HYPERSPACES OF COMPACT CONVEX SETS

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Motivation

Affine group action on $cb(\mathbb{R}^n)$

Global Slices

The John ellipsoid

Computing $J(n)$

The Banach-Mazur compacta

Equivariant conic structure of $cc(\mathbb{R}^n)$

Orbit spaces of $cb(\mathbb{R}^n)$
Some Motivation

For every $n \geq 1$, let's denote:
- $cc(\mathbb{R}^n)$ the hyperspace of all compact convex subsets of $\mathbb{R}^n$,
- $cb(\mathbb{R}^n)$ the hyperspace of all compact convex bodies of $\mathbb{R}^n$,

equipped with the Hausdorff metric topology:

$$d_H(A, B) = \max \left\{ \sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B) \right\},$$

where $d$ is the Euclidean metric and $d(b, A) = \inf \{ d(b, a) \mid a \in A \}$. 
Theorem (Nadler, Quinn, and Stavrakas (1979))

For \( n \geq 2 \), \( cc(\mathbb{R}^n) \) is homeomorphic to \( Q \setminus \{ pt \} \) where \( Q \) denotes the Hilbert cube.

• **Question.** What is the topological structure of \( cb(\mathbb{R}^n) \), \( n \geq 2 \)?

The subspace

\[
B(n) = \{ A \in cb(\mathbb{R}^n) \mid A = -A \}
\]

was studied earlier in [Ant., Fund. Math., 2000] and [Ant., TAMS, 2003].

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B(n) \cong Q \times \mathbb{R}^{n(n+1)/2}.
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Affine group action on \( cb(\mathbb{R}^n) \)

Our approach is largely based on the study of the natural affine group action on \( cb(\mathbb{R}^n) \).

\( \text{Aff}(n) \) is the group of all nonsingular affine transformations of \( \mathbb{R}^n \).

\( g \in \text{Aff}(n) \) iff \( g(x) = v + \sigma(x) \) for every \( x \in \mathbb{R}^n \), where \( \sigma \in GL(n) \) and \( v \) is a fixed vector.

\( \text{Aff}(n) \) acts on \( cb(\mathbb{R}^n) \) by the following rule:

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\text{Aff}(n) \times cb(\mathbb{R}^n) \to cb(\mathbb{R}^n) \\
(g, A) \to gA = \{g(a) \mid a \in A\}.
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**Orbit spaces of** $cb(\mathbb{R}^n)$

(A. Macbeath, 1951). The orbit space $cb(\mathbb{R}^n)/\text{Aff}(n)$ is a compact metric space.
Theorem

The action of Aff(n) on cb(\(\mathbb{R}^n\)) is proper.

Definition (Palais, 1961)

An action of a locally compact Hausdorff group G on a Tychonoff space X is proper if every point \(x \in X\) has a neighborhood \(V_x\) such that for any point \(y \in X\) there is a neighborhood \(V_y\) with the property that the transporter from \(V_x\) to \(V_y\)

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\langle V_x, V_y \rangle = \{ g \in G \mid gV_x \cap V_y \neq \emptyset \}
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Global Slices

Definition

Let $X$ be a $G$-space, and $H \leq G$ a closed subgroup. A subset $S \subset X$ is called a global \textit{H-slice} if the following conditions hold:

- $G(S) = X$, where $G(S) = \bigcup_{g \in G} gS$.
- $S$ is closed in $G(S)$.
- $S$ is $H$-invariant.
- $gS \cap S = \emptyset$ for all $g \in G \setminus H$. 
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![Diagram of global slices](image-url)
Theorem (Palais, 1961)

Let $G$ be a Lie group, $X$ be a proper $G$-space and $x \in X$. Then there exists a $G$-invariant neighborhood $U$ of $x$ which admits a global $G_x$-slice $S$ for $U$.

Equivalent form: there exists a $G$-map

$$f : U \to G/G_x$$

such that $f^{-1}(eG_x) = S$.

If, in addition, $G$ is a Lie group having finitely many connected components, then a maximal compact subgroup $K \subset G$ exists. In this case $gG_xg^{-1} \subset K$, and hence, there is a $G$-map

$$q : G/G_x \to G/K.$$
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Consider the composition:

\[ U \xrightarrow{f} \frac{G}{G_x} \xrightarrow{q} \frac{G}{K} \]

The inverse image \( Q = F^{-1}(eK) \) is a \( K \)-slice for \( U \).
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This $K$-slices can be pasted together to obtain a global $K$-slice of $X$.

