EXPLICIT METHODS FOR DERIVED CATEGORIES OF SHEAVES

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Abstract. These notes aim to complement the lecture notes of Căldăraru by providing an introduction to explicit methods in the study of derived categories in algebraic geometry. The bounded derived category of coherent sheaves on certain very simple algebraic varieties is equivalent to the bounded derived category of finitely generated modules over an algebra. We provide an introduction to this circle of ideas with a focus on explicit examples.

Contents

1. Lecture 1: The category of bound quiver representations

This lecture introduces an abelian category that plays the key rôle in understanding the bounded derived category of coherent sheaves for certain very simple algebraic varieties.

1.1. A *quiver* Q is a directed graph given by a set Q_0 of vertices, a set Q_1 of arrows, and maps $\mathrm{tl}, \mathrm{hd}\colon Q_1 \to Q_0$ that specify the tail and head of each arrow. We assume that both Q_0, Q_1 are finite sets, and that Q is connected, i.e., the graph obtained by forgetting the orientation of arrows in Q is connected. A nontrivial *path* in Q of length $\ell \in \mathbb{N}$ from vertex $i \in Q_0$ to vertex $j \in Q_0$ is a sequence of arrows $p = a_1 \cdots a_\ell$ with $\mathrm{hd}(a_k) = \mathrm{tl}(a_{k+1})$ for $1 \leq k < \ell$. We set $\mathrm{tl}(p) := \mathrm{tl}(a_1)$ and $\mathrm{hd}(p) := \mathrm{hd}(a_\ell)$. In addition, each vertex $i \in Q_0$ gives a trivial path e_i of length zero, where $\mathrm{tl}(e_i) = \mathrm{hd}(e_i) = i$. A *cycle* is a nontrivial path in which the head and tail coincide, and Q is acyclic if it contains no cycles.

Let $\Bbbk$ be a field. The *path algebra* $\Bbbk Q$ of a quiver Q is the $\Bbbk$-algebra whose underlying $\Bbbk$-vector space has as basis the set of paths in Q, where the product of basis elements equals the basis element defined by concatenation of the paths if possible, or zero otherwise. The algebra $\Bbbk Q$ is graded by path length, i.e.,

$$\Bbbk Q = \bigoplus_{k \in \mathbb{N}} (\Bbbk Q)_k$$

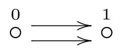
where $(\Bbbk Q)_k$ denotes the vector subspace spanned by paths of length k, and $(\Bbbk Q)_k \cdot (\Bbbk Q)_\ell \subseteq (\Bbbk Q)_{k+\ell}$. The subring $(\Bbbk Q)_0 \subset \Bbbk Q$ spanned by the trivial paths e_i for $i \in Q_0$ is a semisimple ring in which the elements e_i are orthogonal idempotents, that is, $e_i e_j = e_i$ when $i = j$, and 0 otherwise.

Lemma 1.1. *The path algebra $\Bbbk Q$ is an associative algebra with identity $\sum_{i \in Q_0} e_i$. Moreover, $\Bbbk Q$ is finite dimensional over $\Bbbk$ if and only if Q contains no cycles.*

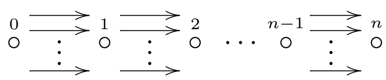
Proof. Composition is tautologically associative, hence so is $\Bbbk Q$. Multiplying a path p on the left and right by $\sum_{i \in Q_0} e_i$ is the same as multiplying on the left and right by $e_{\mathrm{tl}(p)}$ and $e_{\mathrm{hd}(p)}$ respectively; in each case, the result is p. Extending to all of $\Bbbk Q$ by $\Bbbk$-linearity shows that $\sum_{i \in Q_0} e_i$ is the identity. □

Exercise 1.2. Prove the final statement of the lemma.

Example 1.3. The quiver with two vertices and a pair of arrows as shown

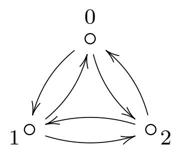


is the *Kronecker quiver*, or the *Beilinson quiver for* $\mathbb{P}^1$. More generally, the *Beilinson quiver for* $\mathbb{P}^n$ is the quiver



with $n+1$ vertices denoted $Q_0 = \{0, 1, \dots, n\}$, that has $n+1$ arrows from the ith vertex to the $(i+1)$-st vertex for each $i = 0, 1, \dots, n-1$.

Example 1.4. The *McKay quiver of the A_2 singularity* provides an example of a quiver that admits drected cycles:



1.2. The category of quiver representations. As is typical in representation theory, our interest lies not just with $\Bbbk Q$, but with modules over $\Bbbk Q$. To help us visualise these modules we use the terminology of quiver representations.

A *representation of the quiver* Q consists of a $\Bbbk$-vector space W_i for each $i \in Q_0$ and a $\Bbbk$-linear map $w_a \colon W_{\mathrm{tl}(a)} \to W_{\mathrm{hd}(a)}$ for each $a \in Q_1$. More compactly, we write $W = \big((W_i)_{i \in Q_0}, (w_a)_{a \in Q_1}\big)$. A representation is finite dimensional if each vector space W_i has finite dimension over $\Bbbk$, and the dimension vector of W is the tuple of nonnegative integers $(\dim_\Bbbk W_i)_{i \in Q_0}$. A map between finite-dimensional representations W and W' is a family $\psi_i\colon W_i \to W'_i$ for $i \in Q_0$ of $\Bbbk$-linear maps that are compatible with the structure maps, that is, such that the diagrams

$$\begin{array}{ccc} W_{\mathrm{tl}(a)} & \xrightarrow{w_a} & W_{\mathrm{hd}(a)} \\ \downarrow{\scriptstyle \psi_{\mathrm{tl}(a)}} & & \downarrow{\scriptstyle \psi_{\mathrm{hd}(a)}} \\ W'_{\mathrm{tl}(a)} & \xrightarrow{w'_a} & W'_{\mathrm{hd}(a)} \end{array}$$

commute for all $a \in Q_1$. With composition defined componentwise, we obtain the category of finite-dimensional representations of Q, denoted $\mathrm{rep}_{\Bbbk}(Q)$.

Example 1.5. For the Kronecker quiver from Example 1.3, consider representations W of Q with dimension vector $(1,1)$. This means that W consists of two one-dimensional vector spaces (W_0, W_1) together with two maps $w_1, w_2\colon W_0 \to W_1$. Any such representation fits into a short exact sequence

$$0 \longrightarrow S \longrightarrow W \longrightarrow Q \longrightarrow 0.$$

To determine the isomorphism classes of such representations we first choose bases and thus identify W_0 and W_1 with $\Bbbk$; the maps (w_1, w_2) then determine an element of $\Bbbk^2$. Rescaling the bases just gives the scaling action of $\Bbbk^*$ on $\Bbbk^2$. If both the maps w_1 and w_2 are zero then $W = S \oplus Q$. For all other points of $\Bbbk^2$ the corresponding representation of Q is indecomposable, and the isomorphism classes of these representations are parameterised by the orbits of $\Bbbk^*$ in $\Bbbk^2 \smallsetminus \{0\}$, i.e., by $\mathbb{P}^1$.

Let $\mathrm{mod}(A)$ denote the category of finitely generated left A-modules. If A^{op} denotes the opposite algebra where the product satisfies $a \cdot b := ba$, then $\mathrm{mod}(A^{\mathrm{op}})$ is the category of finitely generated right A-modules. If Q^{op} denotes the quiver obtained from Q be reversing the orientation of the arrows then $(\Bbbk Q)^{\mathrm{op}} \cong \Bbbk Q^{\mathrm{op}}$.

Proposition 1.6. *The category* $\mathrm{rep}_{\Bbbk}(Q)$ *is equivalent to the category of finitely-generated left* $\Bbbk Q$*-modules.*

Proof. Let $W = \big((W_i)_{i\in Q_0}, (w_a)_{a\in Q_1}\big)$ be a finite-dimensional representation of Q. Define a $\Bbbk$-vector space $M := \bigoplus_{i\in Q_0} W_i$, and define a $\Bbbk Q$-module structure on M by extending linearly from

$$e_i m = \begin{cases} m, & m \in W_i, \\ 0, & m \in W_j \text{ for } j \neq i, \end{cases} \quad \text{and} \quad a{\cdot}m = \begin{cases} w_a(m_{\mathrm{tl}(a)}), & m \in W_{\mathrm{tl}(a)}, \\ 0, & m \in W_j \quad \text{for } j \neq \mathrm{tl}(a), \end{cases}$$

for $i \in Q_0$ and $a \in Q_1$. This construction can be inverted as follows: given a left $\Bbbk Q$-module M we set $W_i := e_i M$ for $i \in Q_0$ and define maps $w_a\colon W_{\mathrm{tl}(a)} \to W_{\mathrm{hd}(a)}$ by sending m to $a(m)$ for each $a \in Q_1$. One easily checks that maps of representations of Q correspond to $\Bbbk Q$-module homomorphisms. □

An immediate consequence of the proposition is that both $\mathrm{rep}_{\Bbbk}(Q)$ and $\mathrm{rep}_{\Bbbk}(Q^{\mathrm{op}})$ are abelian categories.

Exercise 1.7. Prove directly from the definition that $\mathrm{rep}_{\Bbbk}(Q)$ is an abelian category.

1.3. In most geometric contexts, the algebra of interest is not isomorphic to the path algebra of a quiver Q, but is isomorphic to the quotient of a path algebra by an ideal of relations. Formally, a *relation* in a quiver Q (with coefficients in $\Bbbk$) is a $\Bbbk$-linear combination of paths of length at least two, each with the same head and tail. Our interest lies only with relations of the form $p - p' \in \Bbbk Q$, where p, p' are paths in Q with the same head and tail. Any finite set of relations R in Q determines a two-sided ideal $\langle R\rangle$ in the algebra $\Bbbk Q$.

A *bound quiver* (Q, R), or equivalently a *quiver with relations*, is a quiver Q together with a finite set of relations R. A *representation* of (Q, R) is a representation of Q where each relation $p - p' \in R$ is satisfied in the sense that the corresponding linear combination of homomorphisms from $W_{\mathrm{tl}(p)}$ to $W_{\mathrm{hd}(p)}$ arising from the arrows in p coincides with the homomorphisms arising from arrows in p'. As before, finite-dimensional representations of (Q, R) form a category denoted $\mathrm{rep}_{\Bbbk}(Q, R)$.

Proposition 1.8. *The category* $\mathrm{rep}_{\Bbbk}(Q, R)$ *is equivalent to the category of left* $\Bbbk Q/\langle R\rangle$*-modules.*

Exercise 1.9. Prove Proposition 1.8.

Example 1.10. The quivers from Example 1.3 for which $n \geq 3$ admit a set of relations that have geometric significance. Consider the case with three vertices. List the arrows from vertex 0 to vertex 1 as a_1, a_2, a_3 and the arrows from vertex 1 to vertex 2 as a_4, a_5, a_6. Set $R = \{a_1a_5 - a_2a_4, a_1a_6 - a_3a_4, a_2a_6 - a_3a_5\}$. The quotient algebra $\Bbbk Q/\langle R\rangle$ is isomorphic to the endomorphism algebra of the bundle $\mathscr{O}_{\mathbb{P}^2_{\Bbbk}} \oplus \mathscr{O}_{\mathbb{P}^2_{\Bbbk}}(1) \oplus \mathscr{O}_{\mathbb{P}^2_{\Bbbk}}(2)$.

1.4. Given an arbitrary abelian category $\mathcal{A}$ it is natural to start with the *simple* objects, i.e., those for which any subobject is either a zero object or is isomorphic to W. For example, the vertices in a quiver Q determine an obvious collection of simple objects, namely, the representations $S(i)$ for $i \in Q_0$ for which the only nonzero vector space is $\Bbbk$ placed at the ith vertex. One can use short exact sequences in $\mathcal{A}$ to build new objects. For example, if A, B are (say, simple) objects in $\mathcal{A}$ then a short exact sequence

$$0 \longrightarrow A \xrightarrow{\;f\;} C \xrightarrow{\;g\;} B \longrightarrow 0$$

gives an extension C of A by B. The object C will not usually be unique (in fact, the abelian group $\mathrm{Ext}^1_{\mathcal{A}}(B, A)$ classifies such extensions). Note that the simple objects are precisely those objects that cannot be obtained by taking extensions of other objects in this way. Bound states of more than two objects are encoded in the notion of a filtration. A *Jordan-Hölder filtration* of an object E in an abelian category $\mathcal{A}$ is a finite filtration

$$0 = E_0 \subset E_1 \subset \cdots \subset E_{n-1} \subset E_n = E$$

such that each factor object $F_i = E_i/E_{i-1}$ is simple. If such a filtration exists, it will not in general be unique, but one can check that the simple factors F_i are uniquely determined up to isomorphism and reordering. Note that we obtain E by repeatedly gluing simple objects using the short exact sequences

$$0 \longrightarrow E_{i-1} \longrightarrow E_i \longrightarrow F_i \longrightarrow 0.$$

The category $\mathcal{A}$ is *of finite length* if every object $E \in \mathcal{A}$ has a Jordan-Hölder filtration. In an abelian category of finite length the simple objects can be thought of as the basic building blocks; all other objects can be made by repeatedly glueing simple objects together by extensions.

