Partitions of unity in coarse geometry

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Abstract

I will outline how to use partitions of unity to explain **amenability via Følner sequences**, **Property A of Yu**, and **asymptotic dimension of Gromov**.

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1 Large scale versus small scale

We allow the values of metrics (pseudo-metrics, semi-metrics?) to be 0 or infinity.

Two metrics d and ρ are **uniformly equivalent** (the identity $id_X : (X, d) \to (X, \rho)$ is a **uniform homeomorphism**) provided $d(x_n, y_n) \to 0$ is equivalent to $\rho(x_n, y_n) \to 0$.

Two metrics d and ρ are **coarsely equivalent** (or **large scale uniformly equivalent**) provided $d(x_n, y_n) \to \infty$ is equivalent to $\rho(x_n, y_n) \to \infty$.

Basic example: Two word metrics on the same finitely generated group G are coarsely equivalent.

The simplest case is that of QI-equivalent metrics.

Definition. Given $f : X \to (Y, d_Y)$ one induces a new metric d_f on X defined by

$$d_f(x, y) = d_Y(f(x), f(y)).$$

Definition. f is **large scale uniform** if d_X is coarsely equivalent to $d_f + d_X$.

f is a **large scale embedding** if d_f is coarsely equivalent to d_X.

2 Uniform dimension versus asymptotic dimension

Defining the uniform dimension.

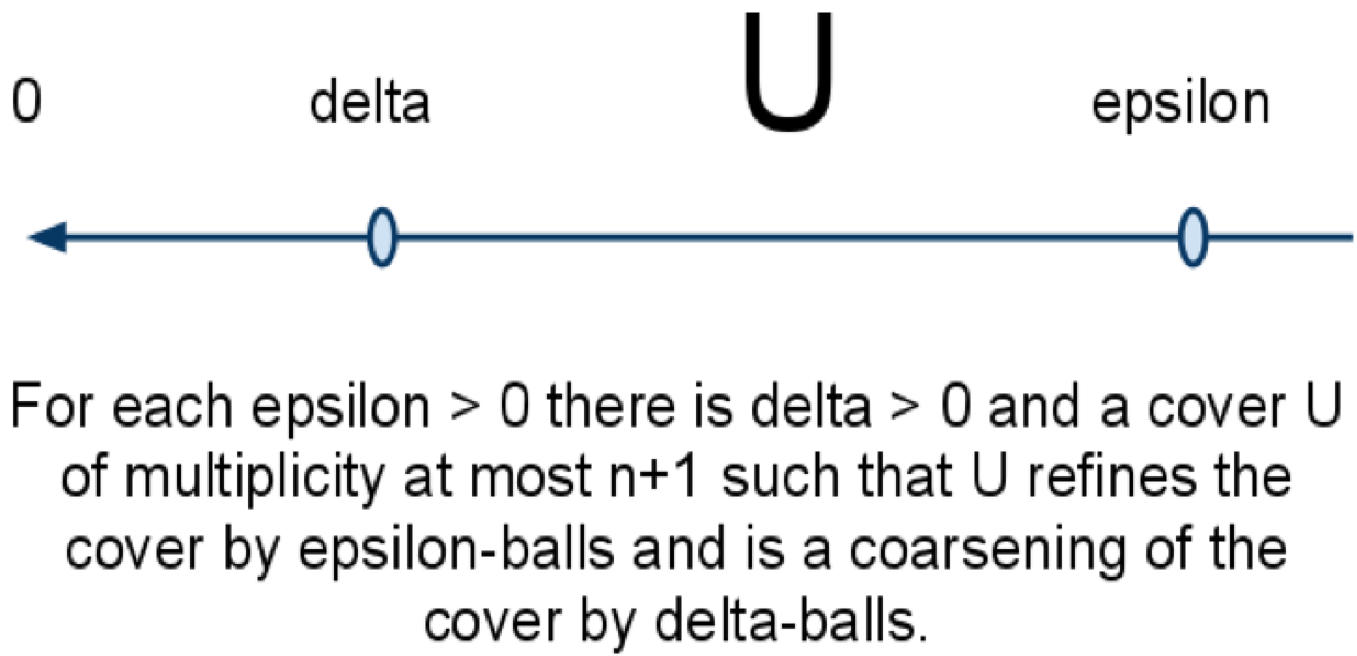


Figure 1: Uniform dimension

Defining the asymptotic dimension.