**Theorem (Abels, 1974)**

Let $G$ be a Lie group having finitely many connected components, $K$ a maximal compact subgroup and $X$ a proper $G$-space. If the orbit space $X/G$ is paracompact then

1. $X$ admits a global $K$-slice $S$.
2. $X$ is $K$-equivariantly homeomorphic to the product $S \times G/K$. 

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Affine group action on $cb(\mathbb{R}^n)$

- Aff($n$) has two connected components.
- $O(n)$, the orthogonal group, is a maximal compact subgroup of $\text{Aff}(n)$.
- Aff($n$) acts properly on $cb(\mathbb{R}^n)$.
- The orbit space $cb(\mathbb{R}^n)/\text{Aff}(n)$ is metrizable and compact.

Hence, there exists a global $O(n)$-slice $S$ for $cb(\mathbb{R}^n)$ and

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$\text{Aff}(n)/O(n) \cong \mathbb{R}^n \times \text{GL}(n)/O(n)$. 

**RQ-Decomposition Theorem**

Every invertible matrix can be uniquely represented as the product of an orthogonal matrix and an upper triangular matrix with positive elements in the diagonal.

$\text{GL}(n)/O(n)$ is homeomorphic to $\mathbb{R}^{n(n+1)/2}$

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The John ellipsoid

For every compact convex body $A \in cb(\mathbb{R}^n)$ there exists a unique minimal volume ellipsoid $j(A)$ containing $A$. The ellipsoid $j(A)$ is called the John (sometimes also the Löwner) ellipsoid of $A$. 
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The map
\[ j : cb(\mathbb{R}^n) \to E(n) = \text{Aff}(n)(\mathbb{B}^n) \subset cb(\mathbb{R}^n) \]
is Aff\((n)\)-equivariant, i.e.,
\[ j(gA) = gj(A) \quad \text{for every} \quad g \in \text{Aff}(n), \text{ and } A \in cb(\mathbb{R}^n). \]
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For every $A \in cb(\mathbb{R}^n)$ there exists an affine transformation $g \in Aff(n)$ such that

$$j(A) = g\mathbb{B}^n$$

where $\mathbb{B}^n$ is the closed Euclidean unit ball.
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$$J(n) = \{ A \in cb(\mathbb{R}^n) \mid j(A) = B^n \}.$$
1. \( J(n) \) is \( O(n) \)-invariant,

2. \( \text{Aff}(n)(J(n)) = cb(\mathbb{R}^n) \),

3. \( J(n) \) is closed in \( cb(\mathbb{R}^n) \),

4. If \( A \in J(n) \) and \( g \in \text{Aff}(n) \setminus O(n) \) then

   \[ B^n \neq gB^n = gj(A) = j(gA) \]
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4. If $A \in J(n)$ and $g \in \text{Aff}(n) \setminus O(n)$ then

$$\mathbb{B}^n \neq g\mathbb{B}^n = gj(A) = j(gA).$$
1. $J(n)$ is $O(n)$-invariant,

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**Theorem**

$J(n)$ is a global $O(n)$-slice for $cb(\mathbb{R}^n)$. 

**HYPERSPACES OF COMPACT CONVEX SETS**
1. $J(n)$ is $O(n)$-invariant,

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1. \( J(n) \) is \( O(n) \)-invariant,

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**Theorem**

\( J(n) \) is a global \( O(n) \)-slice for \( cb(\mathbb{R}^n) \).
\begin{enumerate}
  \item $J(n)$ is $O(n)$-invariant,
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\end{enumerate}

Theorem

$J(n)$ is a global $O(n)$-slice for $\text{cb}(\mathbb{R}^n)$. 
Hence,

\[ cb(\mathbb{R}^n) \cong J(n) \times \mathbb{R}^{n(n+3)/2} . \]
Computing $J(n)$

**Theorem**

$J(n)$ is an $O(n)$-AR (and hence, it is an AR).

**Proof.**

Being a global $O(n)$-slice, $J(n)$ is an $O(n)$-retract of $cb(\mathbb{R}^n)$. But $cb(\mathbb{R}^n) \in O(n)$-AR since

$$\Lambda_k(A_1, \ldots A_k, t_1, \ldots t_k) = \sum_{i=1}^{k} t_i A_i, \quad k = 1, 2, \ldots$$

defines an $O(n)$-equivariant convex structure on $cb(\mathbb{R}^n)$. \[\square\]
We will show that

$$J(n)$$ is a Hilbert cube.