Recall from Proposition 1.6 that $\mathrm{rep}_{\Bbbk}(Q,R)$ is an abelian category for any bound quiver (Q,R).

Proposition 1.11. *The category* $\mathrm{rep}_{\Bbbk}(Q,R)$ *has finite length.*

Proof. Given a finite dimensional representation W of a quiver Q, either W is simple or there is a nontrivial short exact sequence

$$0 \longrightarrow S \longrightarrow W \longrightarrow Q \longrightarrow 0.$$

Now if Q is not simple, then we can break it up into pieces in the same way. This process must stop since W has finite dimension over $\Bbbk$. Eventually, we obtain a simple object B and a surjection $f\colon W \to B$. Take $W^1 \subset W$ to be the kernel of f and repeat the argument with W^1. In this way, we obtain a filtration

$$\cdots \subset W^3 \subset W^2 \subset W^1 \subset W,$$

where each quotient object W^{i-1}/W^i is simple. Once again, this filtration cannot continue indefinitely, so after a finite number of steps we get $W^n = 0$. Renumbering by setting $W_i := W^{n-i}$ for $1 \leq i \leq n$ gives a Jordan-Hölder filtration $W_\bullet$. □

1.5. The fact that $\mathrm{rep}_{\Bbbk}(Q,R)$ has finite length has implications for its Grothendieck group. For an abelian category $\mathcal{A}$ the *Grothendieck group* $K(\mathcal{A})$ is defined to be the free abelian group generated by the isomorphism classes of objects, modulo a relation of the form $C = A + B$ for every short exact sequence

$$0 \longrightarrow A \longrightarrow C \longrightarrow B \longrightarrow 0$$

in the category $\mathcal{A}$.

Theorem 1.12. *Let* (Q,R) *be an acyclic bound quiver and set* $\mathcal{A} := \mathrm{rep}_{\Bbbk}(Q,R) \cong \mathrm{mod}(\Bbbk Q/R)$. *Then the Grothendieck group* $K(\mathcal{A})$ *is isomorphic to the free abelian group* $\mathbb{Z}^{Q_0}$ *generated by the vertex set of* Q.

Proof. The assumption that Q admits no cycles ensures that the vertex simples $\{S(i) : i \in Q_0\}$ form a complete set of representatives for the isomorphism classes of the simple objects in $\mathcal{A}$. Taking the Jordan–Hölder filtration of an arbitrary object $W \in \mathcal{A}$ and applying an easy inductive argument gives

$$[W] = \sum_{k=1}^{n} [W_k/W_{k-1}] = \sum_i \alpha_i [S(i)],$$

where $S(i)$ are the vertex simples and $\alpha_i \in \mathbb{Z}_{\geq 0}$. The isomorphism classes of the vertex simplex therefore generate the Grothendieck group. The map sending a representation of Q to its dimension vector induces a $\mathbb{Z}$-linear map $\dim\colon K(\mathcal{A}) \to \mathbb{Z}^{Q_0}$ under which the image of the set of vertex simples is the standard basis of $\mathbb{Z}^{Q_0}$. It follows that $\{S(i) : i \in Q_0\}$ are independent in $K(\mathcal{A})$. □

This result shows that $\mathrm{rep}_{\Bbbk}(Q)$ is much simpler than the abelian category $\mathrm{coh}(X)$ of coherent sheaves on an algebraic variety of positive dimension. Indeed, the only simple objects of $\mathrm{coh}(X)$ are the skyscraper sheaves of points of X. Since only sheaves supported in dimension zero can have a filtration by finitely many such sheaves, it follows that $\mathrm{coh}(X)$ cannot be of finite length.

2. Lecture 2: Tilting sheaves and exceptional collections

This lecture describes how, in certain very nice cases, the bounded derived category of coherent sheaves on an algebraic variety can be described via the category of finitely-generated representations of a bound quiver (Q, R). When this is the case, we may understand calculations in $D^b(\mathrm{coh}(X))$ using the quiver Q.

2.1. Let $\mathcal{D}$ be a triangulated category. A triangulated subcategory of $\mathcal{D}$ is *epaisse* (thick) if it is closed under isomorphisms, shifts, taking cones of morphisms, and direct summands of objects. The *epaisse envelope* of an object E in $\mathcal{D}$ is the smallest epaisse triangulated subcategory of $\mathcal{D}$ containing E. If the epaisse envelope of E in $\mathcal{D}$ is equal to $\mathcal{D}$ itself, then we say that E *classically generates* $\mathcal{D}$.

2.2. Tilting sheaves. Let X be a smooth projective variety over $\Bbbk$, and let $\mathrm{coh}(X)$ denote the abelian category of coherent sheaves on X. Recall that the *global dimension* of a $\Bbbk$-algebra A is defined to be the maximal projective dimension of any object in $\mathrm{mod}(A)$.

A coherent sheaf T on X is a *tilting sheaf* if

(T1) the algebra $A := \mathrm{End}_{\mathscr{O}_X}(T)$ has finite global dimension;

(T2) we have $\mathrm{Ext}^k_{\mathscr{O}_X}(T, T) = 0$ for all $k > 0$; and

(T3) the sheaf T classically generates $\mathcal{D}^b(\mathrm{coh}(X))$.

If T is locally-free then it is called a *tilting bundle*.

Theorem 2.1 (Baer [1], Bondal [4]). *Let T be a tilting sheaf on a smooth projective variety X, with associated tilting algebra $A = \mathrm{End}_{\mathscr{O}_X}(T)$. Then the functors*

$$F(-) := \mathrm{Hom}_{\mathscr{O}_X}(T, -)\colon\ \mathrm{coh}(X) \to \mathrm{mod}(A^{\mathrm{op}})$$

and

$$G(-) := - \otimes_A T\colon\ \mathrm{mod}(A^{\mathrm{op}}) \to \mathrm{coh}(X)$$

induce equivalences of triangulated categories

$$\mathbf{R}F(-) = \mathbf{R}\mathrm{Hom}_{\mathscr{O}_X}(T,-)\colon \mathcal{D}^b(\mathrm{coh}(X)) \to \mathcal{D}^b(\mathrm{mod}(A^{\mathrm{op}}))$$

and

$$\mathbf{L}G(-) = - \overset{\mathbf{L}}{\otimes}_A T\colon \mathcal{D}^b(\mathrm{mod}(A^{\mathrm{op}})) \to \mathcal{D}^b(\mathrm{coh}(X))$$

that are quasi-inverse to each other.

Proof. The first step is to construct the functors $\mathbf{R}F$ and $\mathbf{L}G$. Let E be a quasicoherent sheaf on X. The vector space $\mathrm{Hom}_{\mathscr{O}_X}(T,E)$ becomes a right A-module by precomposition, that is, for $a \in \mathrm{Hom}_{\mathscr{O}_X}(T,T)$ and $f \in \mathrm{Hom}_{\mathscr{O}_X}(T,E)$ set $a \cdot f = f \circ a \in \mathrm{Hom}_{\mathscr{O}_X}(T,E)$. Since $\mathrm{Hom}_{\mathscr{O}_X}(T,-)$ is a covariant left-exact functor and since the category of quasicoherent sheaves has enough injectives, one obtains a right-derived functor

$$\mathbf{R}F(-) = \mathbf{R}\mathrm{Hom}_{\mathscr{O}_X}(T,-)\colon \mathcal{D}^b(\mathcal{Q}\mathrm{coh}(X))) \to \mathcal{D}(\mathrm{Mod}(A^{\mathrm{op}}))$$

from the bounded derived category of quasicoherent sheaves on X to the derived category of the category of (not necessarily finitely-generated) right A-modules. The cohomology modules of the image are

$$H^i(\mathbf{R}\mathrm{Hom}_{\mathscr{O}_X}(T,E)) = \mathbf{R}^i\,\mathrm{Hom}_{\mathscr{O}_X}(T,E) = \mathrm{Ext}^i_{\mathscr{O}_X}(T,E).$$

Smoothness of X implies that $\mathrm{Ext}^i_{\mathscr{O}_X}(T,E) = 0$ for $i < 0$ and $i > \dim(X)$, so the image of the functor $\mathbf{R}F$ lies in $\mathcal{D}^b(\mathrm{Mod}(A^{\mathrm{op}}))$. Since $\mathcal{D}^b(\mathrm{coh}(X))$ is equivalent to the full subcategory of $\mathcal{D}^b(\mathcal{Q}\mathrm{coh}(X)))$ whose objects have coherent cohomology sheaves, we may consider the restriction of $\mathbf{R}F$ to $\mathcal{D}^b(\mathrm{coh}(X))$. If E is a coherent sheaf then $\mathrm{Ext}^i_{\mathscr{O}_X}(T,E)$ is finite dimensional as a $\Bbbk$-vector space and hence as an A-module, so the cohomology modules of the image actually lie in $\mathrm{mod}(A)$. Thus, we obtain by restriction a functor

$$\mathbf{R}F(-) = \mathbf{R}\mathrm{Hom}_{\mathscr{O}_X}(T,-)\colon \mathcal{D}^b(\mathrm{coh}(X))) \to \mathcal{D}^b(\mathrm{mod}(A^{\mathrm{op}})).$$

Similarly, since the module category $\mathrm{Mod}(A^{\mathrm{op}})$ has enough projectives and the functor $- \otimes_A T$ is right-exact, we obtain a left-derived functor

$$\mathbf{L}G(-) = - \overset{\mathbf{L}}{\otimes}_A T\colon \mathcal{D}^b(\mathrm{Mod}(A^{\mathrm{op}})) \to \mathcal{D}(\mathcal{Q}\mathrm{coh}(X))$$

to the a priori unbounded derived category of quasicoherent sheaves. For an A-module B, the cohomology sheaves of the image are

$$H^j\left(B \overset{\mathbf{L}}{\otimes}_A T\right) = \mathcal{T}or^A_{-j}(B,T),$$

and these vanish outwith a finite range because A has finite global dimension. Furthermore, restricting to finitely generated A-modules ensures that these cohomology sheaves are coherent, giving

$$\mathbf{L}G(-) = - \overset{\mathbf{L}}{\otimes}_A T \colon \mathcal{D}^b(\mathrm{mod}(A^{\mathrm{op}})) \to \mathcal{D}^b(\mathrm{coh}(X)).$$

Since T satisfies property (T2), the composite functor satisfies

$$\mathbf{R}F \circ \mathbf{L}G(A) = \mathbf{R}F\Big(A \overset{\mathbf{L}}{\otimes}_A T\Big) = \mathbf{R}\mathrm{Hom}_{\mathscr{O}_X}(T,T) = \mathrm{Hom}_{\mathscr{O}_X}(T,T) = A,$$

so $\mathbf{R}F \circ \mathbf{L}G$ is the identity on A. The classical envelope of A contains finitely-generated free A-modules, and hence finitely-generated projective A-modules. Since A has finite global dimension, every finitely generated A-module admits a finite projective resolution. This implies that A classically generates $\mathcal{D}^b(\mathrm{mod}(A))$, so $\mathbf{R}F\circ\mathbf{L}G$ is the identity on the whole of $\mathcal{D}^b(\mathrm{mod}(A^{\mathrm{op}}))$. Thus, the left-derived functor $\mathbf{L}G(-)$ identifies $\mathcal{D}^b(\mathrm{mod}(A^{\mathrm{op}}))$ with the triangulated subcategory of $\mathcal{D}^b(\mathrm{coh}(X))$ generated by $\mathbf{L}G(A) = A \overset{\mathbf{L}}{\otimes}_A T = T$. Property (T3) now gives the derived equivalences. □

Corollary 2.2. *Suppose that T is a coherent sheaf on X satisfying (T1) and (T2). Then T satisfies (T3) if and only if*

$$\mathbf{R}\mathrm{Hom}_{\mathscr{O}_X}(T,E) \overset{\mathbf{L}}{\otimes}_A T \cong E$$

for every object $E \in \mathcal{D}^b(\mathrm{coh}(X))$.

Proof. If (T1) and (T2) hold then one needs only $\mathbf{L}G \circ \mathbf{R}F(E) \cong E$ for every object $E \in \mathcal{D}^b(\mathrm{coh}(X))$. □

Example 2.3. For $X = \{x\}$, a point, the trivial bundle $\mathscr{O}_x$ determines both the algebra $A = \mathrm{Hom}(\mathscr{O}_x, \mathscr{O}_x) \cong \Bbbk$ and the equivalence of abelian categories

$$F(-) = \Gamma(x,-)\colon\ \mathrm{coh}(x) \to \mathrm{mod}(A),$$

where $\mathrm{mod}(A)$ is the category of finite dimensional $\Bbbk$-vector spaces. Clearly $\mathbf{R}F$ is a derived equivalence.

