL}_n(\mathbb{Z}[x_1, \ldots x_k])\) for \(n \geq 4; X = L_p\) for all \(p > 1\) (Mimura, 2010)
   
   [both use a representation-theoretic Howe-Moore property]

5. Gromov’s random groups containing expanders for \(X = L_p, p\)-uniformly convex Banach lattices for all \(p > 1\) (Naor – Silberman, 2010)
   
   [Some of these arguments also apply to Shatten \(p\)-class operators]
Some groups with property (T) admit fixed point free actions on certain $L_p$.

- $Sp(n, 1)$ admits fixed point free actions on $L_p(G)$, $p \geq 4n + 2$ (Pansu 1995)

- hyperbolic groups admit fixed point free actions on $\ell_p(G)$ for $p \geq 2$ sufficiently large (Bourdon and Pajot, 2003)

- for every hyperbolic group $G$ there is a $p > 2$ (sufficiently large) such that $G$ admits a metrically proper action by affine isometries on $\ell_p(G \times G)$ (Yu, 2006)
Consider e.g. a hyperbolic group $G$ with property (T).

Let $\mathcal{P} = \{p : H^1(G, \pi) = 0 \text{ for every isometric rep. } \pi \text{ on } L_p\}$

The only thing we know about $\mathcal{P}$ is that it is open.

Question: Is $\mathcal{P}$ connected?
Values of $p$ (after C. Drutu)

Consider e.g. a hyperbolic group $G$ with property (T).

Let $P = \{ p : H^1(G, \pi) = 0 \text{ for every isometric rep. } \pi \text{ on } L_p \}$

The only thing we know about $P$ is that it is open.

Question: Is $P$ connected?
Spectral conditions for property (T)

Based on the work of Garland, used by Ballmann – Świątkowski, Dymara – Januszkiewicz, Pansu, Żuk ...

**Theorem (General form of the theorems)**

Let $G$ be acting properly discontinuously and cocompactly on a 2-dimensional contractible simplicial complex $K$ and denote by $\lambda_1(x)$ the smallest positive eigenvalue of the discrete Laplacian on the link of a vertex $x \in K$. If

$$\lambda_1(x) > \frac{1}{2}$$

for every vertex $x \in K$ then $G$ has property (T).
Link graphs on generating sets

$G$ - group, $S = S^{-1}$ - finite generating set of $G$, $e \not\in S$.

Definition

The link graph $\mathcal{L}(S) = (V, E)$ of $S$:

- vertices $V = S$,
- $(s, t) \in S \times S$ is an edge $\in E$ if $s^{-1}t \in S$.

Laplacian on $\ell_2(S, \text{deg})$:

$$\Delta f(s) = f(s) - \frac{1}{\text{deg}(s)} \sum_{t \sim s} f(t)$$

$\lambda_1$ denotes the smallest positive eigenvalue

Theorem (Žuk)

If $\mathcal{L}(S)$ connected and $\lambda_1(\mathcal{L}(S)) > \frac{1}{2}$ then $G$ has property (T).
Let $Mf = \sum_{x \in V} f(x) \frac{\deg(x)}{\#E}$ be the mean value of $f$

**Definition (p-Poincaré inequality for the norm of $X$)**

For every $f : V \rightarrow X$ in an $X$-Banach space, $p \geq 1$, $\Gamma = (V, E)$ - finite graph.

\[
\left( \sum_{s \in V} \|f(s) - Mf\|_X^p \deg(s) \right)^{1/p} \leq \kappa \left( \sum_{(s,t) \in E} \|f(s) - f(t)\|_X^p \right)^{1/p}.
\]

The inf of $\kappa$ for $L(S)$, giving the optimal constant, is denoted $\kappa_p(S, X)$

The classical $p$-Poincaré inequality when $X = \mathbb{R}$.

1. $\kappa_1(S, \mathbb{R}) \simeq$ Cheeger isoperimetric const
2. $\kappa_2(S, \mathbb{R}) = \sqrt{\lambda_1^{-1}}$;
3. for $1 \leq p < \infty$ we have $\kappa_p(S, L_p) = \kappa_p(S, \mathbb{R})$
Let $Mf = \sum_{x \in V} f(x) \frac{\deg(x)}{\#E}$ be the mean value of $f$.