**Theorem**

The singleton \( \{ \mathbb{B}^n \} \) is a Z-set in \( J(n) \). Moreover, if \( K \subset O(n) \) is a closed subgroup that acts nontransitively on the sphere \( \mathbb{S}^{n-1} \), then for every \( \varepsilon > 0 \), there exists a \( K \)-map, \( \chi_\varepsilon : J(n) \to J_0(n) \), \( \varepsilon \)-close to the identity map of \( J(n) \).

Lets denote by \( J_0(n) = J(n) \setminus \{ \mathbb{B}^n \} \).
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Let's denote by \( J_0(n) = J(n) \setminus \{ B^n \} \).
Motivation  Affine group action on $cb(\mathbb{R}^n)$  Global Slices  The John ellipsoid  Computing $J(n)$  The Banach-Mazur compacta

**Theorem**

$J_0(n)$ satisfies the equivariant DDP: for every $\varepsilon > 0$, there exist $O(n)$-maps, $f_\varepsilon, h_\varepsilon : J_0(n) \to J_0(n)$, $\varepsilon$-close to the identity map of $J_0(n)$ and such that $\text{Im} f_\varepsilon \cap \text{Im} h_\varepsilon = \emptyset$. 

HYPERSPACES OF COMPACT CONVEX SETS
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HYPERSPACES OF COMPACT CONVEX SETS
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According to Toruńczyk’s Characterization Theorem, we have:

**Corollary**

\( J_0(n) \) is a Q-manifold and hence \( J(n) \) is a Hilbert cube.

**Corollary**

\( cb(\mathbb{R}^n) \) is homeomorphic to \( Q \times \mathbb{R}^{n(n+3)/2} \).
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Motivation

Affine group action on \( cb(\mathbb{R}^n) \)

Global Slices

The John ellipsoid

Computing \( J(n) \)

The Banach-Mazur compacta

Equivariant conic structure of \( cc(\mathbb{R}^n) \)

Orbit spaces of \( cb(\mathbb{R}^n) \)

\[
X^H = \{ x \in X \mid hx = x, \ \forall h \in H \}
\]

Corollary

(c) for a closed subgroup \( H \subset O(n) \) that acts nontransitively on \( S^{n-1} \), the \( H \)-fixed point set \( J(n)^H \) is homeomorphic to the Hilbert cube.

(d) for a closed subgroup \( H \subset O(n) \) that acts nontransitively on \( S^{n-1} \), the \( H \)-orbit space \( J(n)/H \) is homeomorphic to the Hilbert cube.

(e) for any closed subgroup \( H \subset O(n) \), the \( H \)-orbit space \( J_0(n)/H \) is a Q-manifold.

HYPERSPACES OF COMPACT CONVEX SETS
Motivation

Affine group action on \( \mathcal{G}(\mathbb{R}^n) \)

Global Slices

The John ellipsoid

Computing \( J(n) \)

The Banach-Mazur compacta

Equivariant conic structure of \( \mathcal{C}(\mathbb{R}^n) \)

Orbit spaces of \( \mathcal{G}(\mathbb{R}^n) \)

\[
X^H = \{ x \in X \mid hx = x, \quad \forall h \in H \}
\]

Corollary

(c) for a closed subgroup \( H \subset O(n) \) that acts nontransitively on \( \mathbb{S}^{n-1} \), the \( H \)-fixed point set \( J(n)^H \) is homeomorphic to the Hilbert cube.

(d) for a closed subgroup \( H \subset O(n) \) that acts nontransitively on \( \mathbb{S}^{n-1} \), the \( H \)-orbit space \( J(n)/H \) is homeomorphic to the Hilbert cube.

(e) for any closed subgroup \( H \subset O(n) \), the \( H \)-orbit space \( J_0(n)/H \) is a \( Q \)-manifold.
The Banach-Mazur compacta

In his 1932 book *Théorie des Opérations Linéaires*, S. Banach introduced the space of isometry classes $[X]$, of $n$-dimensional Banach spaces $X$ equipped with the well-known Banach-Mazur metric:

$$d([X], [Y]) = \ln \inf \left\{ \| T \| \cdot \| T^{-1} \| \mid T : X \to Y \text{ linear isomorphism} \right\}$$

$$BM(n) = \{ [X] \mid \dim X = n \},$$

the Banach-Mazur compactum.

$$BM_0(n) = BM(n) \setminus \{ [E] \},$$

the punctured Banach-Mazur compactum.

