HYPERSPACES OF COMPACT CONVEX SETS

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Some Motivation

For every $n \geq 1$, lets denote:

- $cc(\mathbb{R}^n)$ the hyperspace of all compact convex subsets of $\mathbb{R}^n$,
- $cb(\mathbb{R}^n)$ the hyperspace of all compact convex bodies of $\mathbb{R}^n$,

equipped with the Hausdorff metric topology:

$$d_H(A, B) = \max\left\{\sup_{b\in B} d(b, A), \sup_{a\in A} d(a, B)\right\},$$

where d is the Euclidean metric and $d(b, A) = \inf\{d(b, a) \mid a \in A\}$.

Theorem (Nadler, Quinn, and Stavrakas (1979))

For $n \geq 2$, $cc(\mathbb{R}^n)$ is homeomorphic to $Q \setminus \{pt\}$ where Q denotes the Hilbert cube.

- **Question.** What is the topological structure of $cb(\mathbb{R}^n)$, $n \geq 2$?

The subspace

$$\mathcal{B}(n) = \{A \in cb(\mathbb{R}^n) \mid A = -A\}$$

was studied earliear in [Ant., Fund. Math., 2000] and [Ant., TAMS, 2003].

$$\mathcal{B}(n) \cong Q \times \mathbb{R}^{n(n+1)/2}.$$

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Affine group action on $cb(\mathbb{R}^n)$

Our approach is largely based on the study of the natural affine group action on $cb(\mathbb{R}^n)$.

Aff(n) is the group of all nonsingular affine transformations of $\mathbb{R}^n$.

$g \in \text{Aff}(n)$ iff $g(x) = v + \sigma(x)$ for every $x \in \mathbb{R}^n$, where $\sigma \in GL(n)$ and v is a fixed vector.

Aff(n) acts on $cb(\mathbb{R}^n)$ by the following rule:

$$\text{Aff}(n) \times cb(\mathbb{R}^n) \to cb(\mathbb{R}^n)$$

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(A. Macbeath, 1951). The orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ is a compact metric space.

Theorem

The action of $\mathrm{Aff}(n)$ *on* $cb(\mathbb{R}^n)$ *is proper.*

Definition (Palais, 1961)

An action of a locally compact Hausdorff group G on a Tychonoff space X is proper if every point $x \in X$ has a neighborhood V_x such that for any point $y \in X$ there is a neighborhood V_y with the property that the *transporter from* V_x *to* V_y

$$\langle V_x, V_y \rangle = \{g \in G \mid gV_x \cap V_y \neq \emptyset\}$$

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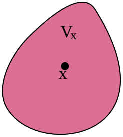

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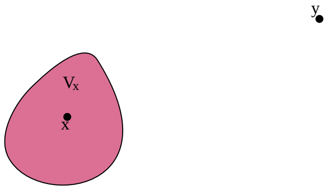

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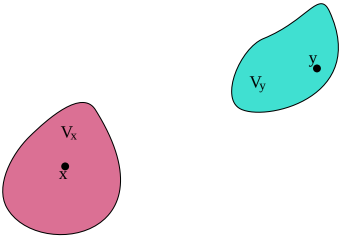

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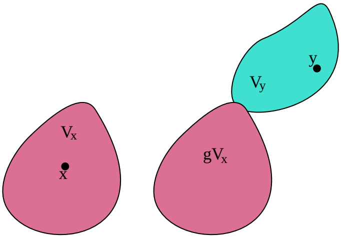

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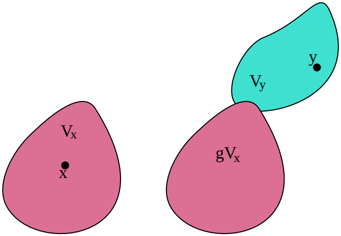

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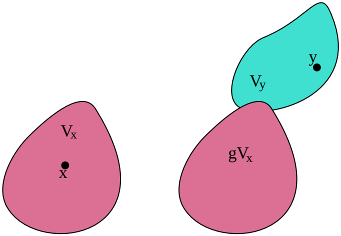

