Generalized Cayley graphs for fundamental groups of one-dimensional spaces

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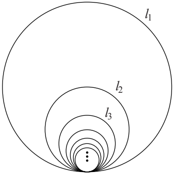
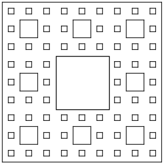
Fundamental groups of general 1-dimensional Peano continua are notoriously difficult to analyze:



Hawaiian Earring



Sierpiński carpet

Menger curve

Theorems [Eda 2002-2010]

Let X and Y be 1-dimensional Peano continua.

A. $\pi_1(\textit{Sierpiński carpet}) \not\hookrightarrow \pi_1(\textit{Hawaiian Earring})$,

$\pi_1(\textit{Menger curve}) \not\hookrightarrow \pi_1(\textit{Hawaiian Earring})$.

B. If X and Y are **not** locally simply-connected at **any** point and if $\pi_1(X) \cong \pi_1(Y)$, then X and Y are homeomorphic.

C. If $\pi_1(X) \cong \pi_1(Y)$, then X and Y are homotopy equivalent.

Theorems [Curtis-Fort 1959]

A. Suppose X is a 1-dimensional Peano continuum.
Then $\pi_1(X)$ is free $\Leftrightarrow$ X is (semi)locally simply-connected
$\Leftrightarrow$ $\pi_1(X)$ is finitely presented
$\Leftrightarrow$ $\pi_1(X)$ is countable

B. Suppose X is a 1-dimensional separable metric space.
Then every finitely generated subgroup of $\pi_1(X)$ is free.

C. The homotopy class of every loop in a 1-dimensional separable metric space has an essentially unique shortest representative.

Question [Cannon-Conner 2006]

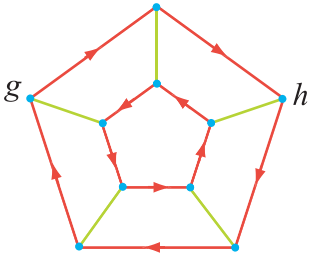
Given a 1-dimensional path-connected compact metric space X, is there a tree-like object that might be considered the topological Cayley graph for $\pi_1(X)$?

Solution

- A combinatorial description of an $\mathbb{R}$-tree (i.e. a uniquely arcwise connected geodesic space),
- along with a combinatorial description of $\pi_1(X)$, which
- to the extend possible, functions like a Cayley graph for $\pi_1(X)$

Functionality of a classical Cayley graph:

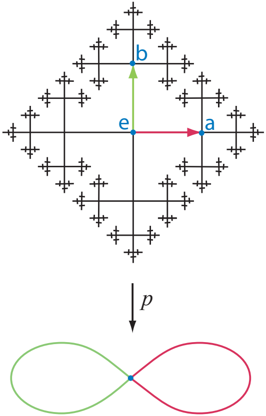
$$G = \langle a, b \mid a^5 = e,\ b^2 = e,\ ab = ba^{-1} \rangle$$



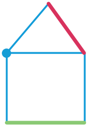
$$g^{-1}h = aba^{-1}b$$
$$g^{-1}h = aa$$
$$g^{-1}h = baaab$$

- There is a natural distance based on word length: $d(g,h) = 2$
- G acts on the Cayley graph by graph automorphism
- G acts freely and transitively on the vertex set

The tame case:



$X=$ 1-dimensional simplicial complex

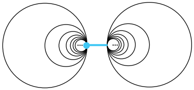
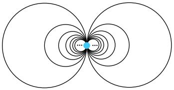


Collapsing all translates of a maximal tree in the universal covering space yields a Cayley graph for the free fundamental group $\pi_1(X)$

Obstacles

In general, we are facing the following obstacles:

- $\pi_1(X)$ might be uncountable
- There might not be a universal covering space
- Collapsing contractible subsets of X might change $\pi_1(X)$

$\pi_1($  $) \ncong \pi_1($  $)$

Theorem [Curtis-Fort '59, Eda-Kawamura '98]

Let X be a 1-dimensional separable metric space, or
let X be a 1-dimensional compact Hausdorff space.
Then the natural homomorphism $\varphi : \pi_1(X) \hookrightarrow \check{\pi}_1(X)$ is injective.

Suppose $X = \varprojlim \left(X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow} X_3 \overset{f_3}{\leftarrow} \cdots \right)$ with finite graphs X_n.