While our interest in these lectures lies primarily with smooth projective varieties, we observe in passing that the previous example generalises to any affine variety X. The trivial bundle $\mathscr{O}_X$ determines both the $\Bbbk$-algebra $A = \mathrm{Hom}(\mathscr{O}_X, \mathscr{O}_X) \cong H^0(\mathscr{O}_X)$ and the equivalence of abelian categories

$$F(-) = \Gamma(X,-)\colon\ \mathrm{coh}(X) \to \mathrm{mod}(A),$$

where $\mathrm{mod}(A)$ is the category of finite dimensional $H^0(\mathscr{O}_X)$-modules. Again, $\mathbf{R}F$ is a derived equivalence.

2.3. If X admits a tilting sheaf then understanding $\mathcal{D}^b(\mathrm{coh}(X))$ via the bounded derived category of $\mathrm{mod}(A^{\mathrm{op}})$ is a significant simplification. For example, we have already seen that the module category $\mathrm{mod}(A)$ has finite length, and can be visualised via the corresponding bound quiver (Q, R). The key question, then, becomes existence: which varieties admit tilting bundles? We now present a strong necessary condition for the existence of a tilting bundle.

The Grothendieck group of an abelian category was introduced in lecture 1. The map sending an object $E \in \mathcal{D}^b(\mathcal{A})$ in the bounded derived category to the sum $[E] := \sum_i (-1)^i [E^i] \in K(\mathcal{A})$ in the Grothendieck group gives a well-defined map

$$[\quad]\colon \mathcal{D}^b(\mathcal{A}) \to K(\mathcal{A})$$

that is compatiable with the additive structure, in the sense that $[E \oplus F] = [E] + [F]$. It is well-known that the Grothendieck group $K(\mathrm{coh}(X))$ of a smooth projective variety admits a bilinear form

$$\big\langle [E], [F] \big\rangle := \sum_i (-1)^i \dim_{\Bbbk} \mathrm{Ext}^i_{\mathscr{O}_X}(E, F).$$

called the *Mukai pairing.* As for $K(\mathrm{mod}(A))$, one can show that the global dimension of a $\Bbbk$-algebra A is the minimal number $d \in \mathbb{N}$ such that $\mathrm{Ext}^k(M, N) = 0$ for all $k > d$ and all $M, N \in \mathrm{mod}(A)$. In particular, if A has finite global dimension then a similar formula determines a bilinear form on $K(\mathrm{mod}(A))$, namely

$$\big\langle [M], [N] \big\rangle := \sum_i (-1)^i \dim_{\Bbbk} \mathrm{Ext}^i_A(M, N).$$

Proposition 2.4. *Let T be a tilting sheaf on a smooth projective variety X with $A := \mathrm{End}_{\mathscr{O}_X}(T)$. The homomorphism $[RF(-)]\colon K(\mathrm{coh}(X)) \longrightarrow K(\mathrm{mod}(A^{\mathrm{op}}))$ defined by*

$$[RF(-)] = \sum_i (-1)^i \,\mathrm{Ext}^i_{\mathscr{O}_X}(T, -)$$

is an isomorphism that preserves the natural bilinear forms on either side.

Exercise 2.5. Prove it.

Corollary 2.6. *If a smooth projective variety X is to admit a tilting sheaf, then its Grothendieck group of coherent sheaves must be finitely generated and free.*

Proof. Combine Proposition 2.4 with Theorem 1.12. □

2.4. Strongly exceptional sequences. We now relate the notion of tilting sheaf to the more algebro-geometric notion of full strongly exceptional sequence. First, recall that for coherent sheaves E, F, we have

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(X))}(E, F[k]) = \mathrm{Ext}^k_{\mathscr{O}_X}(E, F).$$

An object E in a triangulated category $\mathcal{D}$ is *exceptional* if

$$\mathrm{Hom}_{\mathcal{D}}(E, E) = \Bbbk \text{ and } \mathrm{Hom}_{\mathcal{D}}(E, E[k]) = 0 \text{ for } k \neq 0.$$

A sequence $(E_0, E_1, \dots, E_m)$ of exceptional objects is called *exceptional* if

$$\mathbf{R}\mathrm{Hom}_{\mathcal{D}}(E_i, E_j) = 0 \text{ for } i > j,$$

and *strongly exceptional* if in addition

$$\mathrm{Hom}_{\mathcal{D}}(E_i, E_j[k]) = 0 \text{ for } i < j \text{ and } k \neq 0.$$

The sequence is *full* (or *complete*) if $E_0, E_1, \dots, E_m$ generate $\mathcal{D}$ as a triangulated category.

Proposition 2.7. *Let T be a locally-free sheaf on X, and $T = \bigoplus_{i=0}^m E_i$ a decomposition into locally-free sheaves satisfying $\mathrm{Hom}_{\mathscr{O}_X}(E_i, E_i) = 0$ for all $i = 0, \dots, m$, e.g., each E_i is a line bundle. Then*

(i) *If T satisfies conditions (T1) and (T2) then, by reordering if necessary, the list $(E_0, E_1, \dots, E_m)$ forms a strongly exceptional sequence.*
(ii) *If, in addition, T satisfies (T3) then $(E_0, E_1, \dots, E_m)$ is a full strongly exceptional sequence.*

Conversely, every full strongly exceptional sequence defines a tilting sheaf.

Proof. Since T satisfies (T2), we have

$$0 = \mathrm{Ext}^k_{\mathscr{O}_X}(T, T) = \bigoplus_{i,j} \mathrm{Ext}^k_{\mathscr{O}_X}(E_i, E_j) \tag{2.1}$$

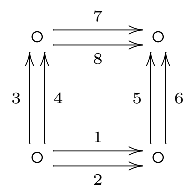
for $k > 0$, which implies all relevant higher Ext-vanishing statements. Each E_i is then exceptional because projectivity of X gives $\mathrm{Hom}_{\mathscr{O}_X}(E_i, E_i) = H^0(\mathscr{O}_X) = \Bbbk$. We now show that we may list the sheaves so that $\mathrm{Hom}_{\mathscr{O}_X}(E_i, E_j) = 0$ for $i > j$. For $0 \leq i \neq j \leq m$, one of the vector spaces $\mathrm{Hom}_{\mathscr{O}_X}(E_i, E_j), \mathrm{Hom}_{\mathscr{O}_X}(E_j, E_i)$ must be trivial, otherwise $\mathrm{Hom}_{\mathscr{O}_X}(E_i, E_i) \neq \Bbbk$ which is absurd, and we relabel to ensure that $i < j$ whenever $\mathrm{Hom}_{\mathscr{O}_X}(E_i, E_j) \neq 0$. Repeat, employing the condition $\mathrm{Hom}_{\mathscr{O}_X}(E_i, E_i) = \Bbbk$ to see that the procedure can be carried out consistantly. This proves part (i), while part (ii) is immediate.

For the converse, equation (2.1) guarantees that a strongly exceptional sequence $(E_0, E_1, \dots, E_m)$ gives a sheaf $T := \bigoplus_i E_i$ satisfying (T2). Define the algebra $A := \mathrm{End}_{\mathscr{O}_X}\left(\bigoplus_i E_i\right)$. The conditions imposed on the Hom-groups of the E_i imply that A is isomorphic to the quotient algebra $\Bbbk Q/\langle R\rangle$ of an acyclic bound quiver (Q, R). This implies that A is isomorphic to an algebra of lower triangular matrices, from which it follows that A has finite global dimension. Given (T1) and (T2), the fullness of the sequence then implies that $T = \bigoplus_i E_i$ satisfies (T3). □

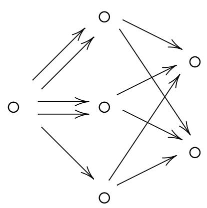
2.5. In the course of proving the previous result we constructed the endomorphism algebra of a tilting bundle as an acyclic bound quiver. For the examples of interest in these lectures, the relations that arise are of the form $p - p' \in R$.

Example 2.8. The algebra $A := \Bbbk Q/\langle R\rangle$ arising from the bound quiver in Example 1.10 is isomorphic to the endomorphism algebra of $T := \bigoplus_{i=0}^{n} \mathscr{O}_{\mathbb{P}^n}(i)$. It is not hard to show that $(\mathscr{O}_{\mathbb{P}^n}, \mathscr{O}_{\mathbb{P}^n}(1), \dots, \mathscr{O}_{\mathbb{P}^n}(n))$ form a strongly exceptional sequence on $\mathbb{P}^n$. It is much harder to show that the sequence is full or, equivalently, that T satisfies (T3). We'll take this up shortly.

Example 2.9. On $\mathbb{P}^1 \times \mathbb{P}^1$, consider the line bundles $\mathscr{O}, \mathscr{O}(1,0), \mathscr{O}(0,1), \mathscr{O}(1,1)$, where $\mathscr{O}(1,0)$ is the pullback of $\mathscr{O}(1)$ via the first projection, and $\mathscr{O}(0,1)$ is the pullback of $\mathscr{O}(1)$ via the second projection. The quiver of sections that encodes the endomorphism algebra of the direct sum of these bundles is shown:



Example 2.10. Let $X = \mathrm{dP}_6$ denote the del Pezzo surface of degree 6 obtained by blowing up $\mathbb{P}^2$ at three points (this variety arises as the apex of the 'roof' in the plane Cremona transformation). The trivial bundle, the triple of line bundles that give morphisms to $\mathbb{P}^1$, and the pair of line bundles defining morphisms to $\mathbb{P}^2$ encode the quiver as shown:



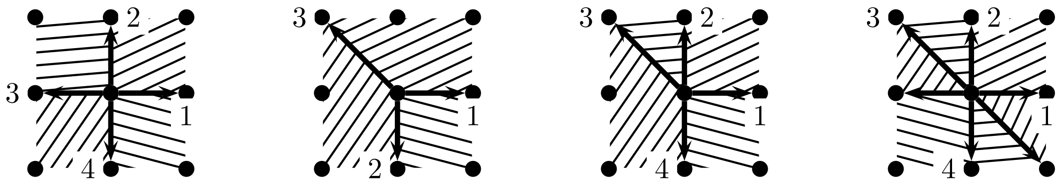
The construction of quivers from collections of line bundles on simple varieties such as these will be taken up in the next lecture.

3. Lecture 3: The derived category of smooth toric del Pezzos

We now investigate the bounded derived category of smooth toric Fano varieties of dimension two.

3.1. Every smooth (projective) toric surface X over $\Bbbk$ arises as follows. For a rank two lattice $N \cong \mathbb{Z}^2$, let $\Sigma \subset N \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}^2$ be a strongly convex rational polyhedral fan (= collection of cones that intersect only along faces) whose support is the whole of $\mathbb{Q}^2$, where each two-dimensional cone is generated by a basis of the lattice N. Each cone $\sigma \in \Sigma$ defines a copy of $\mathbb{A}^2_{\Bbbk}$, and the glueing data of these local charts is encoded in the intersections between cones in the fan. The strongly convex assumption ensures that the origin is a face of each cone, which implies that every $\mathbb{A}^2_{\Bbbk}$ contains a common algebraic torus $T_X := N \otimes_{\mathbb{Z}} \Bbbk^* \cong (\Bbbk^*)^2$.