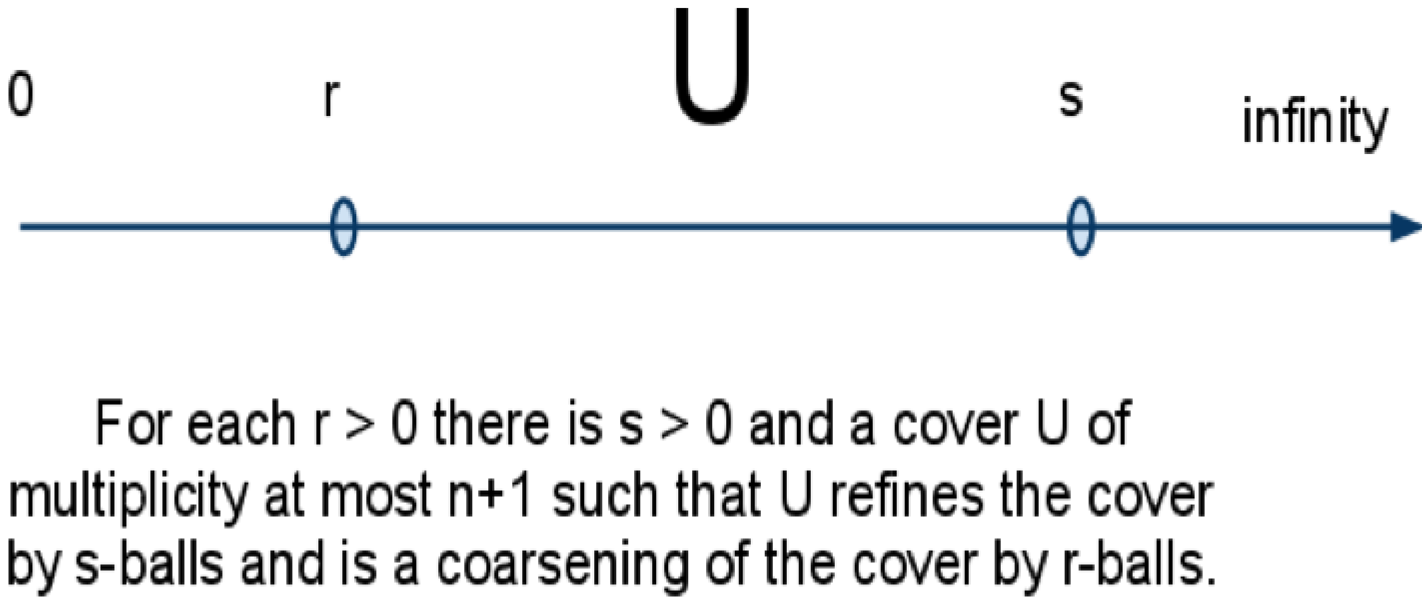


Figure 2: Asymptotic dimension

Philosophy: At scale r points are balls $B(x, r)$ of radius r.

Example: **Multiplicity at a point** changes to **multiplicity at scale** r: $m_r(x, \mathcal{U})$.

It is the number of elements of $\mathcal{U}$ containing $B(x, r)$.

Notice other authors use a **different definition of multiplicity at scale** r: they count all elements of $\mathcal{U}$ intersecting $B(x, r)$. The advantage of our definition is that we do not have to introduce the concept of the Lebesgue number of a cover: the condition $1 \leq m_r(x, \mathcal{U})$ for all $x \in X$ is equivalent to the Lebesgue number of $\mathcal{U}$ being at least r.

Alternative definition of uniform dimension: for each $\epsilon > 0$ there is an ϵ-bounded family $\mathcal{U}$ such that $1 \leq m_{\delta}(x, \mathcal{U}) \leq n + 1$ for all $x \in X$ for some $\delta > 0$.

Alternative definition of asymptotic dimension: for each $r > 0$ there is a uniformly bounded family $\mathcal{U}$ such that $1 \le m_r(x, \mathcal{U}) \le n + 1$ for all $x \in X$.