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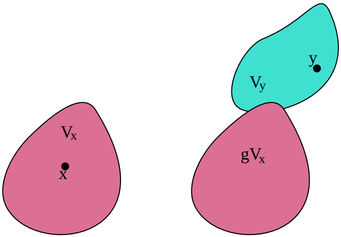

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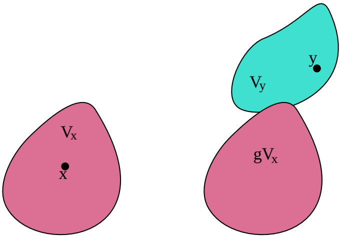

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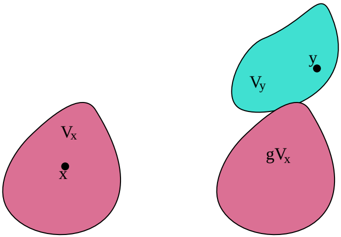

Global Slices

Definition

Let X be a G-space, and $H \leq G$ a closed subgroup. A subset $S \subset X$ is called a global *H-slice* if the following conditions hold:

- $G(S) = X$, where $G(S) = \bigcup_{g \in G} gS$.
- S is closed in $G(S)$.
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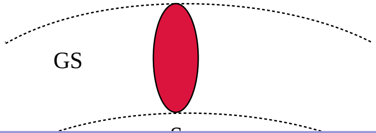

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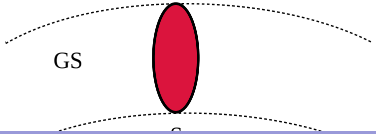

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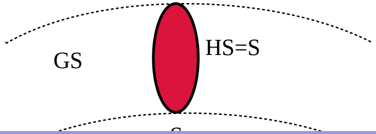

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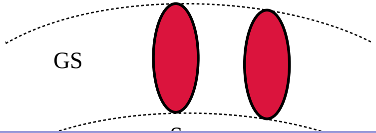

Theorem (Palais, 1961)

Let G be a Lie group, X be a proper G-space and $x \in X$. Then there exists a G-invariant neighborhood U of x which admits a global G_x-slice S for U.

Equivalent form: there exists a G-map

$$f : U \to G/G_x \quad \text{such that} \quad f^{-1}(eG_x) = S.$$

If, in addition, G is a Lie group having finitely many connected components, then a maximal compact subgroup $K \subset G$ exists. In this case $gG_xg^{-1} \subset K$, and hence, there is a G-map

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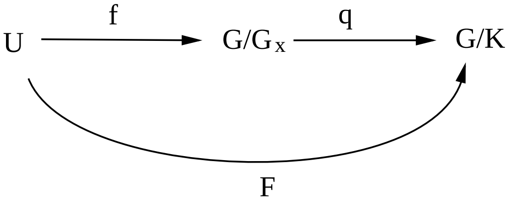

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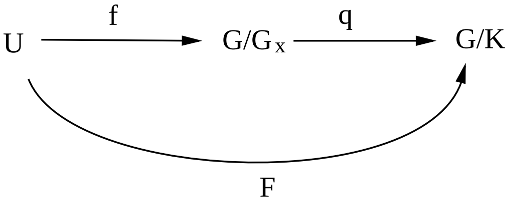

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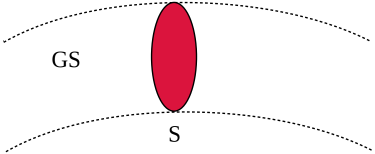

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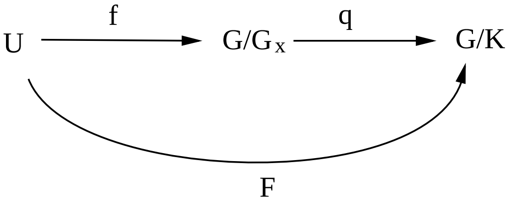

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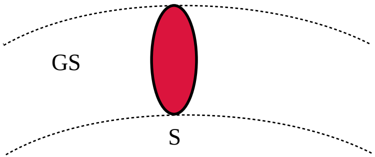