Let \( L(n) = \{ A \in J(n) \mid A = -A \} \). Then

\[
BM(n) \cong L(n) / O(n).
\]
Theorem (Ant., 2005, Fundamentalnaya i Prikladnaya Matematika)

Let the orthogonal group $O(n)$ act on a Hilbert cube $Q$ in such a way that:

(a) $Q$ is an $O(n)$-AR with a unique $O(n)$-fixed point $\ast$,
(b) $Q$ is strictly $O(n)$-contractible to $\ast$,
(c) for a closed subgroup $H \subset O(n)$, $Q^H = \{\ast\}$ if and only if $H$ acts transitively on the unit sphere $S^{n-1}$ and, $Q^H$ is homeomorphic to the Hilbert cube whenever $Q^H \neq \{\ast\}$,
(d) for any closed subgroup $H \subset O(n)$, the $H$-orbit space $Q_0/H$ is a $Q$-manifold, where $Q_0 = X \setminus \{\ast\}$.

Then for every $K < O(n)$, $Q_0/K \cong L_0(n)/K$. In particular, $Q_0/O(n) \cong BM_0(n)$, and hence, $Q/O(n) \cong BM(n)$. 

HYPERSPACES OF COMPACT CONVEX SETS
A $G$-space $X$ is called strictly $G$-contractible, if there exist a $G$-homotopy $f_t : X \to X$, $t \in 0, 1$ and a $G$-fixed point $a \in X$ such that $f_0$ is the identity map of $X$, and $f_t(x) = a$ if and only if $(x, t) \in \{(x, 1), (a, t)\}$. The corresponding nonequivariant notion was introduced by Michael.

Corollary

1. $J(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $BM(n)$.
2. $cb(\mathbb{R}^n)/\text{Aff}(n) \cong J(n)/O(n) \cong BM(n)$. 

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Special Case of $exp S^1$

Denote $exp_0 S^1 = (exp S^1) \setminus \{S^1\}$.

Corollary (Ant., 2007, Topology Appl.)

$$(exp_0 S^1)/S^1 \simeq L_0(2)/S^1.$$  

Corollary (Toruńczyk-West, 1978)

$(exp_0 S^1)/S^1$ is a Q-manifold Eilenberg-MacLane space $K(Q, 2)$.

Proof [Ant., 2007, Topology Appl.]

Since $(exp_0 S^1)/S^1 \simeq L_0(2)/S^1$ and $L_0(2)/S^1$ is a Q-manifold Eilenberg-MacLane space $K(Q, 2)$ (Ant., 2000, Fund. Math.)
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Corollary (Ant., 2007, Topology Appl.)

- \((\exp_0 S^1)/\mathcal{O}(2) \cong L_0(2)/\mathcal{O}(2)\).
- \((\exp S^1)/\mathcal{O}(2) \cong BM(2)\).
Corollary (Ant., 2007, Topology Appl.)

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Equivariant conic structure of $cc(\mathbb{R}^n)$

$$OC(X) = X \times [0, \infty)/X \times \{0\},$$

the open cone over $X$.

$$\mathbb{R}^n = OC(S^{n-1})$$

$$cc(\mathbb{R}^n) = OC(?)$$

$$M(n) := \{ A \in cc(B^n) \mid A \cap S^{n-1} \neq \emptyset \}.$$
Equivariant conic structure of $cc(\mathbb{R}^n)$

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$$M(n) := \{ A \in cc(\mathbb{B}^n) \mid A \cap \mathbb{S}^{n-1} \neq \emptyset \}.$$
Proposition $cc(\mathbb{R}^n)$ is $O(n)$-homeomorphic to the open cone over $M(n)$.

Proposition $cc(\mathbb{R}^n)/K$ is homeomorphic to the open cone over $M(n)/K$. 
Proposition

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Proposition

\( cc(\mathbb{R}^n)/K \) is homeomorphic to the open cone over \( M(n)/K \).
Motivation Affine group action on $cb(\mathbb{R}^n)$ Global Slices The John ellipsoid Computing $J(n)$ The Banach-Mazur compacta

Proposition

$cc(\mathbb{R}^n)$ is $O(n)$-homeomorphic to the open cone over $M(n)$.