**Definition (p-Poincaré inequality for the norm of $X$)**

Let $X$ be a Banach space, $p \geq 1$, $\Gamma = (V, E)$ - finite graph. For every $f : V \to X$:

$$\left( \sum_{s \in V} \|f(s) - Mf\|_X^p \deg(s) \right)^{1/p} \leq \kappa \left( \sum_{(s,t) \in E} \|f(s) - f(t)\|_X^p \right)^{1/p}.$$

The inf of $\kappa$ for $L^p(S)$, giving the optimal constant, is denoted $\kappa_p(S, X)$.

The classical $p$-Poincaré inequality when $X = \mathbb{R}$:

1. $\kappa_1(S, \mathbb{R}) \approx$ Cheeger isoperimetric const
2. $\kappa_2(S, \mathbb{R}) = \sqrt{\lambda_1^{-1}}$
3. for $1 \leq p < \infty$ we have $\kappa_p(S, L_p) = \kappa_p(S, \mathbb{R})$
The Main Theorem

Given $p > 1$ denote by $p^*$ the adjoint index: \[ \frac{1}{p} + \frac{1}{p^*} = 1. \]

Main Theorem

Let $X$ be a reflexive Banach space, $G$ a group generated by $S$ as earlier. If for some $p > 1$

\[ \max\left\{ 2^{-\frac{1}{p}} \kappa_p(S, X), 2^{-\frac{1}{p^*}} \kappa_{p^*}(S, X^*) \right\} < 1 \]

then

\[ H^1(G, \pi) = 0 \]

for any isometric representation $\pi$ of $G$ on $X$.

Remark 1. By reflexivity, the same conclusion holds for actions on $X^*$

Remark 2. The roles of the two constants in the proof are different.
Sketch of proof

Difficulty: lack of self-duality when $X$ is not a Hilbert space

For any Hilbert space $\mathcal{H}^* = \mathcal{H}$, every subspace has an orthogonal complement

For $Y \subseteq X$ Banach spaces, $Y$ might not have a complement,

$$Y^* = X^*/\text{Ann}(Y)$$

with the quotient norm

$$\| [y] \|_{Y^*} = \inf_{x \in \text{Ann}(Y)} \|y - x\|_{Y^*}$$

Example: Every separable Banach space is a quotient of $\ell_1(\mathbb{N})$. 
Sketch of proof

Difficulty: lack of self-duality when $X$ is not a Hilbert space

For any Hilbert space $\mathcal{H}^* = \mathcal{H}$, every subspace has an orthogonal complement.

For $Y \subseteq X$ Banach spaces, $Y$ might not have a complement,

$$Y^* = X^*/\text{Ann}(Y)$$

with the quotient norm

$$\| [y] \|_{Y^*} = \inf_{x \in \text{Ann}(Y)} \|y - x\|_{Y^*}$$

Example: Every separable Banach space is a quotient of $\ell_1(\mathbb{N})$. 
$X^*$ is equipped with the adjoint representation, $\overline{\pi} g = \pi^* g^{-1}$.

We want to show that $\delta$ is onto.

This is equivalent to $\delta^*$ having closed range.

The first step is to identify $(\text{cochains}_\pi)^*$. 

$\delta v(s) = v - \pi_s v$
Theorem

If $X$-reflexive, $\pi$ – isometric representation. Then

$$(\text{cochains}_\pi)^*$$ is isometrically isomorphic to $\text{cochains}_{\bar{\pi}}$.

Sketch of proof: we view $\text{cochains}_\pi$ as a complemented subspace of a larger Banach space, $\mathcal{Y}$:

$$\text{cochains}_\pi \oplus \mathcal{Z} = \mathcal{Y},$$

$$\text{cochains}_{\bar{\pi}} \oplus \overline{\mathcal{Z}} = \mathcal{Y}^*.$$

Compute to get

$$(\text{cochains}_\pi)^* = \frac{\mathcal{Y}^*}{\overline{\mathcal{Z}}} \text{ isomorphic to } \text{cochains}_{\bar{\pi}}$$

This is not sufficient – we need an isometric isomorphism.
Theorem

If $X$-reflexive, $\pi$ – isometric representation. Then

$$(\text{cochains}_\pi)^*$$ is isometrically isomorphic to $\text{cochains}_{\overline{\pi}}$.