(Example: If X is the boundary of a CAT(0) 2-complex, we can take metric spheres for X_n and geodesic retraction for f_n.)

Then $\check{\pi}_1(X) = \varprojlim \left(\pi_1(X_1) \overset{f_{1\#}}{\leftarrow} \pi_1(X_2) \overset{f_{2\#}}{\leftarrow} \pi_1(X_3) \overset{f_{3\#}}{\leftarrow} \cdots \right)$.

$\check{\pi}_1(X)$ = coherent sequences of reduced words in free groups.

Problem: How do we identify the image of $\pi_1(X)$ in $\check{\pi}_1(X)$?

Theorem [F-Zastrow 2007]

Suppose X is a path-connected topological space.
If the natural homomorphism $\varphi : \pi_1(X) \hookrightarrow \check{\pi}_1(X)$ is injective, then there is a **generalized universal covering** $p : \widetilde{X} \to X$, that is, a continuous surjection characterized by the usual lifting criterion:

$\widetilde{X}$ = path-conn, loc path-conn, simply conn.

Y = path-conn, loc path-conn.

$$\begin{tikzcd} & (\widetilde{X}, \widetilde{x}) \arrow[d, "p"] \\ (Y, y) \arrow[r, "\forall f"'] \arrow[ur, dotted, "\exists ! g"] & (X, x) \end{tikzcd} \iff f_\#(\pi_1(Y, y)) = 1$$

- $\pi_1(X) \cong Aut(\widetilde{X} \overset{p}{\to} X)$ acts freely and transitively on $p^{-1}(x)$;
- If X is 1-dimensional separable metric, then $\widetilde{X}$ is an $\mathbb{R}$-tree. (There is no $\mathbb{R}$-tree metric for which $\pi_1(X)$ acts by isometry.)

Problem: How do we combinatorially describe $\widetilde{X}$?

General Assumption

Let X be a 1-dimensional path-connected compact metric space.

Express $X = \varprojlim \left(X_1 \overset{f_1}{\leftarrow} X_2 \overset{f_2}{\leftarrow} X_3 \overset{f_3}{\leftarrow} \cdots \right)$ with finite graphs X_n.

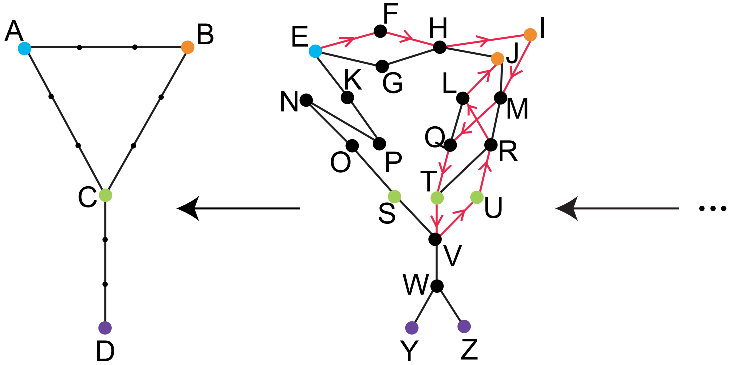
Arrange that $f_n : X_{n+1} \to X_n^*$ maps each edge linearly onto an edge of a regular subdivision X_n^* of X_n and fix a base point $(x_n)_n \in X$.

Let $\mathcal{W}_n = \left\{ \begin{matrix} \text{all words } v_1 v_2 \cdots v_k \text{ over the vertex alphabet of } X_n \\ \text{which describe paths starting at the base vertex } x_n \end{matrix} \right\}$

Set of **word sequences**: $\mathcal{W} = \varprojlim \left(\mathcal{W}_1 \xleftarrow{\phi_1} \mathcal{W}_2 \xleftarrow{\phi_2} \mathcal{W}_3 \xleftarrow{\phi_3} \cdots \right)$

where $\phi_n : \mathcal{W}_{n+1} \to \mathcal{W}_n$ is the natural combinatorial projection.