If one requires that X is Fano ($= -K_X$ is ample) then there are only five examples: $\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_1, \mathrm{Bl}_x(\mathbb{F}_1)$ and dP_6 whose fans are shown below:



(a) $\mathbb{P}^1 \times \mathbb{P}^1$ (b) $\mathbb{P}^2$ (c) $\mathbb{F}_1$ (d) dP_6

FIGURE 1. Fans for four of the five smooth toric del Pezzos

For a global description of X and coherent sheaves on X, write $\Sigma(1)$ for the set of one-dimensional cones in Σ. Each $\rho \in \Sigma(1)$ corresponds to a T_X-invariant Weil divisor D_ρ on X, and these divisors generate the free abelian group $\mathbb{Z}^{\Sigma(1)}$ of T_X-invariant Weil divisors, and there is an exact sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^{\Sigma(1)} \xrightarrow{\deg} \mathrm{Pic}(X) \longrightarrow 0 \tag{3.1}$$

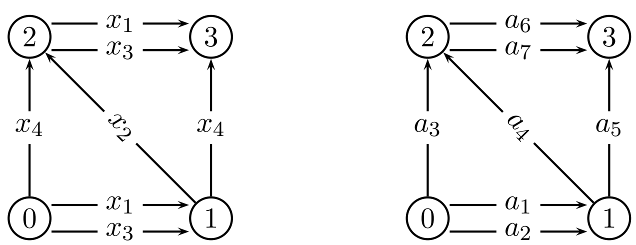
where $M := \mathrm{Hom}(N, \mathbb{Z})$ is the dual lattice. For a line bundle L on X and a global section $s \in H^0(X, L)$, $\mathrm{div}(s)$ denotes the effective Cartier divisor determined by s. The total coordinate ring of X is the polynomial ring $S := \Bbbk[x_\rho : \rho \in \Sigma(1)]$ obtained as the semigroup algebra of the semigroup $\mathbb{N}^{\Sigma(1)}$ of T_X-invariant effective divisors. The map deg induces a $\mathrm{Pic}(X)$-grading of S, where induced by $\deg(x^u) := \deg(u)$. In particular, the algebraic torus $\mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^*)$ acts on S and hence on $\mathbb{A}^{\Sigma(1)} := \mathrm{Spec}(S)$. For a cone $\sigma \in \Sigma$, write $\widehat{\sigma}$ for the set of one-dimensional cones in Σ that are not contained in σ, and write $x^{\widehat{\sigma}} = \prod_{\rho \in \widehat{\sigma}} x_\rho$ for the associated monomial in S. Consider the monomial ideal $B := \big(x^{\widehat{\sigma}} : \sigma \in \Sigma\big)$. Cox [10, Theorem 2.1] shows that the toric variety X is isomorphic to the geometric of the complement $\mathbb{A}^{\Sigma(1)} \smallsetminus \mathbb{V}(B)$ by the action of the torus $\mathrm{Hom}(\mathrm{Pic}(X), \Bbbk^*)$ that is induced by the $\mathrm{Pic}(X)$-grading of S. In addition, Cox [10, Theorem 3.2] estabishes that every coherent sheaf on X arises from a finitely-generated $\mathrm{Pic}(X)$-graded S-module.

3.2. Quivers of sections. Let $(L_0, \dots, L_r)$ be a list of distinct line bundles on the projective toric variety X. A T_X-invariant section $s \in H^0(X, L_j \otimes L_i^{-1})$ is *indecomposable* if the divisor $\mathrm{div}(s)$ cannot be expressed as a sum $\mathrm{div}(s') + \mathrm{div}(s'')$ where $s' \in H^0(X, L_k \otimes L_i^{-1})$ and $s'' \in H^0(X, L_j \otimes L_k^{-1})$ are nonzero T_X-invariant sections and $0 \le k \le r$. The (complete) *quiver of sections* of the list is the quiver Q in which the vertices $Q_0 = \{0, \dots, r\}$ correspond to the line bundles, and the arrows from i to j correspond to the indecomposable T_X-invariant sections in $H^0(X, L_j \otimes L_i^{-1})$. Since X is projective, at least one of $H^0(X, L_j \otimes L_i^{-1})$ and $H^0(X, L_j^{-1} \otimes L_i)$

is zero for $j \neq i$, so Q is acyclic. Note that Q depends only on the line bundles $L_j \otimes L_i^{-1}$ where $0 \leq i, j \leq r$, so we normalise by setting $L_0 = \mathscr{O}_X$. If we assume that $H^0(X, L_i) \neq 0$ for $0 \leq i \leq r$ then Q is connected and has a unique source at $0 \in Q_0$.

Since each arrow $a \in Q_1$ corresponds to a T_X-invariant section $s \in H^0(X, L_j \otimes L_i^{-1})$, we simply write $\mathrm{div}(a) := \mathrm{div}(s) \in \mathbb{Z}^{\Sigma(1)}$. More generally, for a path $p = a_1 \cdots a_\ell$ in Q, we set $\mathrm{div}(p) := \mathrm{div}(a_1) + \cdots + \mathrm{div}(a_\ell)$. This labelling of paths induces relations R on Q, where a difference of paths $p - p'$ lies in R if $\mathrm{tl}(p) = \mathrm{tl}(p')$, $\mathrm{hd}(p) = \mathrm{hd}(p')$ and $\mathrm{div}(p) = \mathrm{div}(p')$. The pair (Q, R) is the *bound quiver of sections* of the line bundles.

Example 3.1. Let $X = \mathbb{F}_1 = \mathbb{P}\big(\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1)\big)$ be the Hirzebruch surface obtained as the one-point blow-up of $\mathbb{P}^2$. For $(k, \ell) \in \mathbb{Z}^2$, we write $\mathscr{O}_X(k, \ell) := \mathscr{O}_X(kD_1 + \ell D_4) \in \mathrm{Pic}(X)$. The complete quiver of sections for $\big(\mathscr{O}_X, \mathscr{O}_X(1,0), \mathscr{O}_X(0,1), \mathscr{O}_X(1,1)\big)$ appears in Figure 2 (a). If we order the arrows as in Figure 2 (b), then the set of



(a) Quiver of sections (b) Listing the arrows

FIGURE 2. Hirzebruch surface $\mathbb{F}_1$

relations is $R = \{a_3a_6 - a_1a_5, a_3a_7 - a_2a_5, a_2a_4a_6 - a_1a_4a_7\}$. These relations arise from pairs of paths from vertex 0 to vertex 3 labelled with the divisors $D_1 + D_4$, $D_3 + D_4$ and $D_1 + D_2 + D_3$.

Proposition 3.2. *If (Q, R) is the complete bound quiver of sections for $(L_0, \ldots, L_r)$ then the quotient algebra $\Bbbk Q/\langle R\rangle$ is isomorphic to $\mathrm{End}\big(\bigoplus_{i=0}^r L_i\big)$.*

Proof. The map sending a path $p = a_1 \cdots a_\ell$ in Q to the product of the corresponding sections $s_1 \cdots s_\ell \in H^0(X, L_{\mathrm{hd}(p)} \otimes L_{\mathrm{tl}(p)}^{-1}) = \mathrm{Hom}(L_{\mathrm{tl}(p)}, L_{\mathrm{hd}(p)})$ determines a homomorphism of $\Bbbk$-algebras η from the path algebra of Q to the endomorphism algebra of $\bigoplus_{i=0}^r L_i$. The map is surjective because Q is a complete quiver. Moreover, η sends paths p, p' in Q satisfying $\mathrm{tl}(p) = \mathrm{tl}(p')$ and $\mathrm{hd}(p) = \mathrm{hd}(p')$ to the same element in $\mathrm{Hom}(L_{\mathrm{tl}(p)}, L_{\mathrm{hd}(p)})$ if and only if $\mathrm{div}(p) = \mathrm{div}(p')$. Thus, we have $\mathrm{Ker}(\eta) = \langle R\rangle$. □

Under mild positivity assumptions on the line bundles, one can reconstruct the variety X directly from the algebra $\Bbbk Q/\langle R\rangle$ (see Craw–Smith [13]).

3.3. We now reconsider condition (T3) for a tilting sheaf. Write $\pi_1, \pi_2 \colon X \times X \to X$ for the first and second projections and let $\iota\colon \Delta \hookrightarrow X \times X$ denote the diagonal embedding.

Lemma 3.3. *The functor*

$$\mathbf{R}(\pi_2)_*\big(\pi_1^*(-) \overset{\mathbf{L}}{\otimes} \mathscr{O}_\Delta\big) \colon D^b(\mathrm{coh}(X)) \to D^b(\mathrm{coh}(X))$$

is naturally isomorphic to the identity.

Proof. For $E \in D^b(\mathrm{coh}(X))$, consider $\mathbf{R}(\pi_2)_*\big(\pi_1^*(E) \overset{\mathbf{L}}{\otimes} \mathscr{O}_\Delta\big)$. The projection formula for ι gives

$$\mathbf{R}\iota_*\big(\mathbf{L}\iota^*(F) \overset{\mathbf{L}}{\otimes} \mathscr{O}_X\big) \cong F \overset{\mathbf{L}}{\otimes} \mathbf{R}\iota_*(\mathscr{O}_X) \cong F \overset{\mathbf{L}}{\otimes} \mathscr{O}_\Delta,$$

for $F \in D^b(\mathrm{coh}(X \times X))$, therefore

$$\begin{aligned}\mathbf{R}(\pi_2)_*\big(\pi_1^*(E) \overset{\mathbf{L}}{\otimes} \mathscr{O}_\Delta\big) &\cong \mathbf{R}(\pi_2)_*\big(\mathbf{R}\iota_*\big(\mathbf{L}\iota^*(\pi_1^*(E)) \overset{\mathbf{L}}{\otimes} \mathscr{O}_X\big)\big)\\ &\cong \mathbf{R}(\pi_2 \circ \iota)_*\big(\mathbf{L}(\pi_1 \circ \iota)^*(E) \overset{\mathbf{L}}{\otimes} \mathscr{O}_X\big)\big)\end{aligned}$$

This is isomorphic to E since both $\pi_2 \circ \iota$ and $\pi_1 \circ \iota$ coincide with the identity. □

Exercise 3.4. Show that $\mathbf{R}\Gamma \circ \mathbf{R}\mathcal{H}om(T, -) \cong \mathbf{R}\mathrm{Hom}(T, -)$ for $T \in \mathcal{D}^b(\mathrm{coh}(X))$.

Lemma 3.5. *There is a natural isomorphism* $\mathbf{R}(\pi_2)_* \circ \pi_1^*(-) \cong \mathbf{R}\Gamma(-) \otimes \mathscr{O}_X$.

Proof. Apply flat base change for the square

$$\begin{array}{ccc} X \times X & \xrightarrow{\pi_2} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow{f} & \mathrm{Spec}(\Bbbk) \end{array}$$

to obtain an isomorphism of functors $\mathbf{R}(\pi_2)_* \circ \pi_1^*(-) \cong g^*\,\mathbf{R}f_*(-)$. Since both f and g are the structure morphism for X, we have $g^*(-) \cong - \otimes \mathscr{O}_X$ and $\mathbf{R}f_*(-) \cong \mathbf{R}\Gamma(-)$. □

3.4. Resolution of the diagonal. We now restrict to the case where the coherent sheaf T is locally-free, so the (derived) dual is simply $T^\vee := \mathcal{H}om(T, \mathscr{O}_X)$. Set $A = \mathrm{Hom}_{\mathscr{O}_X}(T, T)$. Note that T is a left A-module and $T^\vee$ is a right A-module. By pulling back via the first and second projections, $\pi_2^*(T)$ and $\pi_1^*(T^\vee)$ become left and right A-modules respectively, so

$$T^\vee \overset{\mathbf{L}}{\boxtimes}_A T := \pi_1^*(T^\vee) \overset{\mathbf{L}}{\otimes}_A \pi_2^*(T)$$

is an object of $D^b(\mathrm{coh}(X \times X))$. The main result of this lecture is due to King [20], generalising the celebrated result of Beilinson [2]:

Proposition 3.6. *Let T be a locally-free sheaf on X satisfying (T1) and (T2). If there is an isomorphism*

$$T^\vee \overset{\mathbf{L}}{\boxtimes}_A T \longrightarrow \mathscr{O}_\Delta \tag{3.2}$$

in $D^b(\mathrm{coh}(X \times X))$ then T is a tilting bundle, and (3.2) *is a* resolution of the diagonal.

Proof. Suppose that $T^\vee \boxtimes_A T \longrightarrow \mathscr{O}_\Delta$ is an isomorphism. Corollary 2.2 shows that we need only prove that $\mathbf{L}G \circ \mathbf{R}F$ is isomorphic to the identity. The previous lemma and the exercise above gives

$$(\mathbf{L}G \circ \mathbf{R}F)(E) = \mathbf{R}\mathrm{Hom}(T,E) \overset{\mathbf{L}}{\otimes}_A T \cong \mathbf{R}\Gamma(E \otimes T^\vee) \overset{\mathbf{L}}{\otimes}_A T \cong \mathbf{R}(\pi_2)_*\Big(\pi_1^*(E \otimes T^\vee)\Big) \overset{\mathbf{L}}{\otimes}_A T.$$

A (modified) projection formula combined with the exercise above gives

$$\begin{aligned} \mathbf{R}(\pi_2)_*\Big(\pi_1^*(E \otimes T^\vee)\Big) \overset{\mathbf{L}}{\otimes}_A T \;&\cong\; \mathbf{R}(\pi_2)_*\Big(\pi_1^*(E \otimes T^\vee) \overset{\mathbf{L}}{\otimes}_A \pi_2^*(T)\Big) \\ &\cong\; \mathbf{R}(\pi_2)_*\Big(\pi_1^*(E) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X\times X}} \pi_1^*(T^\vee) \overset{\mathbf{L}}{\otimes}_A \pi_2^*(T)\Big). \end{aligned}$$

Thus, we now have

$$\mathbf{R}\mathrm{Hom}(T,E) \overset{\mathbf{L}}{\otimes}_A T \cong \mathbf{R}(\pi_2)_*\Big(\pi_1^*(E) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X\times X}} T^\vee \overset{\mathbf{L}}{\boxtimes}_A T\Big).$$

Compare this with the isomorphism from Lemma 3.3:

$$E \cong \mathbf{R}(\pi_2)_*\Big(\pi_1^*(E) \overset{\mathbf{L}}{\otimes}_{\mathscr{O}_{X\times X}} \mathscr{O}_\Delta\Big),$$

The map $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T \overset{\sim}{\longrightarrow} \mathscr{O}_\Delta$ forces the natural map $\mathbf{R}\mathrm{Hom}(T,E) \overset{\mathbf{L}}{\otimes}_A T \to E$ to be an isomorphism. This means that T satisfies (T3) as required. □

3.5. We conclude this lecture by describing the bounded derived category on the smooth toric del Pezzo surfaces. Let us suppose for now that we have chosen a collection of line bundles on X whose direct sum satisfies (T1) and (T2). The next step is to describe $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T$ explicitly in terms of the bound quiver of sections (Q,R) of the line bundles.