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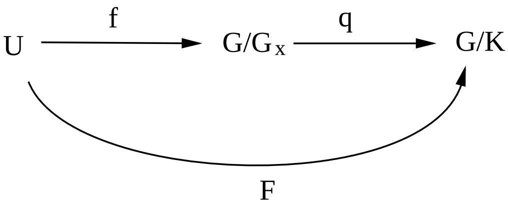

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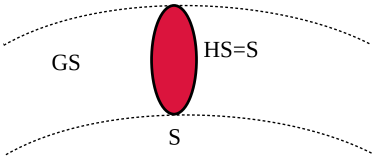

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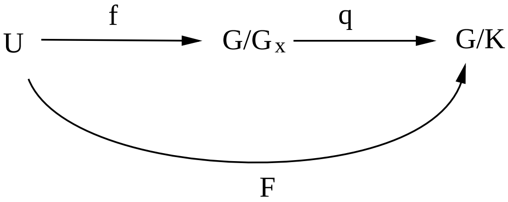

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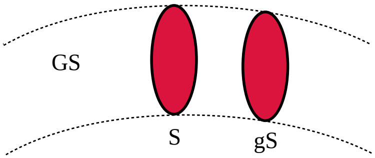

This K-slices can be pasted together to obtain a global K-slice of X.

Theorem (Abels, 1974)

Let G be a Lie group having finitely many connected components, K a maximal compact subgroup and X a proper G-space. If the orbit space X/G is paracompact then

(1) *X admits a global K-slice S.*

(2) *X is K-equivariantly homeomorphic to the product $S \times G/K$.*

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- $\operatorname{Aff}(n)$ has two connected components.
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- $\operatorname{Aff}(n)$ acts properly on $cb(\mathbb{R}^n)$.
- The orbit space $cb(\mathbb{R}^n)/\operatorname{Aff}(n)$ is metrizable and compact.

Hence, there exists a global $O(n)$-slice S for $cb(\mathbb{R}^n)$ and

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$$\mathrm{Aff}(n)/O(n) \cong \mathbb{R}^n \times GL(n)/O(n).$$

RQ-Decomposition Theorem

Every invertible matrix can be uniquely represented as the product of an orthogonal matrix and an upper triangular matrix with positive elements in the diagonal.

$GL(n)/O(n)$ is homeomorphic to $\mathbb{R}^{n(n+1)/2}$

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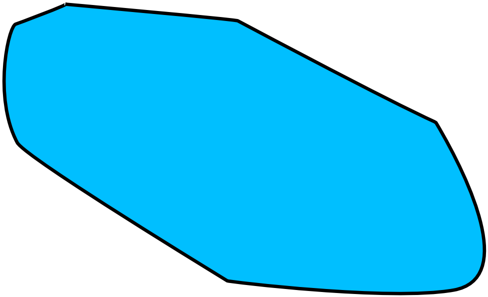
The John ellipsoid

For every compact convex body $A \in cb(\mathbb{R}^n)$ there exists a unique minimal volume ellipsoid $j(A)$ containing A. The ellipsoid $j(A)$ is called the John (sometimes also the Löwner) ellipsoid of A.



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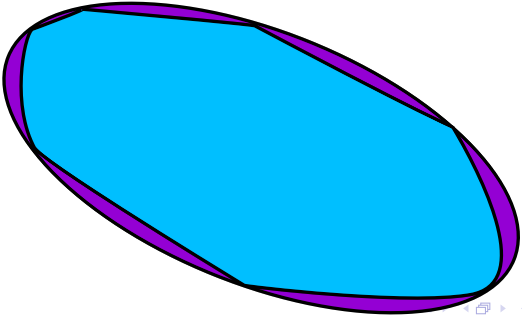
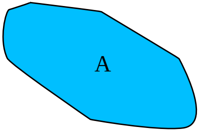


The map

$$j : cb(\mathbb{R}^n) \to E(n) = \mathrm{Aff}(n)\big(\mathbb{B}^n\big) \subset cb(\mathbb{R}^n)$$

is $\mathrm{Aff}(n)$-equivariant, i.e.,

$$j(gA) = gj(A) \qquad \text{for every} \quad g \in \mathrm{Aff}(n), \text{ and } A \in cb(\mathbb{R}^n).$$