Example:

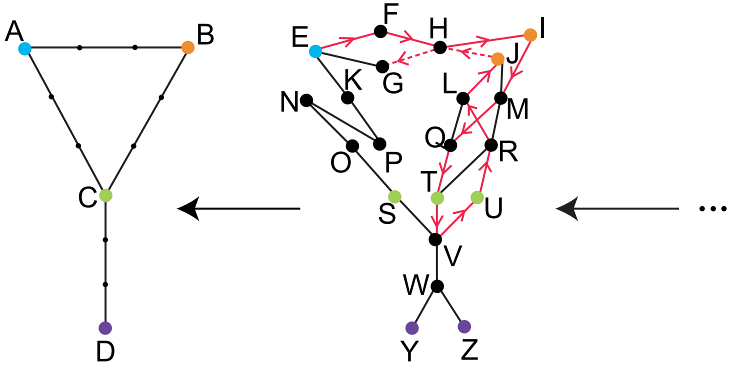


$\omega_2 =$ E F H I M Q T V U R L J

A B C C B

$\omega_1 = \phi_2(\omega_2) =$ ABCB

Example:



$\omega_2 =$ E F H I M Q T V U R L J H G

A B C C B

$\omega_1 = \phi_2(\omega_2) =$ ABCB/A

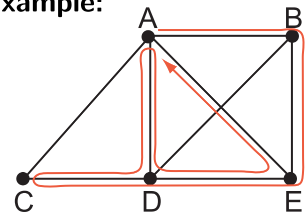
Formally, we allow for words of the form "$v_1 v_2 \cdots v_k / v_{k+1}$" in $\mathcal{W}_n$, unless this can eventually be avoided. ("$0.999\ldots = 1.000\ldots$")

Combinatorial Reduction:
Given a word ω_n, repeatedly apply the following replacements

$$\begin{array}{ccc} \ldots uvu \ldots & \rightsquigarrow & \ldots u \ldots \\ \ldots uv/u & \rightsquigarrow & \ldots u/v \end{array}$$

until this is no longer possible. Denote the resulting word by ω_n'.

Example:



$\omega_n = \mathrm{ABEDCDADEA}$

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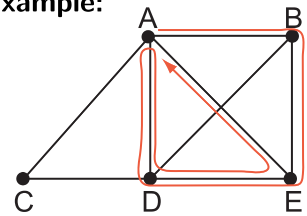

$$\begin{array}{rl} \omega_n = & \mathrm{ABEDCDADEA} \\ & \mathrm{ABED\quad ADEA} \end{array}$$

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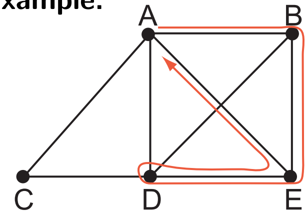

$$\begin{array}{rll} \omega_n = & \mathrm{ABEDCDADEA} & \\ & \mathrm{ABED} & \mathrm{ADEA} \\ & \mathrm{ABED} & \quad\;\; \mathrm{EA} \end{array}$$

Combinatorial Reduction:

Given a word ω_n, repeatedly apply the following replacements

$$\begin{array}{ccc} \ldots uvu \ldots & \rightsquigarrow & \ldots u \ldots \\ \ldots uv/u & \rightsquigarrow & \ldots u/v \end{array}$$

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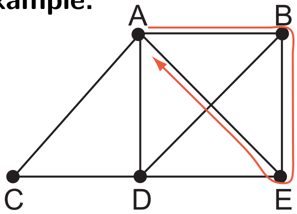


$$\begin{array}{rl} \omega_n = & \mathrm{ABEDCDADEA} \\ & \mathrm{ABED\quad ADEA} \\ & \mathrm{ABED\qquad\quad EA} \\ & \mathrm{ABE\qquad\qquad\ \ A} \end{array}$$

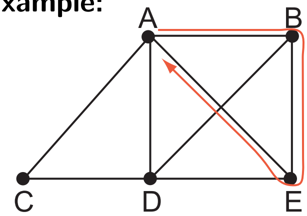
Combinatorial Reduction:

Given a word ω_n, repeatedly apply the following replacements

$$\begin{array}{ccc} \ldots uvu \ldots & \rightsquigarrow & \ldots u \ldots \\ \ldots uv/u & \rightsquigarrow & \ldots u/v \end{array}$$

until this is no longer possible. Denote the resulting word by ω'_n.

Example:



$$\begin{array}{rl} \omega_n = & \text{ABEDCDADEA} \\ & \text{ABED} \quad \text{ADEA} \\ & \text{ABED} \quad \quad \text{EA} \\ & \text{ABE} \quad \quad \quad \text{A} \end{array}$$

$$\omega'_n = \text{ABEA}$$

Let $\Omega_n = \{\text{words in } \mathcal{W}_n \text{ that start and end at } x_n\}$.