Proposition

$cc(\mathbb{R}^n)/K$ is homeomorphic to the open cone over $M(n)/K$. 
Theorem

(a) for a closed subgroup $K \subset O(n)$ that acts nontransitively on $S^{n-1}$, the $K$-fixed point set $M(n)^K$ is homeomorphic to the Hilbert cube.

(b) for a closed subgroup $K \subset O(n)$ that acts nontransitively on $S^{n-1}$, the $K$-orbit space $M(n)/K$ is homeomorphic to the Hilbert cube.

(c) for any closed subgroup $K \subset O(n)$, the $K$-orbit space $M_0(n)/K$ is a $Q$-manifold.

Corollary

$M(n)/O(n) \cong BM(n)$
## Theorem

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## Corollary

$M(n)/O(n) \cong BM(n)$
Theorem

- \( cc(\mathbb{B}^n)/O(n) \) is the cone over the Banach-Mazur compactum BM\((n)\).
- \( cc(\mathbb{R}^n)/O(n) \) is the open cone over the Banach-Mazur compactum BM\((n)\).

Theorem

For every closed subgroup \( K \subset O(n) \) that acts non-transitively on \( S^{n-1} \), \( cc(\mathbb{R}^n) \) satisfies the \( K \)-equivariant DDP: for every \( \varepsilon > 0 \), there exist \( K \)-maps, \( f_\varepsilon, h_\varepsilon : cc(\mathbb{R}^n) \rightarrow cc(\mathbb{R}^n) \), \( \varepsilon \)-close to the identity map of \( cc(\mathbb{R}^n) \) and such that \( \text{Im} f_\varepsilon \cap \text{Im} h_\varepsilon = \emptyset \).
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Motivation

Affine group action on $cb(\mathbb{R}^n)$

Global Slices

The John ellipsoid

Computing $J(n)$

The Banach-Mazur compacta

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Theorem

For every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$, $cc(\mathbb{R}^n)$ satisfies the $K$-equivariant DDP: for every $\varepsilon > 0$, there exist $K$-maps, $f_\varepsilon, h_\varepsilon : cc(\mathbb{R}^n) \rightarrow cc(\mathbb{R}^n)$, $\varepsilon$-close to the identity map of $cc(\mathbb{R}^n)$ and such that $\text{Im} f_\varepsilon \cap \text{Im} h_\varepsilon = \emptyset$. 
Theorem

For every closed subgroup $K \subset O(n)$ that acts non transitively on $\mathbb{S}^{n-1}$, the $K$-orbit space

$$cc(\mathbb{R}^n)/K$$

is homeomorphic to the punctured Hilbert cube $Q_0 = Q \setminus \{\ast\}$. 
Motivation  Affine group action on $cb(\mathbb{R}^n)$  Global Slices  The John ellipsoid  Computing $J(n)$  The Banach-Mazur compacta

**Proof**

- $cc(\mathbb{R}^n)$ is a $K$-AR since it admits an equivariant convex structure (Ant., Topol. Appl., 2005)
- If $X \in G$-AR then $X / G \in$ AR (Ant., Math. USSR Sbornik, 1988)
- $cc(\mathbb{R}^n)/K$ satisfies the DDP (the preceding theorem).
- Thus, $cc(\mathbb{R}^n)/K$ is a contractible $Q$-manifold.
- The map $\nu : cc(\mathbb{R}^n) \to [0, \infty)$ defined by $\nu(A) = \max_{a \in A} \|a\|$ is an $O(n)$-invariant CE-map.
- The induced map $\tilde{\nu} : cb(\mathbb{R}^n)/K \to [0, \infty]$ is a CE-map.
- If there is a CE-map $f : M \to Y$ from a $Q$-manifold to an ANR, then $M \cong Q \times Y$ (R.D. Edwards).
- $cc(\mathbb{R}^n)/K \cong Q \times [0, \infty) \cong Q \setminus \{\ast\}$. 

HYPERSPACES OF COMPACT CONVEX SETS
**Proof**

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**Proof**

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Orbit spaces of $cb(\mathbb{R}^n)$

**Theorem**

For every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$, the $K$-orbit space $cb(\mathbb{R}^n)/K$ is a $Q$-manifold homeomorphic to the product $Q \times \frac{\text{Aff}(n)/O(n)}{K}$. 
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Motivation  Affine group action on $cb(\mathbb{R}^n)$  Global Slices  The John ellipsoid  Computing $J(n)$  The Banach-Mazur compacta

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