Simplicial volume and CAT(-1) filling

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1. Simplicial volume

- Let X be a space, and $c = \sum_{i=1}^{k} a_i \sigma_i$, $a_i \in \mathbb{R}$, a singular real chain. Define the ℓ^1-norm by $||c||_1 = \sum_i |a_i|$.
  For $\omega \in H_*(X; \mathbb{R})$, the simplicial norm is defined by

  $$||\omega|| = \inf\{||z||_1 \mid \partial z = 0, [z] = \omega\}.$$

  This is a semi-norm.
- If M is a closed oriented n-manifold, the simplicial volume is defined by

  $$||M|| = ||[M]||,$$

  where $[M]$ is the fundamental class. If M is non-orientable, define $||M|| = ||M'||/2$ for the double cover M'.

2. Motivation – minimal volume

Let M be a closed manifold.
Want to find an extremal Riemannian metric g on M, e.g.,
$\mathrm{vol}(M,g)$ is smallest with $|K_g| \leq 1$.
$\mathrm{vol}(cM) \to 0$ as $c \to 0$, but $K \to \infty$ unless $K = 0$.
Gromov defined the minimal volume of M by

$$\mathrm{Minvol}(M) = \inf_{|K_g| \leq 1} \mathrm{vol}(M,g) \geq 0$$

Question When is Minvol $M > 0$? Is Minvol M attained ? What is an extremal metric ?

Example

If $\dim M = 2$, then by Gauss-Bonnet thm, for any metric g,

$$\int_M K dv = 2\pi\chi(M)$$

It follows $|2\pi\chi(M)| \leq \int_M 1 dv = \mathrm{vol}\, M$ if $|K| \leq 1$, therefore
Minvol $M = 2\pi|\chi(M)|$, and extremal metrics satisfy $K = -1$ if $\chi(M) < 0$.

3. Lower bound of Minvol M

Theorem (Gromov)

For an n-manifold M,

$$C_n||M|| \leq \text{Minvol}(M),$$

where $C_n > 0$ is a constant which depends only on the dimension n.

- Question: When $||M|| > 0$
- For a continuous map $f : M^n \to N^n$,

  $$||M|| \geq |deg(f)|||N||$$

  Therefore if there is $f : M \to M$ with $deg(f) \neq 0, \pm 1$, then $||M|| = 0$. For example, $||S^n|| = 0$, $||T^n|| = 0$.

4. $K < 0$ implies $||M|| > 0$

By "straightening" of a simplex,

Theorem (Gromov-Thurston)

If M^n is a closed R-manifold with $K \leq -1$, then

$$\mathrm{vol}\, M \leq c_n ||M||,$$

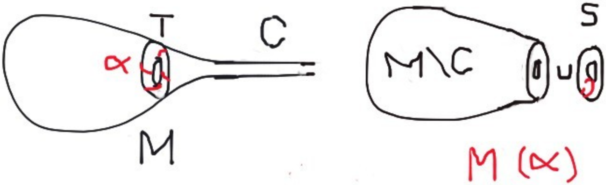
where c_n is a constant which depends only on the dimension n.
In particular $0 < ||M||$.
Moreover, if $K = -1$, then $\mathrm{vol}\, M = T_n ||M||$, where $0 < T_n < \infty$ is the sup of the volume of a geodesic n-simplex in $\mathbb{H}^n$.

- Combined with the previous thm, if $K = -1$, then $C_n \mathrm{vol}\, M / T_n \leq \mathrm{Minvol}\, M$.
- By now, for a closed hyperbolic manifold M, we know $\mathrm{Minvol}\, M = \mathrm{vol}(M)$ and the extremal metric is hyperbolic (Besson-Courtois-Gallot).

5. Dehn filling

Let M be a non-compact hyperbolic 3-manifold of finite volume. M has finitely many cusps. For simplicity, let's assume it has only one cusp, $C = T^2 \times [0, \infty)$.

Let $\alpha \subset T^2$ be a simple (geodesic) loop. We remove C from M and glue a solid torus along T^2 to kill $[\alpha] \in \pi_1(T) \simeq \mathbb{Z}^2$. We get a closed manifold $M(\alpha)$. This is Dehn filling.



6. Hyperbolic Dehn filling, 2π-theorem

Theorem (Thurston)

$M(\alpha)$ has a hyperbolic structure except for finitely many α (in terms of $\pi_1(\alpha)$).
If $M(\alpha)$ is hyperbolic, then $\operatorname{vol} M(\alpha) < \operatorname{vol} M$.

Thurston deforms the representation $\pi_1(M) \to PSL(2, \mathbb{C})$ such that the image of $[\alpha] = 1$, and obtain a representation $\pi_1(M(\alpha)) \to PSL(2, \mathbb{C})$.

Theorem (Gromov, 2π-theorem)

$M(\alpha)$ has a Riemannian metric of negative curvature if $\ell(\alpha) > 2\pi$.