Since $M \otimes_A N = M \otimes_A A \otimes_A N$ for a left A-module N and a right A-module M, the derived tensor product $\pi_1^*(T^\vee) \overset{\mathbf{L}}{\boxtimes}_A \pi_2^*(T)$ can be computed as the double complex $\pi_1^*(T^\vee) \otimes_A P \otimes_A \pi_2^*(T)$, where P is the minimal projective resolution of A in the category of modules over $A^e := A^{\mathrm{op}} \otimes_{\Bbbk} A$. To describe P, we write p^{op} for the element in A^{op} corresponding to $p \in A$. If we write $\{e_i : i \in Q_0\}$ for the set of primitive orthogonal idempotents in A then the elements $e_i^{\mathrm{op}} \otimes e_j$ for $i,j \in Q_0$ form a complete set of primitive orthogonal idempotents in A^e, and $(e_i^{\mathrm{op}} \otimes e_j)A^e$ is a complete set of representatives from the isomorphism classes of indecomposable projective A^e-modules. With this notation, the first terms in a minimal resolution P of A over A^e have the form

$$\to \bigoplus_{r\in Q_2}(e^{\mathrm{op}}_{\mathrm{hd}(p)}\otimes e_{\mathrm{tl}(p)})A^e \xrightarrow{\delta_2} \bigoplus_{a\in Q_1}(e^{\mathrm{op}}_{\mathrm{hd}(a)}\otimes e_{\mathrm{tl}(a)})A^e \xrightarrow{\delta_1} \bigoplus_{i\in Q_0}(e^{\mathrm{op}}_{i}\otimes e_{i})A^e \xrightarrow{\delta_0} A \to 0$$

where Q_2 is a well-chosen basis of $\langle R\rangle$ consisting of differences $p-p'\in R$, and the maps are determined by

$$\delta_0(e_i^{\mathrm{op}}\otimes e_i) = e_i, \quad \delta_1(e^{\mathrm{op}}_{\mathrm{hd}(a)}\otimes e_{\mathrm{tl}(a)}) = a^{\mathrm{op}}\otimes e_{\mathrm{hd}(a)} - e^{\mathrm{op}}_{\mathrm{tl}(a)}\otimes a,$$

and

$$\delta_2(e_h^{\mathrm{op}}\otimes e_t) = \sum_{j=1}^{\ell} a_{1,1}^{\mathrm{op}}\cdots a_{1,j-1}^{\mathrm{op}}\otimes a_{1,j+1}\cdots a_{1,\ell} - \sum_{j=1}^{k} a_{2,1}^{\mathrm{op}}\cdots a_{2,j-1}^{\mathrm{op}}\otimes a_{2,j+1}\cdots a_{2,k},$$

where $r = p-p' = a_{1,1}\cdots a_{1,\ell} - a_{2,1}\cdots a_{2,k}\in Q_2$ and $h=\mathrm{hd}(r)$, $t=\mathrm{tl}(r)$.

The divisors labelling the arrows in Q allows one to lift P to a complex of $\mathrm{Pic}(X)$-graded free modules over the total coordinate ring $S\times S = \Bbbk[x_\rho, w_\rho : \rho\in\Sigma(1)]$ of $X\times X$. The assignment sending $a\in Q_1$ to $\mathrm{div}(a)\in\mathbb{Z}^{\Sigma(1)}$ extends to a $\Bbbk$-linear map $\mathrm{div}\colon A^e\to S\times S$ where $\mathrm{div}(p^{\mathrm{op}}\otimes q) = w^{\mathrm{div}(p)}x^{\mathrm{div}(q)}$. Applying this to P, we obtain a $\mathrm{Pic}(X\times X)$-graded complex E of $S\times S$-modules of the form

$$\dots\to \bigoplus_{r\in Q_2}(S\times S)(r) \longrightarrow \bigoplus_{a\in Q_1}(S\times S)(a) \longrightarrow \bigoplus_{i\in Q_0}(S\times S)(e_i)$$

where $(S\times S)(q)$ is the free $\mathrm{Pic}(X\times X)$-graded $S\times S$-module with one generator q in degree $(-[L_{\mathrm{hd}(q)}],[L_{\mathrm{tl}(q)}])$. In general this complex is not a resolution of the graded $S\times S$-module that defines $\mathscr{O}_\Delta$. However, one can say the following: extend scalars over $\mathbb{Q}$ in (3.1) and set $\square := \{u=\sum_\rho\lambda_\rho e_\rho\in\mathbb{Q}^{\Sigma(1)} : 0\le\lambda_\rho<1\}$.

Theorem 3.7. *Let X be one of the five smooth toric del Pezzo surface. Let T be the direct sum of the line bundles $\{L\in\mathrm{Pic}(X) : L=\deg(u)$ for some $u\in\square\}$. Then T is a tilting bundle on X, so the functor*

$$\mathbf{R}\mathrm{Hom}(T,-)\colon \mathcal{D}^b(\mathrm{coh}(X)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(A^{\mathrm{op}}))$$

is a derived equivalence for $A=\mathrm{End}(T)$.

3.6. The projective plane. The resolution of the diagonal for $\mathbb{P}^n$ a celebrated result of Beilinson [2] (see Căldăraru's lecture notes for a very nice account). In the case of $\mathbb{P}^2$ we have $(0,0,0),(\frac12,\frac12,0),(\frac23,\frac23,\frac23)\in\square$, and $\deg\big((0,0,0)\big) = \mathscr{O}_{\mathbb{P}^2}$, $\deg\big((\frac12,\frac12,0)\big) = \mathscr{O}_{\mathbb{P}^2}(1)$ and $\deg\big((\frac23,\frac23,\frac23)\big) = \mathscr{O}_{\mathbb{P}^2}(2)$. Set $T=\bigoplus_{i=0}^2\mathscr{O}_{\mathbb{P}^2}(i)$. Next, calculate the resolution $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T\to\mathscr{O}_\Delta$ for $\mathbb{P}^2$ using the projective resolution of A

given above:

$$\big((e_2^{\mathrm{op}}\otimes e_0)A^e\big)^{\oplus 3} \xrightarrow{\delta_2} \begin{matrix}\big((e_1^{\mathrm{op}}\otimes e_0)A^e\big)^{\oplus 3}\\ \oplus \\ \big((e_2^{\mathrm{op}}\otimes e_1)A^e\big)^{\oplus 3}\end{matrix} \xrightarrow{\delta_1} \begin{matrix}(e_0^{\mathrm{op}}\otimes e_0)A^e\\ \oplus\\ (e_1^{\mathrm{op}}\otimes e_1)A^e\\ \oplus\\ (e_2^{\mathrm{op}}\otimes e_2)A^e\end{matrix}$$

Having $T = \mathscr{O}_{\mathbb{P}^2}\oplus\mathscr{O}_{\mathbb{P}^2}(1)\oplus\mathscr{O}_{\mathbb{P}^2}(2)$ gives $T^\vee = \mathscr{O}_{\mathbb{P}^2}\oplus\mathscr{O}_{\mathbb{P}^2}(-1)\oplus\mathscr{O}_{\mathbb{P}^2}(-2)$, and hence (or by the complex of $S\times S$-modules given above), the complex $\pi_1^*(T^\vee)\otimes_A P\otimes_A\pi_2^*(T)$ is

$$\big(\mathscr{O}(-2)\boxtimes\mathscr{O}\big)^{\oplus 3} \xrightarrow{\delta_2} \begin{matrix}\big(\mathscr{O}(-1)\boxtimes\mathscr{O}\big)^{\oplus 3}\\ \oplus\\ \big(\mathscr{O}(-2)\boxtimes\mathscr{O}(1)\big)^{\oplus 3}\end{matrix} \xrightarrow{\delta_1} \begin{matrix}\mathscr{O}\boxtimes\mathscr{O}\\ \oplus\\ \mathscr{O}(-1)\boxtimes\mathscr{O}(1)\\ \oplus\\ \mathscr{O}(-2)\boxtimes\mathscr{O}(2)\end{matrix}$$

One can verify that the given complex of $S\times S$-modules gives a resolution of the diagonal. To compare this with Beilinson's resolution, use the Euler sequences

$$0\longrightarrow\mathscr{O}(-1)\boxtimes\Omega^1(1)\longrightarrow\mathscr{O}(-1)\boxtimes\mathscr{O}^{\oplus 3}\longrightarrow\mathscr{O}(-1)\boxtimes\mathscr{O}(1)\longrightarrow 0$$

and

$$0\longrightarrow\mathscr{O}(-2)\boxtimes\Omega^2(2)\longrightarrow\mathscr{O}(-2)\boxtimes\mathscr{O}^{\oplus 3}\longrightarrow\mathscr{O}(-2)\boxtimes\Omega^1(2)\longrightarrow 0$$

to see that the resolution $T^\vee \overset{\mathbf{L}}{\boxtimes}_A T \xrightarrow{\sim} \mathscr{O}_\Delta$ is quasi-isomorphic to the resolution

$$0\longrightarrow\mathscr{O}(-2)\boxtimes\Omega^2(2)\longrightarrow\mathscr{O}(-1)\boxtimes\Omega^1(1)\longrightarrow\mathscr{O}\boxtimes\mathscr{O}\longrightarrow\mathscr{O}_\Delta\longrightarrow 0.$$

This is the classical statement of Beilinson's resolution on $\mathbb{P}^2$.

Exercise 3.8. The goal of this exercise is to compute the derived category for each of the remaining smooth toric del Pezzo surfaces.

(1) Compute the set $\{L\in\mathrm{Pic}(X) : L = \deg(u) \text{ for some } u\in\square\}$ for each of the five toric del Pezzo surfaces;
(2) Calculate the quiver of sections of the resulting line bundles.

As a check, compare with the pictures of the quivers that appeared at the end of lecture 2 (see also the case for $\mathbb{F}_1$ from Figure 2).

4. Lecture 4: On the derived McKay correspondence conjecture

In this section we apply derived category methods to provide an elegant explanation for the McKay correspondence in dimension $n\leq 3$. This correspondence arises naturally in mathematics via the geometry and representation theory of Gorenstein quotient singularities, and in physics in the context of D-branes on certain Calabi–Yau orbifolds.

4.1. The classical statement. Finite subgroups of $\mathrm{SL}(2,\mathbb{C})$ are classified up to conjugacy as

- the cyclic group of order $n \geq 2$ generated by the transformations $(x, y) \to (\varepsilon x, \varepsilon^{n-1} y)$ for $\varepsilon^n = 1$;
- the binary dihedral group of order $4n$ (for $n \geq 2$) generated by the pair $(x, y) \to (-y, x)$ and $(x, y) \to (\varepsilon x, \varepsilon^{2n-1} y)$ for $\varepsilon^{2n} = 1$;
- one of three exceptional cases obtained as the lift under the double cover $\mathrm{SU}(2) \to \mathrm{SO}(3)$ of the symmetry group of a Platonic solid: the binary tetrahedral, binary octahedral or binary icosahedral groups of order 24, 48 and 120 respectively.

In each case, the quotient singularity $X = \mathbb{C}^2/G$ defined by its ring of functions $\mathbb{C}[x, y]^G$ can be embedded as a hypersurface $X \subset \mathbb{C}^3$ with an isolated singularity at the origin. This singular affine variety admits a unique resolution $\tau\colon Y \to X = \mathbb{C}^3/G$ with the property that Y has trivial canonical bundle. The exceptional locus of τ is a tree of rational curves $C \cong \mathbb{P}^1$ intersecting transversally, and we construct a graph from this tree as follows: introduce one vertex for each irreducible exceptional curve C, and join a pair of vertices by an edge if the corresponding curves intersect in Y. The resulting graph is a Dynkin graph of ADE-type. The data of the group, the defining equation and the ADE graph is recorded in Table 1.