Then $\Omega'_n \cong \pi_1(X_n)$ under $g_n * h_n = (g_n h_n)'$ and

$$\check{\pi}_1(X) = \varprojlim \left(\Omega'_1 \xleftarrow{\phi'_1} \Omega'_2 \xleftarrow{\phi'_2} \Omega'_3 \xleftarrow{\phi'_3} \cdots \right).$$

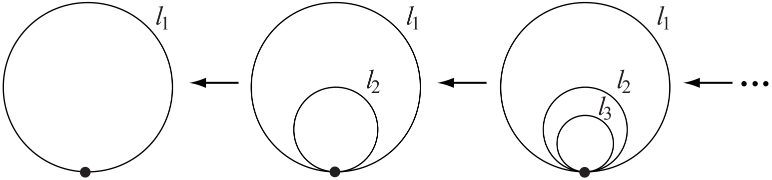
Recall the injective homomorphism $\varphi : \pi_1(X) \hookrightarrow \check{\pi}_1(X)$.

Proposition

An element of $(g_n)_n \in \check{\pi}_1(X)$ is in $\mathcal{G} = \varphi(\pi_1(X))$ if and only if $(g_n)_n$ is **locally eventually constant**, i.e., iff for every n the sequence $(\phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{k-1}(g_k))_{k>n}$ is eventually constant in Ω_n.

For $(g_n)_n \in \mathcal{G}$ we define the **stabilization** $\overleftarrow{(g_n)_n} = (\omega_n)_n \in \mathcal{W}$ by $\omega_n = \phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{k-1}(g_k)$ for sufficiently large k.

Example: An element which is **not** locally eventually constant.



$g_1 = \text{const.} \quad \leftarrow\!\shortmid \quad g_2 = l_1 l_2 l_1^{-1} l_2^{-1} \quad \leftarrow\!\shortmid \quad g_3 = l_1 l_2 l_1^{-1} l_2^{-1} l_1 l_3 l_1^{-1} l_3^{-1}$

$$(g_n)_n = (l_1 l_2 l_1^{-1} l_2^{-1} l_1 l_3 l_1^{-1} l_3^{-1} \cdots l_1 l_n l_1^{-1} l_n^{-1})_n \notin \mathcal{G}$$

□

$$\mathcal{G} = \{(g_n)_n \in \check{\pi}_1(X) \mid (g_n)_n \text{ is locally eventually constant}\}$$

$$\overleftarrow{\mathcal{G}} = \{(\omega_n)_n \in \mathcal{W} \mid (\omega_n)_n = \overleftarrow{(g_n)_n} \text{ with } (g_n)_n \in \mathcal{G}\}$$

Theorem 1

$\overleftarrow{\mathcal{G}}$ forms a group under $(\omega_n)_n * (\xi_n)_n = \overleftarrow{(\omega_n \xi_n)'_n}$ and $\overleftarrow{\mathcal{G}} \cong \pi_1(X)$.

This generalizes the description of $\pi_1(\textit{Sierpiński gasket})$ given by [Akiyama-Dorfer-Thuswaldner-Winkler 2009].

Dynamic word length:

We assign weights to the letters of $(\omega_n)_n \in \mathcal{W}$ recursively.

$$\omega_1 \overset{\phi_2}{\hookleftarrow} \omega_2 \overset{\phi_3}{\hookleftarrow} \omega_3 \overset{\phi_4}{\hookleftarrow} \cdots$$

Write $\omega_1 = v_1 v_2 \cdots v_s / *$ (either $\omega_1 = v_1 v_2 \cdots v_s$ or $\omega_1 = v_1 v_2 \cdots v_s / v_{s+1}$).
We assign the following weights to the letters $v_1, v_2, \ldots, v_s$.

letter	v_1	v_2	v_3	$\cdots$	v_s
weight	$1/2$	$1/4$	$1/8$	$\cdots$	$1/2^s$

(For words of the form $v_1 v_2 \cdots v_s / v_{s+1}$, we assign no weight to v_{s+1}.)