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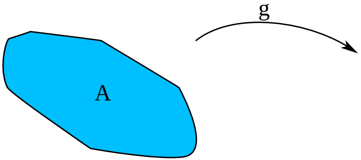

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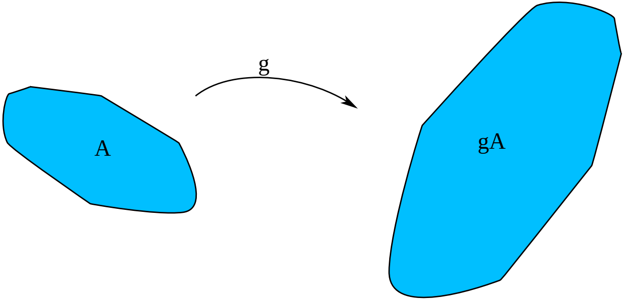

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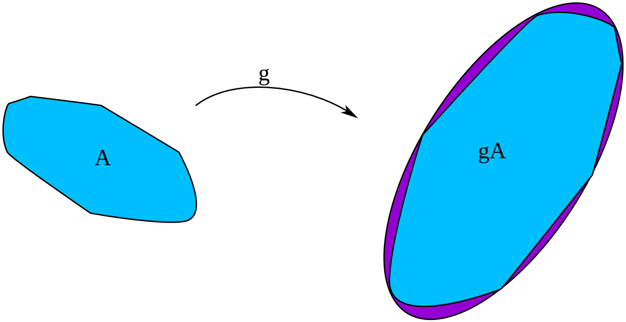

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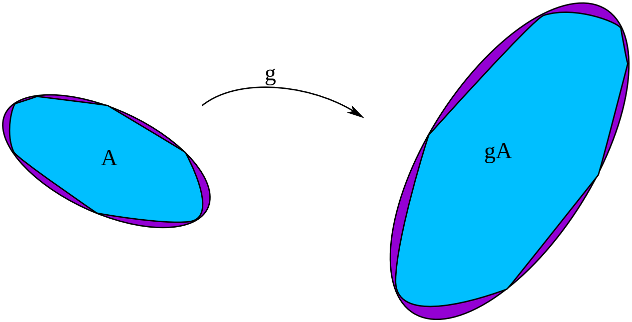

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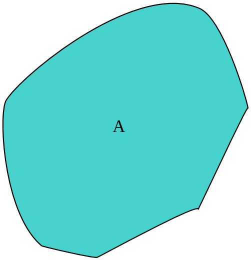

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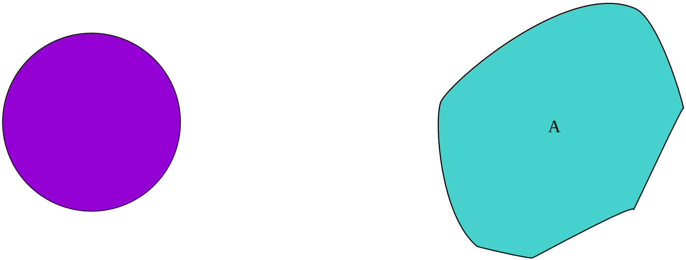

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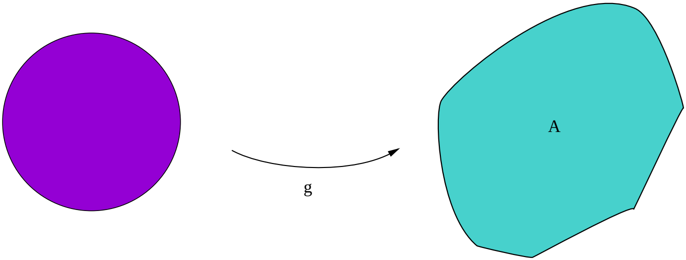

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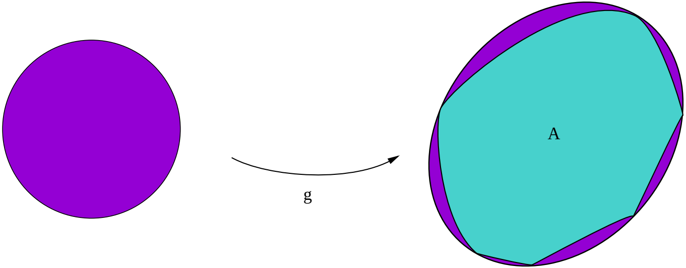