Conjugacy class of G	Defining equation of X	Dynkin graph
cyclic $\mathbb{Z}/n\mathbb{Z}$	$x^2 + y^2 + z^n = 0$	A_{n-1}
binary dihedral $\mathbb{D}_{4n}$	$x^2 + y^2 z + z^{n+1} = 0$	D_{n+2}
binary tetrahedral $\mathbb{T}_{24}$	$x^2 + y^3 + z^4 = 0$	E_6
binary octahedral $\mathbb{O}_{48}$	$x^2 + y^3 + yz^3 = 0$	E_7
binary icosahedral $\mathbb{I}_{120}$	$x^2 + y^3 + z^5 = 0$	E_8

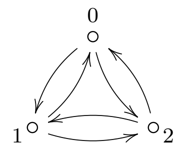
Table 1. Classification of Kleinian singularities.

John McKay [21] observed that the Dynkin graph for G can be constructed using representation theory. Let $V = \mathbb{C}^2$ denote the two-dimensional representation given by the inclusion $G \subset \mathrm{SL}(2,\mathbb{C})$ and write $\mathrm{Irr}(G)$ for the set of isomorphism classes of irreducible representations. The *McKay quiver* Q of $G \subset \mathrm{SL}(2,\mathbb{C})$ has vertex set $Q_0 := \mathrm{Irr}(G)$, and there are $a_{\rho\rho'} := \dim_{\mathbb{C}} \mathrm{Hom}_{\mathbb{C}[G]}(\rho, V \otimes \rho')$ arrows from ρ to ρ'.

Example 4.1. Consider the A_2 case. The given two-dimensional representation of $G \cong \mathbb{Z}/3\mathbb{Z}$ in $\mathrm{SL}(2,\mathbb{C})$ is $V = \rho_1 \oplus \rho_2$, where $\rho_k \in \mathrm{Irr}(G)$ satisfies $\rho_k(g) = \omega^k g$ for $k = 0, 1, 2$ and ω a primitive third root of unity. For $\rho' = \rho_k$ we have

$$V \otimes \rho_k = (\rho_1 \oplus \rho_2) \otimes \rho_k = \rho_{k+1} \oplus \rho_{k-1}.$$

Thus, the McKay quiver has three vertices ρ_0, ρ_1, ρ_2 wth arrows $\rho_{k+1} \to \rho_k$ and $\rho_{k-1} \to \rho_k$ corresponding to the x- and y-coordinates respectively in V.



Exercise 4.2. Draw the McKay quiver for A_n and suitably label the arrows.

For finite subgroups of $\mathrm{SL}(2, \mathbb{C})$, the arrows come in opposite pairs, and replacing every such pair by a single edge produces the McKay graph $\widetilde{\Gamma}_Q$. Let Γ_Q denote the subgraph obtained from the McKay graph by removing the vertex corresponding to the trivial representation and the edges emanating from that vertex.

Theorem 4.3. *The McKay graph $\widetilde{\Gamma}_Q$ is an extended Dynkin graph of ADE type, and the subgraph Γ_Q is the ADE graph of $X = \mathbb{C}^2/G$ given in Table 1, giving a one-to-one* McKay correspondence*:*

$$\textit{basis of } H_*(Y, \mathbb{Z}) \longleftrightarrow \big\{\textit{irreducible representations of } G\big\}.$$

Proof. McKay [21] gives the original observation that forms the first statement. Inspecting the vertices of the graph Γ_Q establishes a one-to-one correspondence between the exceptional curves C of the resolution $\tau \colon Y \to X$ and the nontrivial irreducible representations ρ of G. The exceptional curve classes $[C]$ form a basis for the homology $H_2(Y, \mathbb{Z})$ so that, by adding the homology class of a point on one side and the trivial representation on the other, we obtain the stated one-to-one correspondence. □

The McKay correspondence admits a beautiful geometric explanation that is best described in terms of derived categories as we now describe.

4.2. For a finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$, write $V = \mathbb{C}^n$ for the given representation. The action of G on V induces a dual action of G on the coordinate ring $S := \mathbb{C}[x_1, \dots, x_n]$ of V. The *McKay quiver* Q is defined above, but an explicit description of the appropriate set of relations R is hard to write down in general (see Proposition 4.4 below). If G is abelian, however, then the given representation decomposes into one-dimensional representations $V = \rho_1 \oplus \dots, \oplus \rho_n$, and for every vertex $\rho \in Q$ there are n arrows with head at ρ denoted $a_k^\rho \colon \rho \otimes \rho_k \to \rho$ for $1 \le k \le n$. If we label the arrow a_k^ρ by the monomial x_k, then the set

$$R = \{a_\ell^{\rho \otimes \rho_k} a_k^\rho - a_k^{\rho \otimes \rho_\ell} a_\ell^\rho : \rho \in \mathrm{Irr}(G), 1 \le k \le n\}$$

of relations corresponds to the condition that the labelling monomials commute. We call (Q, R) the *bound McKay quiver* of the abelian subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$.

The set of relations R is chosen to ensure that the quotient algebra $\Bbbk Q/\langle R\rangle$ is isomorphic to the skew group algebra $S * G$. As an S-module, the skew group algebra is the free S-module with basis G, and the ring structure is given by setting $(sg)\cdot(s'g') := s(g\cdot s')gg'$ for $s, s' \in S$ and $g, g' \in G$. Note that since $S = \mathbb{C}[x_1, \dots, x_n]$ has global dimension n by the Hilbert Syzygy Theorem, a well-known result from noncommutative ring theory states that the skew group algebra $S*G$ also has global dimension n.

Proposition 4.4. *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup. There exists a set of relations R in the McKay quiver Q such that the following are equivalent:*

(i) *the category* $\mathrm{rep}_{\mathbb{C}}(Q, R)$*;*
(ii) *the category* $\mathrm{mod}(S * G)$*;*
(iii) *the category* G-$\mathrm{coh}(V)$ *of finitely-generated G-equivariant coherent sheaves on $\mathbb{C}^n$.*

Proof. The set of relations R is constructed so that the skew group algebra $S * G$ is Morita equivalent to the quotient algebra $\Bbbk Q/\langle R\rangle$. It follows that the module categories over these algebras are equivalent. Proposition 1.6 implies that categories (i) and (ii) are equivalent. Coherent sheaves on $\mathbb{C}^n$ are finitely-generated S-modules, and an S-module M is G-equivariant if it has a G-action such that $g \cdot (sm) = (g \cdot s)(g \cdot m)$ for $g \in G, s \in S$ and $m \in M$. This immediately gives the equivalence between (ii) and (iii). □

4.3. Having described the algebraic side of the correspondence, we now consider the geometric side. A natural generalisation for the minimal resolution of the singularity $X = \mathbb{C}^2/G$ arising from a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ is a crepant resolution of an orbifold $X = \mathbb{C}^n/G$ arising from a finite subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$. The special linear condition ensures that X is Gorenstein, i.e., the canonical sheaf ω_X is a line bundle. In fact, the form $dx_1 \wedge \cdots \wedge dx_n$ on $\mathbb{C}^n$ is G-invariant under a special linear action and hence it descends to give a globally defined nonvanishing holomorphic n-form on X, forcing ω_X to be trivial. A resolution $\tau\colon Y \to X$ is said to be *crepant* if $\tau^*(\omega_X) = \omega_Y$; this holds here if and only if ω_Y is also trivial, in which case we call Y a (noncompact) Calabi–Yau manifold. Note that crepant resolutions need not exist, and when they do they are typically nonunique.

4.4. The McKay correspondence conjecture. The guiding principle behind the McKay correspondence was stated by Reid [23] along the following lines:

Principle 4.5. Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite subgroup. Given a crepant resolution $\tau\colon Y \to X = \mathbb{C}^n/G$, the geometry of Y should be equivalent to the G-equivariant geometry of $\mathbb{C}^n$. In particular, any two crepant resolutions of X should have equivalent geometries.

Here, the word ‘geometry’ was left deliberately vague but was known to hold for suitably defined notions of Euler number and Hodge numbers when the principle was first proposed. More significantly, this principle, and indeed any geometric approach to the McKay correspondence owes a great debt to the pioneering work of Gonzalez-Sprinberg–Verdier [15]. For a finite subgroup $G \subset \mathrm{SL}(2, \mathbb{C})$ with minimal resolution $Y \to \mathbb{C}^2/G$, they constructed geometrically an isomorphism $K(\mathrm{coh}(Y)) \to K(\mathrm{mod}(S * G))$ between the Grothendieck group of coherent sheaves on Y and G-equivariant coherent sheaves on $\mathbb{C}^2$.

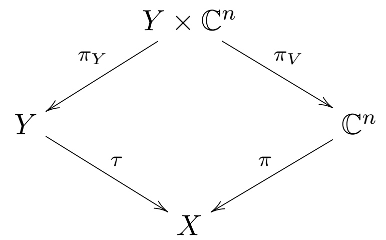
4.5. Reid [22] suggested that one manifestation of Principle 4.5 should be a derived equivalence

$$\Phi\colon \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(S * G)), \tag{4.1}$$

where $\mathcal{D}^b(\mathrm{coh}(Y))$ is the bounded derived categories of coherent sheaves on Y and, following the equivalence of categories from Proposition 4.4, we write $\mathcal{D}^b(\mathrm{mod}(S{*}G))$ for the bounded derived category of G-equivariant S-modules. One approach to this conjecture is to construct a nonprojective analogue of a tilting bundle T on a given crepant resolution Y for which $\mathrm{End}_{\mathscr{O}_Y}(T) = \mathrm{mod}(S * G)$. The fact that the algebra $S * G$ has finite global dimension provides further evidence that this approach might bare fruit.

With the timely publication of Bridgeland's thesis in the summer of 1998, however, Bridgeland, King and Reid together realised that the right approach was to construct the equivalence as a Fourier–Mukai transform. To describe this in more detail, let $\pi\colon \mathbb{C}^n \to X = \mathbb{C}^n/G$ be the quotient morphism and $\tau\colon Y \to X$ a resolution. Consider the commutative diagram

(4.2)



where π_Y and π_V are the projections to the first and second factors. Let G act trivially on both Y and X, so that all morphisms in the above diagram are G-equivariant. By analogy with Mukai's functor on the derived category of the elliptic curve, the key step is to realise the resolution Y as a fine moduli space of certain G-equivariant S-modules. Just as with the Poincaré sheaf for the elliptic curve, this would imply that the product $Y \times \mathbb{C}^n$ comes equipped with a universal sheaf $\mathscr{F}$, such that for each point $y \in Y$, the restriction of $\mathscr{F}$ to the fibre $\pi_Y^{-1}(y) \cong \mathbb{C}^n$ is the G-equivariant coherent sheaf F_y parametrised by the point $y \in Y$. Armed with this universal sheaf, one can define a functor $\Phi^{\mathscr{F}}\colon \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(S * G))$ via

$$\Phi^{\mathscr{F}}(-) = \mathbf{R}(\pi_V)_*\Big(\mathscr{F} \overset{\mathbf{L}}{\otimes} (\pi_Y)^*(- \otimes \rho_0)\Big). \tag{4.3}$$

In this formula: the tensor product with the trivial representation acknowledges that G acts trivially on Y, enabling us to take the G-equivariant pullback via π_Y; and the pullback via π_Y need not be derived since π_Y is flat by virtue of Y being a fine moduli space. Principle 4.5 suggests that Φ is an equivalence of triangulated categories whenever τ is crepant.

4.6. To carry out the programme described above, a given resolution Y must be constructed as a fine moduli space of certain G-equivariant S-modules. We now construct the moduli space $G\text{-Hilb}$ which provided the first examples for which the McKay correspondence was established as a derived equivalence.

For any finite subgroup $G \subset \mathrm{GL}(n, \mathbb{C})$, the *$G$-Hilbert scheme* is the scheme $G\text{-Hilb} := G\text{-Hilb}(\mathbb{C}^n)$ parametrising *G-clusters* in $\mathbb{C}^n$, that is, G-invariant, zero-dimensional subschemes $Z \subset \mathbb{C}^n$ for which the space of global sections $\Gamma(\mathscr{O}_Z)$ is isomorphic as a $\mathbb{C}[G]$-module to the regular representation of G; equivalently, $G\text{-Hilb}$ parametrises the (necessarily G-equivariant) S-modules arising from the structure sheaves $\mathscr{O}_Z$ of these subschemes. There is a projective and surjective Hilbert–Chow morphism

$$\tau \colon G\text{-Hilb} \longrightarrow \mathbb{C}^n/G$$

that sends a G-cluster to its supporting G-orbit. Thus irrespecive of whether $Y := G\text{-Hilb}$ is a resolution of $\mathbb{C}^n/G$, the map τ fits into a commutative diagram of the form (4.2). Any free G-orbit defines a point of $G\text{-Hilb}$, and the set of all such orbits lie in a single irreducible component of $G\text{-Hilb}$ (see [12, §5]).