The weight scheme is then modeled on [Mayer-Overstegen 1990]:

Suppose the letters of $\omega_n = v_1 v_2 \cdots v_k/*$ have the following weights:

letter	v_1	v_2	v_3	$\cdots$	v_k
weight	a_1	a_2	a_3	$\cdots$	a_k

Assign weights to $\omega_{n+1} = u_1 u_2 \cdots u_m/*$ by inductively cutting ω_{n+1} into maximal substrings with $\phi_n(u_{i_t+1} u_{i_t+2} \cdots u_{i_{t+1}}) = v_{t+1}$.

$$\begin{array}{ccccccccc} v_1 & & v_2 & & v_3 & & \cdots & & v_k \\ \uparrow \phi_n & & \uparrow \phi_n & & \uparrow \phi_n & & & & \uparrow \phi_n \\ u_1 \cdots u_{i_1} & | & u_{i_1+1} \cdots u_{i_2} & | & u_{i_2+1} \cdots u_{i_3} & | & \cdots & | & u_{i_{k-1}+1} \cdots u_m \end{array}$$

u_1	u_2	$\cdots$	u_{i_1}	u_{i_1+1}	u_{i_1+2}	$\cdots$	u_{i_2}	u_{i_2+1}	u_{i_2+2}	$\cdots$
$a_1/2$	$a_1/4$	$\cdots$	$a_1/2^{i_1}$	$a_1/2^{i_1}$ $+a_2/2$	$a_2/4$	$\cdots$	$a_2/2^{i_2-i_1}$	$a_2/2^{i_2-i_1}$ $+a_3/2$	$a_3/4$	$\cdots$

For $\omega_n = v_1 v_2 \cdots v_k / *$, we define

$$|\omega_n| = weight(v_1) + weight(v_2) + \cdots + weight(v_k)$$

For $(\omega_n)_n \in \mathcal{W}$, we have $|\omega_1| > |\omega_2| > |\omega_3| > \cdots$ and define

$$\|(\omega_n)_n\| = \lim_{n \to \infty} |\omega_n|$$

We let Γ be the set of all **locally eventually constant** elements of

$$\varprojlim \left(\mathcal{W}_1' \xleftarrow{\phi_1'} \mathcal{W}_2' \xleftarrow{\phi_2'} \mathcal{W}_3' \xleftarrow{\phi_3'} \cdots \right).$$

There is a bijection $\overleftarrow{\varphi} : \widetilde{X} \to \overleftarrow{\Gamma}$ given by $[\alpha] \mapsto \overleftarrow{(r_n)_n}$.

The elements of $\widetilde{X}$ are homotopy classes $[\alpha]$ of paths in X and r_n = reduction of the word spelled by the projection of α into X_n.

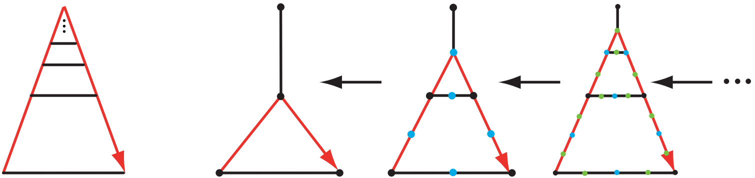
Given $\widetilde{x} \in \widetilde{X}$, there is a unique arc $\widetilde{\alpha}$ in $\widetilde{X}$ from the base point to $\widetilde{x}$. Let $\alpha = p \circ \widetilde{\alpha}$ be the projection into X. Then $\widetilde{x} = [\alpha]$.

One might try measuring the distance between two word sequences $(\omega_n)_n$ and $(\xi_n)_n$ of $\overleftarrow{\Gamma} \subseteq \mathcal{W}$ by

$$\left\|(\omega_n)_n\right\| + \left\|(\xi_n)_n\right\| - 2\left\|(\omega_n)_n \wedge (\xi_n)_n\right\|$$

where $\wedge$ denotes the (stabilized) combinatorial overlap function.

Example:



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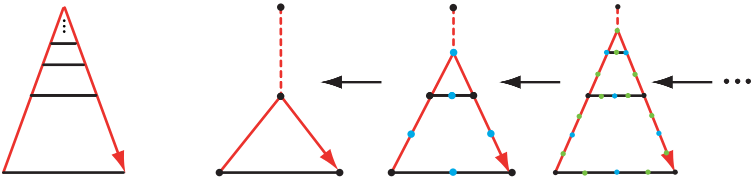


Completion: For $(\omega_n)_n \in \mathcal{W}$ we define a completion $\overline{(\omega_n)_n} \in \mathcal{W}$.