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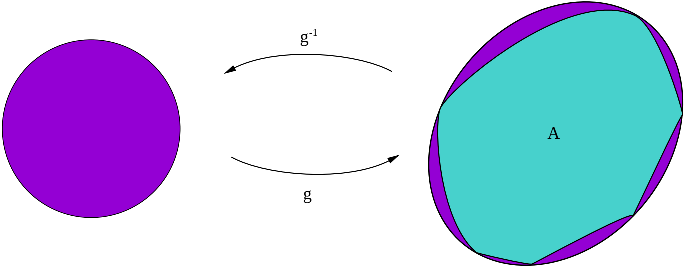

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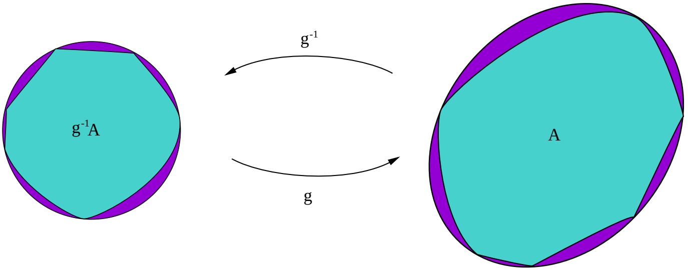

For every $n \geq 2$ lets denote by $J(n)$ the following set:

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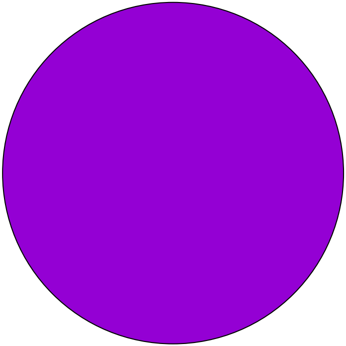

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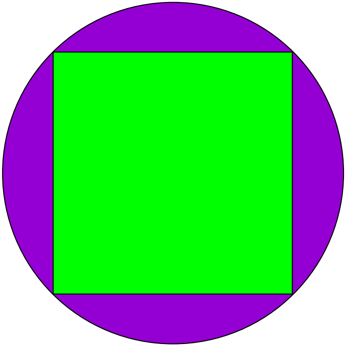

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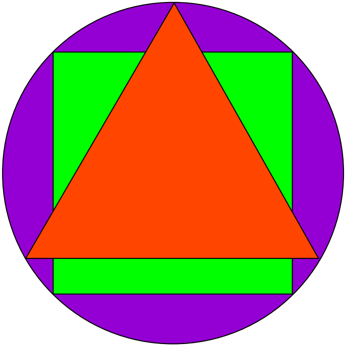

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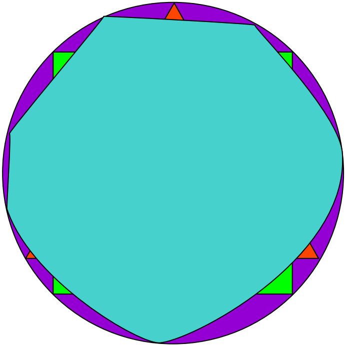

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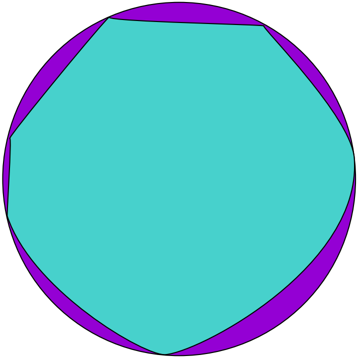

2. $\mathrm{Aff}(n)(J(n)) = cb(\mathbb{R}^n)$,
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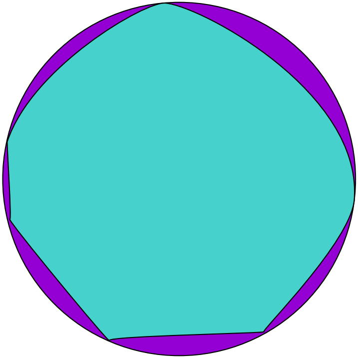

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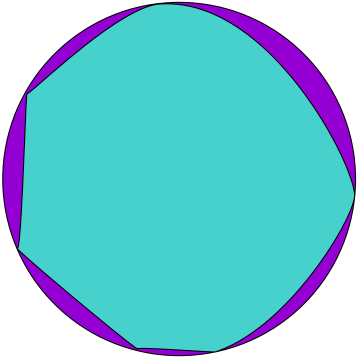

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$J(n)$ is a global $O(n)$-slice for $cb(\mathbb{R}^n)$.