Sketch of proof: we view cochains $\pi$ as a complemented subspace of a larger Banach space, $\mathcal{Y}$:

$$\text{cochains}_\pi \oplus \mathcal{Z} = \mathcal{Y},$$

$$\text{cochains}_{\overline{\pi}} \oplus \overline{\mathcal{Z}} = \mathcal{Y}^*.$$  

Compute to get

$$(\text{cochains}_\pi)^* = \mathcal{Y}^*/\overline{\mathcal{Z}} \text{ isomorphic to } \text{cochains}_{\overline{\pi}}$$

This is not sufficient – we need an isometric isomorphism.
We need an additional geometric condition.

**Theorem**

If \( \pi \) is isometric then

\[
\|c - x\|_Y = \|c + x\|_Y,
\]

for \( c \in \text{cochains}_\pi \), \( x \in \overline{Z} \)

This is an orthogonality-type condition

This implies: \( \delta^* = 2M \), the mean value operator
We need an additional geometric condition.

**Theorem**

If $\pi$ is isometric then

$$\|c - x\|_Y = \|c + x\|_Y,$$

for $c \in \text{cochains}_{\pi}$, $x \in \overline{Z}$

This is an orthogonality-type condition

This implies: $\delta^* = 2M$, the mean value operator
Thm 1. If $2^{1/p^*}\kappa_p(S, X) < 1$ then $\delta^* i^* \bar{i}$ has closed range.

Thm 1 follows from a sequence of inequalities.

It implies $\delta^*$ has closed range on image of $i^* \bar{i}$.

The same argument for the other inequality gives:

$2^{1/p} \kappa_p(S, X) < 1$ then $\delta^* \bar{i}^* i$ has closed range

$\Rightarrow \bar{i}^* i$ has closed range

$\Rightarrow i^* \bar{i}$ is surjective
Thm 1. If $2^{1/p^*} \kappa_{p^*}(S, X) < 1$ then $\delta^* i^* \bar{i}$ has closed range.

Thm 1 follows from a sequence of inequalities.

It implies $\delta^*$ has closed range on image of $i^* \bar{i}$.

The same argument for the other inequality gives:

$2^{1/p} \kappa_p(S, X) < 1$ then $\overline{\delta^* i^*} \ i \ has \ closed \ range$

$\Rightarrow \ i^* \bar{i} \ has \ closed \ range$

$\Rightarrow \ i^* \bar{i} \ is \ surjective$
We want to apply this to $X = L_p, \ p > 2$

Desired outcome: vanishing of cohomology for all $L_p, \ p \in [2, 2 + c)$, where we can say something about $c$.

**Remark.** This cannot be improved, in the sense that we cannot expect vanishing for all $2 < p < \infty$:

1. $p$-Poincaré constants $> 1$ for $p$ sufficiently large
2. the main theorem applies to hyperbolic groups

Difficulties: estimating $p$-Poincaré constants is a hard problem in analysis when $p \neq 1, 2, \infty$. 
Cartwright, Młotkowski and Steger defined finitely presented groups $G_q$ where $q = k^n$ for $k$ - prime such that

$$\mathcal{L}(S) = \text{incidence graph of a projective plane over a finite field}$$

In the 60ies Feit and Higman computed spectra of such incidence graphs, which implies

$$2^{-\frac{1}{2}}\kappa_2(S, \mathbb{R}) = \sqrt{\left(1 - \frac{\sqrt{q}}{q + 1}\right)^{-1}} \quad \rightarrow \quad \frac{1}{\sqrt{2}}.$$  

We now want to estimate $\kappa_p(S, L_p)$ for these graphs.
Estimating the $p$-Poincaré constant

When $p \geq 2$, in finite dimensional spaces: $\|f\|_{\ell_p^n} \leq \|f\|_{\ell_2^n} \leq n^{1/2 - 1/p} \|f\|_{\ell_p^n}$.