The universal sheaf on $Y \times \mathbb{C}^n$ is the structure sheaf $\mathscr{O}_{\mathscr{Z}}$ of the universal closed subscheme $\mathscr{Z} \subset Y \times \mathbb{C}^n$ whose restriction to the fibre over $y \in Y$ is the scheme $Z_y \subset \mathbb{C}^n$ parametrised by the point y. Moreover, since each fibre is isomorphic to the regular representation as a $\mathbb{C}[G]$-module, the pushforward $T := (\pi_Y)_*(\mathcal{O}_Z)$ is a vector bundle on Y that decomposes into a direct sum of vector bundles

$$T = \bigoplus_{\rho \in \mathrm{Irr}(G)} R_\rho^{\oplus \dim(\rho)}$$

according to the decomposition of the regular representation, where $\mathrm{rank}(R_\rho) = \dim(\rho)$. Without loss of generality, we assume that the bundle R_{ρ_0} corresponding to the trivial representation ρ_0 is the trivial bundle.

4.7. Classical McKay revisited. To conclude this lecture we state the derived McKay correspondence in dimension two. In the form stated below, the result is due to Ito–Nakamura [16] and Kapranov–Vasserot [17] (though the later result was known also to Gonzalez–Sprinberg and Verdier [15]).

Theorem 4.6. *Let $G \subset \mathrm{SL}(2, \mathbb{C})$ be a finite subgroup and let $\tau \colon Y \to \mathbb{C}^2/G$ be the unique crepant resolution.*

(i) *The variety Y is isomorphic to the G-Hilbert scheme* $G\text{-Hilb}(\mathbb{C}^2)$*;*

(ii) *The functor $\Phi^{\mathscr{O}_{\mathscr{Z}}}$ with kernel the universal sheaf on* $G\text{-Hilb}$ *is an equivalence of derived categories*

$$\Phi^{\mathscr{O}_{\mathscr{Z}}} \colon \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(S * G)).$$

Corollary 4.7. *The classical McKay correspondence Theorem 4.3 holds.*

Proof of the Corollary given Theorem 4.6. The equivalence $\Phi^{\mathscr{F}}$ induces an isomorphism of Grothendieck groups $\varphi\colon K(\mathrm{coh}(Y)) \to K(\mathrm{mod}(S*G))$ as in Proposition 2.4. The map $K(\mathrm{coh}(Y)) \to H^0(Y,\mathbb{Z}) \oplus H^2(Y,\mathbb{Z})$ sending the class of a sheaf $[\mathscr{E}]$ to $(\mathrm{rank}(\mathscr{E}), c_1(\mathscr{E}))$ is an isomorphism of groups, as is that from $K(\mathrm{mod}(S*G))$ to the underlying the representation ring of G. Now inspect the bases on each side. □

Exercise 4.8. For the A_n case, compute explicitly the monomial ideals that define torus-invariant points of G-Hilb and hence compute the tautological line bundles $\{R_\rho : \rho \in \mathrm{Irr}(G)\}$ on G-Hilb. Observe that the first Chern classes of these bundles (for $\rho \neq \rho_0$) form the basis of $H^2(Y,\mathbb{Z})$ that is dual to the basis of $H_2(Y,\mathbb{Z})$ given by the irreducible components of the exceptional curve.

5. Lecture 5: Mukai $\Longrightarrow$ McKay

5.1. Bridgeland–King–Reid presented a significant generalisation of Theorem 4.6 in the celebrated paper [6]. Given a morphism $\tau\colon Y \to X$, the fibre product of Y with itself over X is a scheme with underlying set given by

$$Y \times_X Y = \big\{(y,y') \in Y \times Y : \tau(y) = \tau(y')\big\}.$$

Theorem 5.1. *Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup. If the irreducible component $Y \subseteq G$-*Hilb *containing the free G-orbits satisfies* $\dim\big(Y \times_X Y\big) \leq n+1$*, then:*

1. *the morphism $\tau\colon Y \to X$ is a crepant resolution; and*
2. *the functor $\Phi^{\mathscr{O}_\mathcal{Z}}$ with kernel the universal sheaf for G-*Hilb *is an equivalence of derived categories*

$$\Phi^{\mathscr{O}_\mathcal{Z}} \colon \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(S*G)).$$

Remark 5.2. The condition on the dimension of the fibre product always holds for finite $G \subset \mathrm{SL}(n,\mathbb{C})$ with $n \leq 3$, because $\dim(Y \times_X Y)$ is at most twice the dimension of the exceptional locus; this equals one for $n=2$ and two for $n=3$. However, for $n \geq 4$ this condition rarely holds.

The approach to the proof of Bridgeland, King and Reid was fundamentally different in spirit from that of Kapranov and Vasserot in proving Theorem 4.6. Rather than beginning with a moduli construction of a fixed crepant resolution, they took as their starting point the moduli space G-Hilb$(\mathbb{C}^n)$ and introduced a necessary condition for the variety G-Hilb$(\mathbb{C}^n)$ to be a crepant resolution of $\mathbb{C}^n/G$. The remarkable point of their construction is that, whenever this necessary condition is satisfied, the morphism $\tau\colon G$-Hilb$(\mathbb{C}^n) \to \mathbb{C}^n/G$ sending a G-cluster to its supporting G-orbit is shown to be a crepant resolution only after the equivalence of derived categories has been established.

5.2. We now present a roadmap for the proof of Theorem 5.1. To simplify our discussion, we make several rather outrageous assumptions: firstly, that Y is projective; and secondly, that $\tau\colon G\text{-Hilb} \to X$ is a crepant resolution. The projectivity assumption is patently false (!), but it simplifies the discussion.

Set $Y = G$-Hilb and consider the integral functor $\Phi^{\mathscr{O}_{\mathcal{Z}}}$ defined by (4.3). The goal is to establish that $\Phi^{\mathscr{O}_{\mathcal{Z}}}$ is a derived equivalence by using the following result of Mukai, Bondal–Orlov and Bridgeland (see Căldăraru [8] for a similar statement and for the interpretation as an isometry):

Theorem 5.3. *Let Y be a smooth projective variety and let $\mathcal{D}$ be an indecomposable triangulated category that admits a Serre functor. Let $\Phi\colon \mathcal{D}^b(\mathrm{coh}(Y)) \to \mathcal{D}$ be an exact functor that admits a left adjoint. The functor Φ is fully faithful if and only if for all*[1] *$i \in \mathbb{Z}$ and $y, y' \in Y$, we have*

$$\mathrm{Hom}_{\mathcal{D}}(\Phi(\mathscr{O}_y), \Phi(\mathscr{O}_{y'})[i]) \cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(Y))}(\mathscr{O}_y, \mathscr{O}_{y'}[i]).$$

If, in addition, Φ commutes with the Serre functors then Φ is a derived equivalence.

In fact, Theorem 5.3 is a special case of a result of Bridgeland that applies to exact functors between (almost) arbitrary triangulated categories. In our case, since Y is smooth and projective, the skyscraper sheaves $\{\mathscr{O}_y : y \in Y\}$ form a spanning class in $\mathcal{D}^b(\mathrm{coh}(Y))$, that is, they form the appropriate notion of an orthogonal basis in $\mathcal{D}^b(\mathrm{coh}(Y))$.

5.3. We ignore indecomposabilty, other than to note that the triangulated category $\mathcal{D}^b(\mathrm{mod}(S * G))$ is indecomposable.

Sketch proof. Since Y is projective (!), we may apply a G-equivariant version of Grothendieck duality due to Neeman, and hence, as in Căldăraru [8, §5.2], the functor $\Psi\colon \mathcal{D}^b(\mathrm{mod}(S * G)) \longrightarrow \mathcal{D}^b(\mathrm{coh}(Y))$ defined by

$$\Psi(-) := \Big[\mathbf{R}(\pi_Y)_*\Big(\pi_V^*(-) \overset{\mathbf{L}}{\otimes} \mathscr{O}_{\mathcal{Z}}^{\vee} \overset{\mathbf{L}}{\otimes} \pi_V^*(\omega_V)[n]\Big)\Big]^G,$$

is left-adjoint to $\Phi^{\mathscr{O}_{\mathcal{Z}}}$ (note that taking the G-invariant part is adjoint to the functor that forms the tensor product by the trivial representation of G). Theorem 5.3 implies that $\Phi^{\mathscr{O}_{\mathcal{Z}}}$ is fully faithful if for $\mathcal{D} = \mathcal{D}^b(\mathrm{mod}(S * G))$ we have

$$\mathrm{Hom}_{\mathcal{D}}\big(\Phi(\mathscr{O}_y), \Phi(\mathscr{O}_{y'})[i]\big) = \mathrm{Ext}^i_{\mathscr{O}_X}(\mathscr{O}_y, \mathscr{O}_{y'}) = \begin{cases} 0 & \text{if } y \neq y' \text{ or } i \notin [0,n] \\ \mathbb{C} & \text{if } y = y' \text{ and } i = 0, \end{cases} \tag{5.1}$$

for $i \in \mathbb{Z}$ and $y, y' \in Y$. The kernel gives $\Phi(\mathscr{O}_y) = \mathscr{O}_{Z_y}$ where $Z_y \subset \mathbb{C}^n$ is the G-cluster corresponding to the point $y \in Y$, and Hom-groups in $\mathcal{D}^b(\mathrm{mod}(S * G))$

[1] Strikingly, we need not verify the isomorphism for the difficult $0 < i \leq \dim Y$ and $y = y'$ cases, when $\mathrm{Ext}^i(\mathscr{O}_y, \mathscr{O}_y) \neq 0$. Take care though: [6] cites a theorem for the equivalence that does not state this crucial point.

are the G-invariant part of Ext-groups, so

$$\mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}(S*G))}\big(\Phi(\mathscr{O}_y),\Phi(\mathscr{O}_{y'})[i]\big) = G\text{-}\mathrm{Ext}^i_S(\mathscr{O}_{Z_y},\mathscr{O}_{Z_{y'}}).$$

It's easy to see that the $i=0$ and $y=y'$ case is simply $G\text{-}\mathrm{Hom}_S(\mathscr{O}_{Z_y},\mathscr{O}_{Z_y}) = G\text{-}\mathrm{Hom}_S(S/I_{Z_y},S/I_{Z_y}) = \mathbb{C}$. Also, the Ext groups $\mathrm{Ext}^i_S(\mathscr{O}_{Z_y},\mathscr{O}_{Z_y})$ vanish for $i\notin[0,n]$, hence so do the G-Ext groups.

It remains to treat the case when $y\neq y'$. The G-clusters Z_y and $Z_{y'}$ are disjoint when their supporting G-orbits $\tau(y),\tau(y')\in\mathbb{C}^n$ satisfy $\tau(y)\neq\tau(y')$, so

(5.2)
$$(y,y')\in Y\times Y \text{ with } \tau(y)\neq\tau(y') \implies G\text{-}\mathrm{Ext}^i_S(\mathscr{O}_{Z_y},\mathscr{O}_{Z_{y'}})=0 \quad \text{for all } i\in\mathbb{Z}.$$

Otherwise $y\neq y'$ and $\tau(y)=\tau(y')$, so the pair $y,y'\in Y$ satisfies $(y,y')\in Y\times_X Y\smallsetminus\Delta$, where Δ is the diagonal. The assumption that $G\subset\mathrm{SL}(n,\mathbb{C})$ implies that $\omega_{\mathbb{C}^n}$ is a (trivial) G-equivariant coherent sheaf. Moreover, our spurious projectivity assumption on Y enables us to apply Serre duality[2] on $\mathbb{C}^n$ to obtain

$$G\text{-}\mathrm{Ext}^n(\mathscr{O}_{Z_y},\mathscr{O}_{Z_{y'}}) \cong G\text{-}\mathrm{Hom}(\mathscr{O}_{Z_{y'}},\mathscr{O}_{Z_y})=0$$

(omitting the dual). Since the global dimension of $S*G$ is n we obtain

(5.3)
$$(y,y')\in Y\times_X Y \text{ with } y\neq y' \implies G\text{-}\mathrm{Ext}^i(\mathscr{O}_{Z_y},\mathscr{O}_{Z_{y'}})=0 \quad \text{unless } 1\le i\le n-1.$$

The assumption $\dim(Y\times_X Y)\le n+1$ in Theorem 5.1 is imposed to ensure that these groups vanish for all $i\in\mathbb{Z}$. To see how this works we pause to recall a result of Bridgeland–Maciocia [7].

5.4. Consider $Q\in\mathcal{D}^b(\mathrm{coh}(Y))$. The *support* of Q, denoted $\mathrm{supp}(Q)$, is defined to be the closed subset of Y obtained as union of the supports of the cohomology sheaves $\mathcal{H}^i(Q)$ of Q. The *homological dimension* of E is the smallest integer d such that Q is quasi-isomorphic to a complex of locally-free sheaves of length d; write $\mathrm{homdim}(Q)=d$.