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Theorem

$J(n)$ is a global $O(n)$-slice for $cb(\mathbb{R}^n)$.

Hence,

$$cb(\mathbb{R}^n) \cong J(n) \times \mathbb{R}^{n(n+3)/2}.$$

Computing $J(n)$

Theorem

$J(n)$ is an $O(n)$-AR (and hence, it is an AR).

Proof.

Being a global $O(n)$-slice, $J(n)$ is an $O(n)$-retract of $cb(\mathbb{R}^n)$. But $cb(\mathbb{R}^n) \in O(n)$-AR since

$$\Lambda_k(A_1, \dots A_k, t_1, \dots t_k) = \sum_{i=1}^{k} t_i A_i, \quad k = 1, 2, \dots$$

defines an $O(n)$-equivariant convex structure on $cb(\mathbb{R}^n)$. □

We will show that

$J(n)$ is a Hilbert cube.

Theorem

The singleton $\{\mathbb{B}^n\}$ is a Z-set in $J(n)$. Moreover, if $K \subset O(n)$ is a closed subgroup that acts nontransitively on the sphere $\mathbb{S}^{n-1}$, then for every $\varepsilon > 0$, there exists a K-map, $\chi_\varepsilon : J(n) \to J_0(n)$, ε-close to the identity map of $J(n)$.

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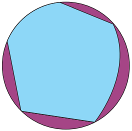

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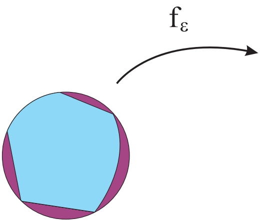

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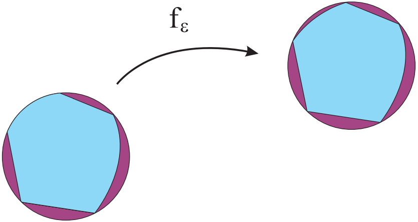

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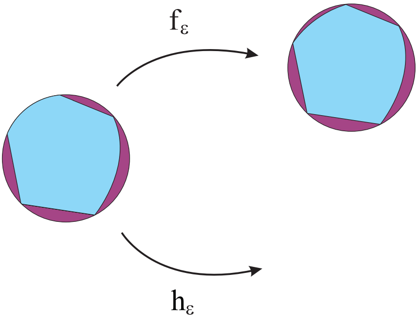

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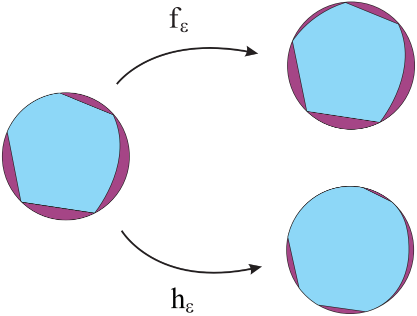

According to Toruńczyk's Characterization Theorem, we have:

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$J_0(n)$ is a Q-manifold and hence $J(n)$ is a Hilbert cube.

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$$X^H = \{x \in X \mid hx = x, \ \forall h \in H\}$$

Corollary

(c) *for a closed subgroup $H \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the H-fixed point set $J(n)^H$ is homeomorphic to the Hilbert cube.*

(d) *for a closed subgroup $H \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the H-orbit space $J(n)/H$ is homeomorphic to the Hilbert cube.*

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The Banach-Mazur compacta

In his 1932 book *Théorie des Opérations Linéaires*, S. Banach introduced the space of isometry classes $[X]$, of n-dimensional Banach spaces X equipped with the well-known Banach-Mazur metric:

$$d([X],[Y]) = \ln \inf \left\{ \|T\| \cdot \|T^{-1}\| \;\middle|\; T : X \to Y \text{ linear isomorphism} \right\}$$

$$BM(n) = \{[X] \mid \dim X = n\},$$

the Banach-Mazur compactum.

$$BM_0(n) = BM(n) \setminus \{[E]\},$$

the punctured Banach-Mazur compactum.