3 Review of paracompactness

Here is the correct definition of **paracompactness**:

For each open cover $\mathcal{U}$ of the topological space X there is a continuous partition of unity $\phi : X \to |K|_m$ such that the family $\{\phi^{-1}(st(v))\}_{v \in K^{(0)}}$ refines $\mathcal{U}$.

By $|K|_m$ we mean the subspace of $l_1(V)$ ($V = K^{(0)}$ is the set of vertices of the simplicial complex $|K|$) consisting of non-negative functions $f : V \longrightarrow [0,1]$ of finite support belonging to K such that $\sum_{v \in V} f(v) = 1$. The **star** of vertex v consists of all $f \in K$ such that $f(v) > 0$.

Large scale paracompactness.

Definition (Cencelj,JD,Vavpetić). X is **large scale paracompact** if for each $r > 0$ there is a $(\frac{1}{r}, \frac{1}{r})$-Lipschitz partition of unity $\phi : X \to |K|_m$ such that the family $\{\phi^{-1}(st(v))\}_{v \in K}$ is uniformly bounded and has positive r-multiplicity at each point of X.

4 Review of Venn diagrams

Venn diagrams at high school level

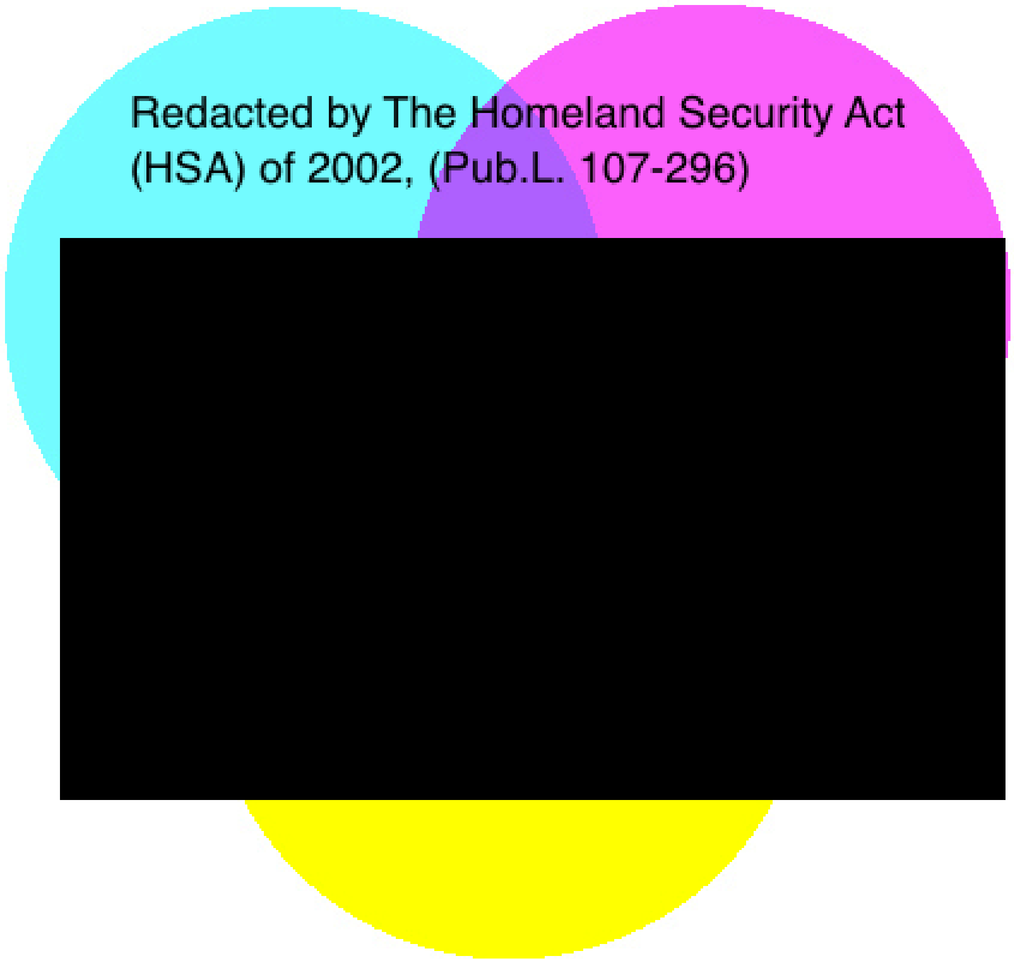
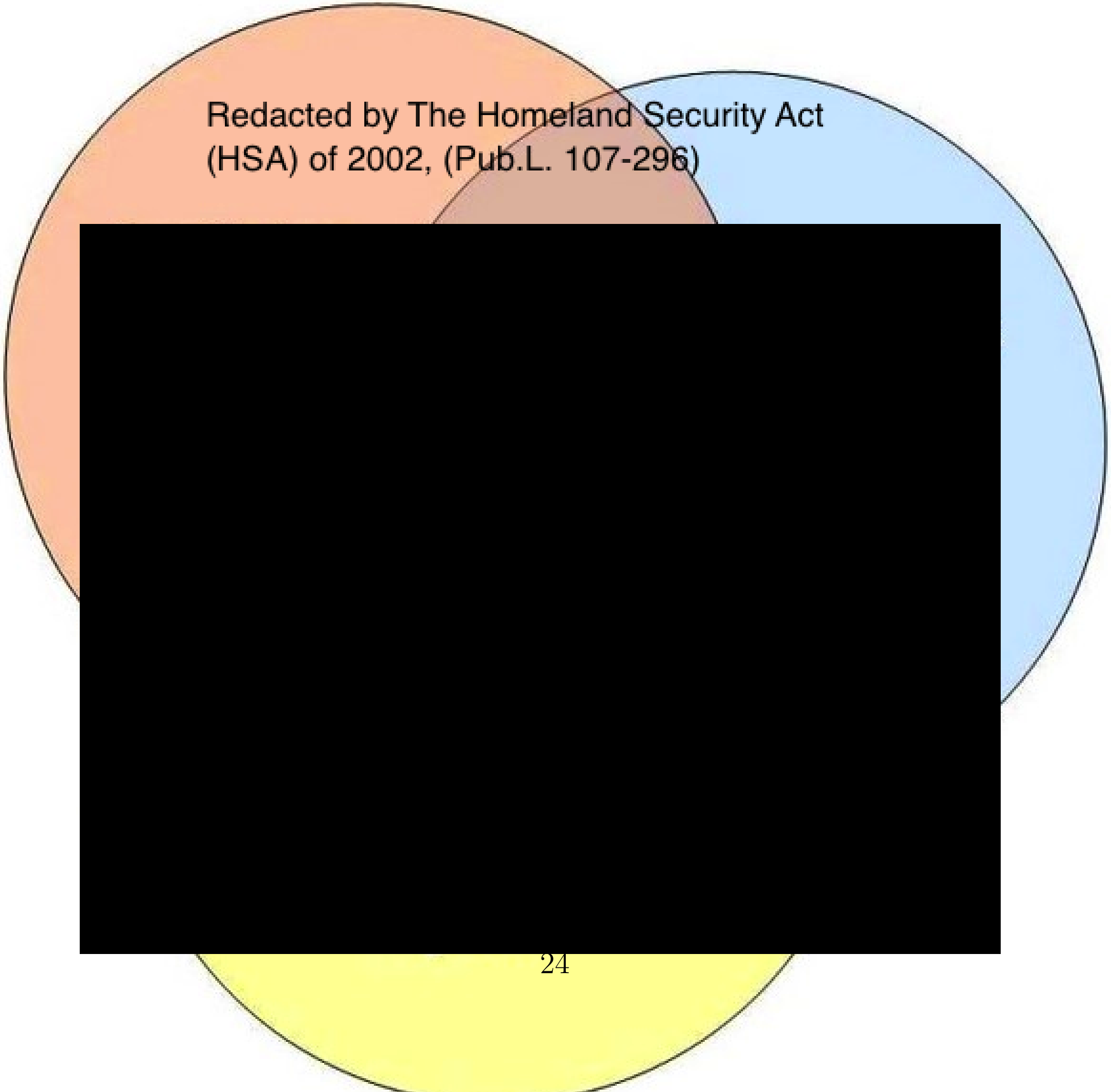


Figure 3: Venn diagrams at high school level

Venn diagrams at university level


Redacted by The Homeland Security Act
(HSA) of 2002, (Pub.L. 107-296)
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Quiz: Find X and Y for which the diagram makes sense.

a. List the string values of X and Y for which the diagram is the **least offensive** to you.

b. List the string values of X and Y for which the diagram is the **most offensive** to you.