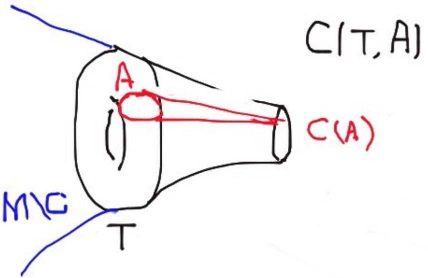
Gromov extends the hyperbolic metric on $M \backslash C$ to the solid torus S, and obtain a metric of negative curvature on $M(\alpha)$.
For each M, there are infinitely many $\pi_1(M(\alpha))$. They approximate M, therefore the diameter $\to \infty$ although the volume is bounded from above and the sectional curvature is pinched between -1 and $-a^2$ for some $a > 0$.

7. Filling in dim ≥ 4

Assume $d = \dim \geq 4$. Let M be a hyperbolic d-manifold of finite volume with toral cusps (let's assume only one cusp) $C = T^{n-1} \times [0, \infty)$. Let $A^{n-2} \subset T$ be a flat subtorus.
Topologically $T = S^1 \times A$.
Let $C(A)$ be the cone over A. We define a partial cone by

$$C(T, A) = C(A) \times S^1$$

Remove C from M and glue $C(T, A)$, and obtain $M(A)$, Dehn



filling.

- The Dehn-filling $M(A)$ is not a manifold, only a pseudo-manifold. The singular set is S^1 (the cone points).
- For $1 \leq \dim A < d - 2$, we can also define the partial cone $C(T, A)$, and the Dehn filling $M(A)$ similarly. The singular set is $T^{d-1-\dim A}$.
- The pair $(\pi_1(M \backslash C), \pi_1(T))$ is relatively hyperbolic. In $M(A)$, we kill $\mathbb{Z}^{n-2} \simeq \pi_1(A) < \pi_1(T) \simeq \mathbb{Z}^{n-1}$, therefore $\pi_1(M(A))$ has a chance to be word-hyperbolic (cf. Grove-Manning-Osin).

8. CAT(-1) filling

We generalize 2π-theorem.

Theorem (Manning-F)

If a shortest non-trivial loop on A has length $> 2\pi$, then we can put a metric on $M(A)$ which is locally CAT(-1).

- We use warped metrics following Gromov. The metric is Riemannian except for the singular set.
- $\pi_1(M(A))$ is word-hyperbolic, and we obtain a family of interesting examples: torsion-free, $\dim G = \dim M$, not Poincare duality groups.
- If T satisfies the 2π-condition, one can cone off T and put locally CAT(-1) metric on $M(T)$(Mosher-Sageev).
- Even if T is small (i.e. does not satisfy the 2π-condition) , we can always find A which satisfies the 2π condition.
- If $\dim A < \dim M - 2$, we can still put a locally CAT(0) metric on $M(A)$ (cf. Schroeder when $M(A)$ is a manifold, i.e. $\dim A = 1$)

9. Upper bound on $||M(A)||$

- $M(A)$ is not a manifold, and there is no canonical metric for $\operatorname{vol} M(A)$, but we can define $||M(A)||$ for a pseudo-manifold by $||[M(A)]||$.
- Remember that in dim $= 3$, $\operatorname{vol} M(\alpha) < \operatorname{vol} M$, therefore $||M(\alpha)|| < ||M||$.

Theorem (Manning-F)

Let M be a hyperbolic d-manifold of finite volume with toral cusps, $d \geq 3$. If $A^{d-2} \subset T^{d-1} \subset M^d$ satisfies 2π-condition, then

$$||M(A)|| \leq ||M||$$

We don't know if $||M(A)|| < ||M||$.

10. Questions on finiteness

Our theorem raises a question. Define for d, V,

$$C(d,V) = \{\pi_1(M) \quad | \quad M : \textit{a closed Riem.mfd}, \\ \dim = d,\ ||M|| \le V,\ (-1 \le)K < 0\}$$

Question: $\sharp C(d,V) = \infty$?

- Finite if $d = 2$. $||M||$ grows linearly on the genus.
- If $d = 3$, then ∞ by hyperbolic Dehn fillings. vol $M(\alpha) <$ vol M.
- Unknown if $d \ge 4$.
  - If we replace $||M|| \le V$ by vol $M \le V$, then finite; since it follows $\text{diam}(M) \le C(V)$ by Gromov, then a finiteness thm by Cheeger applies to M.
  - Or, if we additionally assume $-1 \le K \le -a^2 < 0$ (pinching), and define a subclass $C(d, V, a)$, then finite; since we then have vol $M \le V_1(V, d, a)$ from $||M|| \le V$.

- If we allow pseudo-manifolds with locally CAT(-1) metrics, then ∞ by our theorem for all $d \geq 4$, since $M(A)$ is locally CAT(-1) and $||M(A)|| \leq ||M||$ for all A.
- Approach to $C(d, V)$:
  - To show finiteness by contradiction, let M_i be a sequence, and let it converge to M_∞, then analyze M_∞.
    Don't know how to use $||M_i|| \leq V$.
  - If we expect ∞, since pinching $-1 \leq K \leq -a^2 < 0$ gives finiteness, we need a sequence of manifolds M_i of negative curvature which does not allow pinching.
    Only one example is known using "branch coverings" (Gromov-Thurston), but in that example $||M_i|| \to \infty$.
    Need a new example to show ∞.