Theorem (Ant., 2000, Fund. Math.)

Let $L(n) = \{A \in J(n) \mid A = -A\}$. *Then*

$$BM(n) \cong L(n)/O(n).$$

Theorem (Ant., 2005, Fundamentalnaya i Prikladnaya Matematika)

Let the orthogonal group $O(n)$ act on a Hilbert cube Q in such a way that:

(a) *Q is an $O(n)$-AR with a unique $O(n)$-fixed point $*$,*

(b) *Q is strictly $O(n)$-contractible to $*$,*

(c) *for a closed subgroup $H \subset O(n)$, $Q^H = \{*\}$ if and only if H acts transitively on the unit sphere S^{n-1} and, Q^H is homeomorphic to the Hilbert cube whenever $Q^H \neq \{*\}$,*

(d) *for any closed subgroup $H \subset O(n)$, the H-orbit space Q_0/H is a Q-manifold, where $Q_0 = X \setminus \{*\}$.*

Then for every $K < O(n)$, $Q_0/K \cong L_0(n)/K$. In particular, $Q_0/O(n) \cong BM_0(n)$, and hence, $Q/O(n) \cong BM(n)$.

A G-space X is called strictly G-contractible, if there exist a G-homotopy $f_t : X \to X$, $t \in 0, 1$ and a G-fixed point $a \in X$ such that f_0 is the identity map of X, and $f_t(x) = a$ if and only if $(x, t) \in \{(x, 1), (a, t)\}$. The corresponding nonequivariant notion was introduced by Michael.

Corollary

- *$J(n)/O(n)$ is homeomorphic to the Banach-Mazur compactum $BM(n)$.*
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Special Case of $exp\,\mathbb{S}^1$

Denote $exp_0\,\mathbb{S}^1 = (exp\,\mathbb{S}^1) \setminus \{\mathbb{S}^1\}$.

Corollary (Ant., 2007, Topology Appl.)

$$(exp_0\,\mathbb{S}^1)/\mathbb{S}^1 \cong L_0(2)/\mathbb{S}^1.$$

Corollary (Toruńczyk-West, 1978)

$(exp_0\,\mathbb{S}^1)/\mathbb{S}^1$ *is a Q-manifold Eilenberg-MacLane space* $K(\mathbb{Q}, 2)$.

Proof [Ant., 2007, Topology Appl.]
Since $(exp_0\,\mathbb{S}^1)/\mathbb{S}^1 \cong L_0(2)/\mathbb{S}^1$ and $L_0(2)/\mathbb{S}^1$ is a Q-manifold Eilenberg-MacLane space $K(\mathbb{Q}, 2)$ (Ant., 2000, Fund. Math.)

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Special Case of $exp\, \mathbb{S}^1$

Denote $exp_0\, \mathbb{S}^1 = (exp\, \mathbb{S}^1) \setminus \{\mathbb{S}^1\}$.

Corollary (Ant., 2007, Topology Appl.)

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- $(exp\,\mathbb{S}^1)/\mathbb{O}(2) \cong BM(2)$.

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Equivariant conic structure of $cc(\mathbb{R}^n)$

$$OC(X) = X \times [0, \infty / X \times \{0\},$$

the open cone over X.

$$\mathbb{R}^n = OC(\mathbb{S}^{n-1})$$

$$cc(\mathbb{R}^n) = OC(?)$$

$$M(n) := \{A \in cc(\mathbb{B}^n) \mid A \cap \mathbb{S}^{n-1} \neq \emptyset\}.$$

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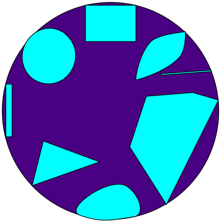

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$cc(\mathbb{R}^n)$ is $O(n)$-homeomorphic to the open cone over $M(n)$.