5 Barycentric partitions of unity

By a **barycentric partition of unity** we mean a partition of unity $\phi : X \to |K|_m$ such that each of $\phi(x)$ is of the form $\dfrac{\chi_{A(x)}}{|A(x)|}$ for some finite subset $A(x) \subset V$.

In other words, each $\phi(x)$ is the **barycenter** of a simplex in K.

Basic Lemma:

$$\frac{|A \triangle B|}{\max(|A|, |B|)} \leq \|\frac{\chi_A}{|A|} - \frac{\chi_B}{|B|}\|_1 \leq 2 \cdot \frac{|A \triangle B|}{\min(|A|, |B|)}$$

Picture for Basic Lemma

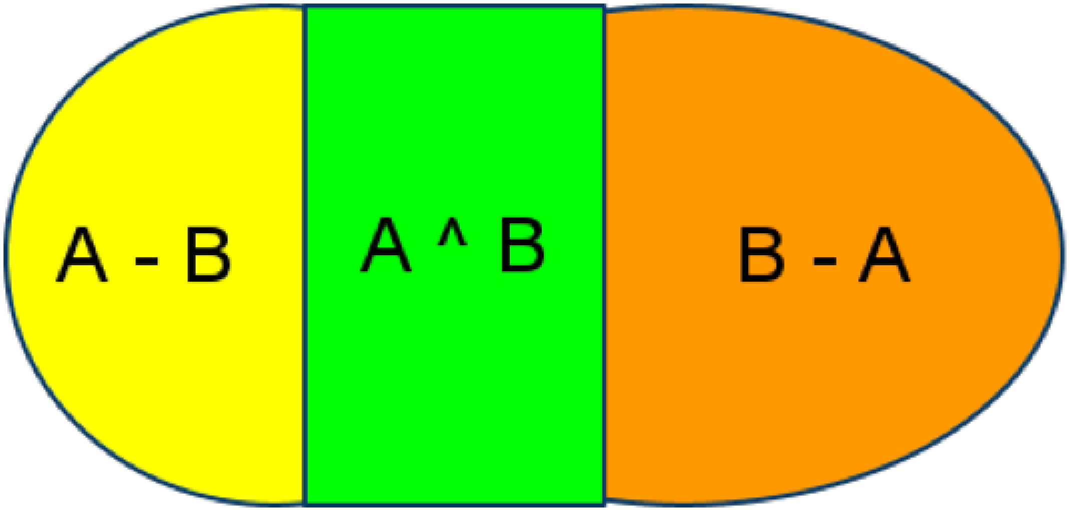

Figure 5: Picture for Basic Lemma

Proof of Basic Lemma:

$$\| |A| \cdot \chi_B - |B| \cdot \chi_A \|_1 =$$

$$|A| \cdot |A \setminus B| + |B| \cdot |B \setminus A| + |A \cap B| \cdot ||A| - |B||.$$

6 Amenability and Folner sequences

One can introduce large scale geometry on a group G by declaring uniformly bounded families to be exactly those refining $\{g \cdot F\}_{g \in G}$ for some finite subset $F \subset G$ of G.

That structure is metrizable if and only if G is countable and, in case of finitely generated groups, is identical with the coarse structure induced by a word metric on G.

It is natural to consider barycentric partitions of unity on G of the form

$$\phi_F(x) = \frac{\chi_{x \cdot F}}{|F|}.$$

When can we find a sequence $\{F(n)\}_{n \geq 1}$ of finite subsets of G such that each $\phi_{F(n)}$ is (ϵ_n, ϵ_n)-Lipschitz and $\epsilon_n \to 0$ as $n \to \infty$?

Basic Lemma says

$$\frac{|x^{-1}yF(n)\triangle F(n)|}{|F(n)|} \leq \|\phi_{F(n)}(x)-\phi_{F(n)}(y)\|_1 \leq 2 \cdot \frac{|x^{-1}yF(n)\triangle F(n)|}{|F(n)|}$$

That means we need

$$\lim_{n\to\infty} \frac{|gF(n)\triangle F(n)|}{|F(n)|} = 0$$

for every $g \in G$. That is the defining condition for a **Følner sequence**.