$$\begin{tikzcd} \overline{\overleftarrow{\Gamma}} \arrow[dr, dotted, "{'}"'] & \textit{bijections} & \overleftarrow{\Gamma} \arrow[ll, dotted, "\text{``}\overline{\phantom{x}}\text{''}"'] \\ & \Gamma \arrow[ur, dotted, "\text{``}\leftarrow\text{''}"'] & \end{tikzcd}$$

Define

$$d((\omega_n)_n, (\xi_n)_n) = \left\| \overline{(\omega_n)_n} \right\| + \left\| \overline{(\xi_n)_n} \right\| - 2 \left\| \overline{(\omega_n)_n} \wedge \overline{(\xi_n)_n} \right\|$$

Theorem 2

(a) The function d defines a metric on $\overleftarrow{\Gamma}$.

(b) The metric space $(\overleftarrow{\Gamma}, d)$ is an $\mathbb{R}$-tree.

(c) The function $\overleftarrow{\varphi} : \widetilde{X} \to \overleftarrow{\Gamma}$ is a homeomorphism.

Summary: Generalized Cayley Graph

- $\overleftarrow{\mathcal{G}} = \{\text{stabilized locally eventually constant closed sequences}\}$ forms a group under $(\omega_n)_n * (\xi_n)_n = \overleftarrow{(\omega_n\xi_n)'_n}$ and $\overleftarrow{\mathcal{G}} \cong \pi_1(X)$.
- $\overleftarrow{\Gamma} = \{\text{stabilized locally eventually constant sequences}\}$ is an $\mathbb{R}$-tree with radial word length metric

$$d((\omega_n)_n, (\xi_n)_n) = \left\|\overline{(\omega_n)_n}\right\| + \left\|\overline{(\xi_n)_n}\right\| - 2\left\|\overline{(\omega_n)_n} \wedge \overline{(\xi_n)_n}\right\|.$$

- Arcs in $\overleftarrow{\Gamma}$ whose endpoints $(\omega_n)_n$ and $(\xi_n)_n$ are in $\overleftarrow{\mathcal{G}}$ generate the labels for the word sequence $\overline{(\omega_n)_n^{-1} * (\xi_n)_n}$.
- $\overleftarrow{\mathcal{G}}$ acts freely and by homeomorphism on $\overleftarrow{\Gamma}$ via its natural action $(\omega_n)_n.(\xi_n)_n = \overleftarrow{(\omega_n\xi_n)'_n}$.

Theorem 3

Suppose the essential multiplicity of every letter is finite.
Then $\overleftarrow{\Gamma}/\overleftarrow{\mathcal{G}}$ is homeomorphic to X.

Essential Multiplicity:

We write $u_1 \overset{v}{\sim} u_2$ if
$$\begin{aligned} \phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{k-1}(u_1) &= v, \\ \phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{k-1}(u_2) &= v, \\ \phi_n \circ \phi_{n+1} \circ \cdots \circ \phi_{k-1}(\omega_k) &= v, \end{aligned}$$

for some word ω_k containing both letters u_1 and u_2.

Let $c_k(v)$ denote the number of $\overset{v}{\sim}$-equivalence classes at level k.

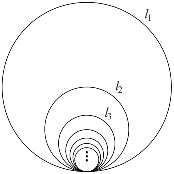
Then $c_{n+1}(v) \leqslant c_{n+2}(v) \leqslant c_{n+3}(v) \leqslant \cdots$

We call $\lim_{k\to\infty} c_k(v)$ the **essential multiplicity** of v.

Proof (of Theorem 3)**:** The essential multiplicity of every letter is finite $\Leftrightarrow$ X is locally path-connected $\Rightarrow$ $\widetilde{X}/\pi_1(X) = X$. $\square$

Natural Limitation: In general, there is no $\mathbb{R}$-tree metric for $\widetilde{X}$ such that the action of $\pi_1(X) \cong Aut(\widetilde{X} \overset{p}{\to} X)$ on $\widetilde{X}$ is by isometry.

Example: X = Hawaiian Earring.



Suppose every lift of a given loop l_i has the same length in $\widetilde{X}$.

Consider a loop $L = l_1^{n_1} l_2^{n_2} l_3^{n_3} \cdots$ with sufficiently large n_i.

Then the lift of L is an arc of infinite length.

In an $\mathbb{R}$-tree: length of an arc = distance between endpoints. $\square$