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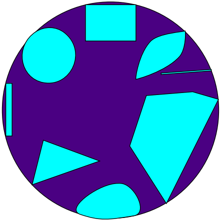
$cc(\mathbb{R}^n)/K$ is homeomorphic to the open cone over $M(n)/K$.



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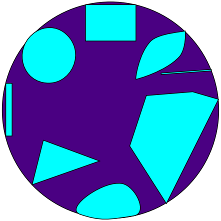

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Theorem

(a) *for a closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-fixed point set $M(n)^K$ is homeomorphic to the Hilbert cube.*

(b) *for a closed subgroup $K \subset O(n)$ that acts nontransitively on $\mathbb{S}^{n-1}$, the K-orbit space $M(n)/K$ is homeomorphic to the Hilbert cube.*

(c) *for any closed subgroup $K \subset O(n)$, the K-orbit space $M_0(n)/K$ is a Q-manifold.*

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$M(n)/O(n) \cong BM(n)$

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- $cc(\mathbb{B}^n)/O(n)$ *is the cone over the Banach-Mazur compactum* $BM(n)$.
- $cc(\mathbb{R}^n)/O(n)$ *is the open cone over the Banach-Mazur compactum* $BM(n)$.

Theorem

For every closed subgroup $K \subset O(n)$ *that acts non-transitively on* $\mathbb{S}^{n-1}$, $cc(\mathbb{R}^n)$ *satisfies the* K*-equivariant DDP: for every* $\varepsilon > 0$, *there exist* K*-maps,* $f_\varepsilon, h_\varepsilon : cc(\mathbb{R}^n) \to cc(\mathbb{R}^n)$, ε*-close to the identity map of* $cc(\mathbb{R}^n)$ *and such that* $\operatorname{Im} f_\varepsilon \cap \operatorname{Im} h_\varepsilon = \emptyset$.

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For every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$, $cc(\mathbb{R}^n)$ satisfies the K-equivariant DDP: for every $\varepsilon > 0$, there exist K-maps, $f_\varepsilon, h_\varepsilon : cc(\mathbb{R}^n) \to cc(\mathbb{R}^n)$, ε-close to the identity map of $cc(\mathbb{R}^n)$ and such that $\mathrm{Im}\, f_\varepsilon \cap \mathrm{Im}\, h_\varepsilon = \emptyset$.

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Theorem

For every closed subgroup $K \subset O(n)$ that acts non transitively on $\mathbb{S}^{n-1}$, the K-orbit space

$$cc(\mathbb{R}^n)/K$$

is homeomorphic to the punctured Hilbert cube $Q_0 = Q \setminus \{*\}$.

Proof

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- $cc(\mathbb{R}^n)$ is a K-AR since it admits an equivariant convex structure (Ant., Topol. Appl., 2005)
- If $X \in G$-AR then $X/G \in$AR (Ant., Math. USSR Sbornik, 1988)
- $cc(\mathbb{R}^n)/K$ satisfies the DDP (the preceding theorem).
- Thus, $cc(\mathbb{R}^n)/K$ is a contractible Q-manifold.
- The map $\nu : cc(\mathbb{R}^n) \to [0, \infty)$ defined by $\nu(A) = \max_{a\in A} \|a\|$ is an $O(n)$-invariant CE-map.
- The induced map $\widetilde{\nu} : cb(\mathbb{R}^n)/K \to [0, \infty]$ is a CE-map.
- If there is a CE-map $f : M \to Y$ from a Q-manifold to an ANR, then $M \cong Q \times Y$ (R.D. Edwards).
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Orbit spaces of $cb(\mathbb{R}^n)$

Theorem

For every closed subgroup $K \subset O(n)$ that acts non-transitively on $\mathbb{S}^{n-1}$, the K-orbit space

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is a Q-manifold homeomorphic to the product $Q \times \frac{\mathrm{Aff}(n)/O(n)}{K}$.

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- Thus, $cb(\mathbb{R}^n)/K$ is a contractible Q-manifold.
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- The induced map $\widetilde{j} : cb(\mathbb{R}^n)/K \to \frac{\text{Aff}(n)/O(n)}{K}$ is a CE-map.
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THANKS