7 Property A of Yu

Suppose (X,d) is a metric space. When is there a sequence $\{\phi_n : X \to l_1(V_n)\}$ of barycentric partitions of unity that are (ϵ_n, ϵ_n)-Lipschitz and $\epsilon_n \to 0$ as $n \to \infty$?

Given a barycentric partition of unity $\phi : X \to l_1(V)$, we can pick $x_v \in \phi^{-1}(st(v))$ for each relevant $v \in V$. Now we can replace V by $X \times V$, and we can replace the carrier $C(x)$ of each $\phi(x)$ by

$$A(x) = \{(x_v, v) | v \in C(x)\}$$

Basic Lemma says: For each $r, \epsilon > 0$ there are finite subsets $A(x)$ of $X \times V$ such that

$$\frac{|A(x)\triangle A(y)|}{|A(x)\cap A(y)|)} < \epsilon$$

if $d(x, y) \leq r$ and the family $\{\{x\} \cup \pi_X(A(x))\}_{x \in X}$ is uniformly bounded.

If one puts $V = N$ this is the defining condition of Property A of Yu.

Theorem (Cencelj,JD,Vavpetić). A metric space X of bounded geometry has Property A if and only if X is large scale paracompact.

8 Large scale paracompactness in terms of covers

Theorem (Cencelj,JD,Vavpetić,Virk). A metric space X of bounded geometry is large scale paracompact if and only if for each $r, \epsilon > 0$ there is a uniformly bounded family $\mathcal{U}$ such that

$$\frac{m(x,\mathcal{U})}{m_r(x,\mathcal{U})} < 1 + \epsilon$$

for all $x \in X$.

In other words, the conditional probability of $B(x, r) \subset U$ given $x \in U \in \mathcal{U}$ can be as large as we want.

Application: Any expander does not have Property A.

An expander is an infinite sequence of d-regular graphs G_k such that $|V(G_k)| \to \infty$ and there is $c > 0$ with the property that for any subset A of $V(G_k)$ with $|A| < |V(G_k)|/2$ the number of points in $V(G_k) \setminus A$ such that their 2-ball intersects A is at least $c \cdot |A|$.

This can be weakened as follows: there is $c > 0$ with the property that for any subset A of $V(G_k)$ with $|A| < |V(G_k)|/2$ the number of points in A such that their 2-ball is not contained in A is at least $c \cdot |A|/d$.

Looking at a uniformly bounded family $\{U_s\}_{s \in S}$ in G_k the conditional probability of $B(x, 2) \subset U_s$ given $x \in U_s$ being bounded by $p < 1$ from below, one can define $n(x)$ as the number of elements of S such that $x \in U_s$.

Now the set of pairs (x, s) such that $x \in U_s$ but $B(x, 2)$ is not contained in U_s is at least

$$(c/d) \cdot \sum_{s \in S} |U_s|$$

and is at most

$$(1 - p) \cdot \sum_{x \in V(G_k)} n(x) = (1 - p) \cdot \sum_{s \in S} |U_s|.$$

Therefore

$$c/d \leq 1 - p$$

and there is a bound on p from above

$$p \leq 1 - c/d.$$

9 Large scale absolute extensors

The material of this section is due to JD and A.Mitra.

Definition. If K is a bounded metric space, then any function $f : X \to K$ has its Lipschitz number $L(f)$ defined as the infimum of all $\epsilon > 0$ such that f is (ϵ, ϵ)-Lipschitz.

Definition. If $f : A \subset X \to K$, we consider $extL(f)$, the infimum of Lipschitz numbers of all extensions of f over X.

Proposition. Suppose $f_n : A_n \subset X \to K$ are functions. If $L(f_n) \to 0$ with respect to a metric d_X on X, then $L(f_n) \to 0$ with respect to any metric ρ on X that is coarsely equivalent to d_X.

Definition. K is a **large scale absolute extensor** of X provided $extL(f_n) \to 0$ if $L(f_n) \to 0$ for any sequence $f_n : A \subset X \to K$.

Proposition. A stronger definition involving $f_n : A_n \subset X \to K$ is equivalent to the weaker condition.

Theorem (JD,Mitra). If $asdim(X) < \infty$, then $asdim(X) \leq n$ if and only if S^n is a large scale extensor of X.