- $\# V = 2(q^2 + q + 1)$,
- $\# E = 2(q^2 + q + 1)(q + 1)$
- $\deg(s) = q + 1$ for every $s \in S$

Similarly for $p^* < 2$.

**Theorem**

For each $q$=power of a prime we have

$$H^1(G_q, \pi) = 0$$

for any isometric representation $\pi$ of $G_q$ on any $L_p$ for all

$$2 \leq p < \frac{2 \ln \left(2(q^2 + q + 1)\right)}{\ln \left(2(q^2 + q + 1)\right) - \ln \sqrt{2 \left(1 - \frac{\sqrt{q}}{q + 1}\right)}}.$$
Numerical values of $p$

We have $2 \leq p \leq 2.106$ and $p \to 2$ as $q \to \infty$. 
Numerical values of $p$

We have $2 \leq p \leq 2.106$ and $p \to 2$ as $q \to \infty$. 
We have $2 \leq p \leq 2.106$ and $p \to 2$ as $q \to \infty$. 
Hyperbolic groups

\[ \dot{\text{Zuk}} \text{ used the spectral conditions to prove that many hyperbolic groups have (T).} \]

Because of randomness we cannot hope for explicit bounds on \( p \).

**Theorem (\dot{\text{Zuk}})**

A group \( G \) in the density model for \( \frac{1}{3} < d < \frac{1}{2} \) is, with probability 1, of the form

\[ H \rightarrow \Gamma \subseteq_{f.i.} G, \]

where \( G \) is hyperbolic and \( H \) has a link graph with \( 2^{-1/2} \kappa_2(S, \mathbb{R}) < 1 \).

Vanishing of cohomology for all isometric representations on \( L_p \) is passed on to quotients and by finite index subgroups, just as (T) is.

**Corollary**

*With probability 1, the main theorem applies to hyperbolic groups.*
Conformal dimension

**Definition (Pansu)**

$G$ hyperbolic, $d_V$ - any visual metric on $\partial G$.

$$\text{confdim}(\partial G) = \inf \{ \dim_{\text{Haus}}(\partial G, d) : d \text{ quasi-conformally equiv. to } d_V \}.$$ 

$\text{confdim}(\partial G)$ is a q.i. invariant of $G$, extremely hard to estimate.

Bourdon-Pajot, 2003: $G$ acts without fixed points on $\ell_p(G)$ for $p \geq \text{confdim}(\partial G)$

**Corollary.** The main theorem gives lower bounds on $\text{confdim}(\partial G)$.

**Corollary**

Let $G$ be a hyperbolic group. Then for $p > \text{confdim}(\partial G)$ we have

$$2^{-1/p} \kappa_p(S, X) \geq 1 \quad \text{or} \quad 2^{-1/p^*} \kappa_{p^*}(S, X^*) \geq 1.$$
Navas studied rigidity properties of diffeomorphic actions on the circle. Vanishing of cohomology for $L_p$ for $p > 2$ improves the differentiability class in his result.

**Corollary**

Let $q$ be a power of a prime number and $G_q$ be the corresponding $\tilde{A}_2$ group. Then every homomorphism $h : G \rightarrow \text{Diff}^{1+\alpha}(S^1)$ has finite image for

$$\alpha > \frac{1}{2} \ln(2(q^2 + q + 1)(q + 1)) - \ln(2) - \ln\left(\sqrt{1 - \frac{\sqrt{q}}{q + 1}}\right) \frac{\ln(q^2 + q + 1) + \ln(q + 1)}{\ln(q^2 + q + 1) + \ln(q + 1)}.$$ 

Here, $\alpha$ is strictly less than $\frac{1}{2}$, improving for these groups the original differentiability class.
One more application to finite dimensional representations allows to estimate eigenvalues of the $p$-Laplacian on finite quotients of groups (some previous estimates using different techniques in joint work with R.I. Grigorchuk)

Q: Do $\widetilde{A}_2$ groups admit an affine isometric action on $L_p$, without fixed points or metrically proper, for $p$ sufficiently large?