Lemma 5.4 (Bridgeland–Maciocia [7]). *For any scheme Y, fix $Q\in\mathcal{D}^b(\mathrm{coh}(Y))$ and $y\in Y$. Then*

$$y\in\mathrm{supp}(Q) \iff \exists\, i\in\mathbb{Z} \text{ such that } \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(Y))}(Q,\mathscr{O}_y[i])\neq 0.$$

The proof of this lemma involves a simple spectral sequence argument. The proof of the next lemma, however, is the geometric interpretation of a deep commutative algebra result called the intersection theorem:

Proposition 5.5 (Bridgeland–Maciocia [7]). *For a quasiprojective scheme Y, let $Q\in\mathcal{D}^b(\mathrm{coh}(Y))$ be a nonzero object and $d\in\mathbb{Z}_{\ge 0}$. Then* $\mathrm{homdim}(Q)\le d$ *if and*

[2]Our assumption that Y is projective is silly enough to imply that $\mathbb{C}^n$ is also projective!!

only if there exists $j \in \mathbb{Z}$ such that for all $y \in Y$,

$$\operatorname{Hom}_{\mathcal{D}^b(\operatorname{coh}(Y))}(Q, \mathscr{O}_y[i]) = 0 \quad \textit{unless } j \leq i \leq j + d.$$

Moreover, the inequality $\operatorname{codim}(\operatorname{supp}(Q)) \leq \operatorname{homdim}(Q)$ *holds.*

This result will be used in the form $\operatorname{codim}(\operatorname{supp}(Q)) > \operatorname{homdim}(Q) \implies Q \cong 0$.

5.5. We now return to the proof of the theorem.

Sketch proof revisited. Suppose for the moment that we can find $Q \in \mathcal{D}^b(\operatorname{coh}(Y \times Y))$ such that

$$\operatorname{Hom}_{\mathcal{D}^b(\operatorname{coh}(Y\times Y))}(Q, \mathscr{O}_{(y,y')}[i]) = G\text{-}\operatorname{Ext}^i_S(\mathscr{O}_{Z_y}, \mathscr{O}_{Z_{y'}}). \tag{5.4}$$

Suppose that the restriction of the object Q to $Y \times Y \smallsetminus \Delta$ is nonzero. Note that

$$(5.2) \text{ with Lemma 5.4} \implies \operatorname{supp}(Q|_{Y\times Y \smallsetminus \Delta}) \subset Y \times_X Y \smallsetminus \Delta.$$

The assumption from the statement of Theorem 5.1 now implies that

$$\dim\big(\operatorname{supp}(Q|_{Y\times Y\smallsetminus\Delta})\big) \leq n+1, \quad \textit{i.e.}, \quad \operatorname{codim}\big(\operatorname{supp}(Q|_{Y\times Y\smallsetminus\Delta})\big) \geq n-1.$$

However, if we compare this with

$$(5.3) \text{ with Proposition 5.5} \implies \operatorname{homdim}(Q|_{Y\times_X Y\smallsetminus\Delta}) \leq n-2.$$

and apply the last statement of the intersection theorem, we obtain $Q|_{Y\times_X Y\smallsetminus\Delta} \cong 0$.

Now chase the logic backwards: since Q is supported on Δ, Lemma 5.4 together with (5.4) imply that

$$G\text{-}\operatorname{Ext}^i_S(\mathscr{O}_{Z_y}, \mathscr{O}_{Z_{y'}}) = 0 \quad \text{for all } (y, y') \in Y \times_X Y \smallsetminus \Delta.$$

This is the final piece of G-Ext-vanishing that we require to apply the result of Mukai, Bondal–Orlov and Bridgeland! Thus, $\Phi^{\mathscr{O}_{\mathcal{Z}}}$ is fully faithful provided we establish that an object Q satisfying (5.4) exists. In passing we note that triviality of $\omega_{\mathbb{C}^n}$ as a G-equivariant sheaf implies that $\Phi^{\mathscr{O}_{\mathcal{Z}}}$ commutes with the Serre functors, so $\Phi^{\mathscr{O}_{\mathcal{Z}}}$ is an equivalence!

5.6. We conclude the proof (given our outrageous assumptions) by constructing Q. As with the derived equivalences arising from a tilting sheaf from Lecture 2, we compose $\Phi := \Phi^{\mathscr{O}_{\mathcal{Z}}}$ with its left-adjoint Ψ to obtain the composite functor

$$\big(\Psi \circ \Phi\big)(-) = \mathbf{R}(\pi_2)_*\big(\mathcal{Q} \overset{\mathbf{L}}{\otimes} \pi_1^*(-)\big),$$

where $\pi_1, \pi_2 \colon Y \times Y \to Y$ are the first and second projections and where $\mathcal{Q} \in \mathcal{D}^b(\mathrm{coh}(Y \times Y))$ is obtained by composition of correspondences (and can be computed explicitly using the formula of Mukai presented in Căldăraru [8, Proposition 5.1]). Observe that for the closed embedding $\iota_y \colon \{y\} \times Y \hookrightarrow Y \times Y$, we have

$$\begin{aligned}
G\text{-}\mathrm{Ext}^i_S(\mathscr{O}_{Z_y}, \mathscr{O}_{Z_{y'}}) &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{mod}(S*G))}(\Phi(\mathscr{O}_y), \Phi(\mathscr{O}_{y'})[i]) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(Y))}(\Psi(\Phi(\mathscr{O}_y)), \mathscr{O}_{y'}[i]) && \text{by adjunction} \\
.\ &\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(Y))}(\mathbf{L}\iota_y^*(\mathcal{Q}), \mathscr{O}_{y'}[i]) \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(Y \times Y))}\big(\mathcal{Q}, \iota_{y,*}(\mathscr{O}_{y'})[i]\big) && \text{by adjunction} \\
&\cong \mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(Y \times Y))}(\mathcal{Q}, \mathscr{O}_{(y,y')}[i])
\end{aligned}$$

Therefore, $Q := \mathcal{Q}$ is precisely the object that we sought to encode the G-Ext-vanishing from (5.4). This concludes the proof given our outrageous assumptions.

5.7. The assumptions imposed to prove the theorem do not hold in general (or indeed, at all!): the scheme G-Hilb is neither smooth nor irreducible in general; the map τ need not be crepant even if G-Hilb is smooth; the scheme G-Hilb is never projective. We consider these problems one-by-one:

(1) Bridgeland, King and Reid avoid the reducibility issue by restricting to the irreducible component $Y \subseteq G$-Hilb that contains the free G-orbit. The Hilbert–Chow morphism $\tau \colon G\text{-Hilb} \to \mathbb{C}^n/G$ that sends a G-cluster to its supporting G-orbit restricts to give a projective birational morphism $\tau \colon Y \longrightarrow X$ which fits into the commutative diagram given earlier.

(2) the assumption that G-Hilb is smooth crucial to the method described above, since it ensures that the skyscraper sheaves $\{\mathscr{O}_y : y \in Y\}$ form a spanning class in $\mathcal{D}^b(\mathrm{coh}(Y))$. To avoid this issue, [6] cites the Mukai, Bondal–Orlov and Bridgeland result only after having established that G-Hilb is smooth. The trick is to establish smoothness by invoking an additional consequence of the intersection theorem. Indeed, after showing that the object $Q \in \mathcal{D}^b(\mathrm{coh}(Y \times Y))$ is supported on the diagonal Δ, the object $\Psi\Phi(\mathscr{O}_y)$ can be shown to satisfy $H^0(\Psi\Phi(\mathscr{O}_y)) = \mathscr{O}_y$ for all $y \in Y$. The G-Ext-vanishing from above gives

$$\mathrm{Hom}^i(\Psi\Phi(\mathscr{O}_y), \mathscr{O}_{y'}) = 0 \text{ unless } y = y' \text{ and } 0 \leq i \leq n.$$

The intersection theorem then implies that Y is smooth at y.

(3) the crepant condition is proven after establishing the derived equivalence. Indeed, triviality of the Serre functor on $\mathcal{D}^b(\mathrm{mod}(S * G))$ is carried across the equivalence to establish that the Serre functor on $\mathcal{D}^b(\mathrm{coh}(Y))$ is trivial. This is certainly not immediate and requires a tricky local argument.

(4) As for the lack of projectivity, an equivariant version of Grothendieck duality is invoked by replacing $\mathcal{D}^b(\mathrm{coh}(Y))$ by the full subcategory $\mathcal{D}^b_0(\mathrm{coh}(Y))$

consisting of objects supported on the subscheme $\tau^{-1}(\pi(0))$. The image under $\Phi^{\mathscr{O}_Z}$ of this subcategory then lies in the full subcategory $\mathcal{D}^b_0(\mathrm{mod}(S*G))$ consisting of complexes of finitely generated nilpotent $S*G$-modules, giving a functor

$$\Phi^{\mathscr{O}_Z}\colon \mathcal{D}^b_0(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b_0(\mathrm{mod}(S*G)).$$

This trick has become standard when working with the derived category of a resolution of an orbifold singularity.

6. Lecture 6: Wall crossing phenomena for moduli of G-constellations

In this lecture we find that the result of Bridgeland, King and Reid holds for a larger class of fine moduli spaces than simply the G-Hilbert scheme.

6.1. Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup. The G-Hilbert scheme parametrises G-clusters, each of which may be regarded as a G-equivariant quotient S-modules S/I for which the $\mathbb{C}[G]$-module structure on S/I is isomorphic to the regular representation of G. We now generalise this notion.

A *G-constellation* is an $S*G$-module that is isomorphic as a $\mathbb{C}[G]$-module to the regular representation of G. Recall that the Grothendieck group of the abelian category of $S*G$-modules is the free abelian group $K(\mathrm{mod}(S*G)) \cong \bigoplus_{\rho\in \mathrm{Irr}(G)} \mathbb{Z}\rho$. Let $R = \sum_\rho \dim(\rho)\rho$ denote the regular representation and consider a parameter $\theta \in \mathrm{Hom}(K(\mathrm{mod}(S*G)),\mathbb{Q})$ satisfying $\theta(R) = 0$. A G-constellation M is said to be *θ-stable* if $\theta(M) = 0$ and if for every proper, nonzero $S*G$-submodule $M' \subset M$ we have $\theta(M') > 0$. The notion of *θ-semistability* is obtained by replacing $>$ with $\geq$. One can recover the set of G-clusters from the set of all G-constellations as follows.

Lemma 6.1. *Consider $\theta_+\colon \bigoplus_{\rho\in\mathrm{Irr}(G)} \mathbb{Z}\rho \to \mathbb{Q}$ satisfying $\theta_+(R) = 0$, with $\theta_+(\rho) > 0$ for all $\rho \neq \rho_0$. Every θ_+-stable G-constellation is a G-cluster, and conversely.*

Proof. The stability condition θ_+ ensures that no proper $S*G$-submodule of a θ_+-stable G-constellation contains the trivial representation. The result follows since a G-constellation M is a G-cluster if and only if M is cyclic as an S-module with generator $\rho_0 \in R \cong M$. □

The G-Hilbert scheme may therefore be regarded as the fine moduli space of θ_+-stable G-constellations. In fact, most parameters in the vector space

$$\Theta := \big\{ \mathrm{Hom}(\textstyle\bigoplus_{\rho\in\mathrm{Irr}(G)} \mathbb{Z}\rho,\mathbb{Q}) : \theta(R) = 0\big\}$$

define a fine moduli space of θ-stable G-constellations. To justify this claim we turn to quiver representations.

6.2. Moduli construction. We assume that $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite abelian subgroup of order $r := |G|$ for simplicity, so that the bound McKay quiver (Q, R) is easy to write down. Recall that the bound McKay quiver of the subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$ has vertex set $Q_0 = \mathrm{Irr}(G)$ and arrow set Q_1 consists of nr arrows.

Given the equivalence of abelian categories established in Proposition 4.4, every finitely-generated $S{*}G$-module corresponds to a representation of the bound McKay quiver and, moreover, the condition that the module is isomorphic as a $\mathbb{C}[G]$-module to the regular representation translates into the condition that the corresponding quiver representation $W = \big(\{W_i\}_{i \in Q_0}, \{w_a\}_{a \in Q_1}\big)$ satisfies $\dim_{\Bbbk}(W_i) = 1$ for $i \in Q_0$. After choosing bases on each W_i for $i \in Q_0$, the representation space is

$$\mathbb{A}^{nr}_{\Bbbk} = \mathrm{Spec}\big(\Bbbk[z_i^\rho : 1 \leq i \leq n, \rho \in \mathrm{Irr}(G)]\big) \cong \bigoplus_{a \in Q_1} \mathrm{Hom}_{\Bbbk}(W_{\mathrm{tl}(a)}, W_{\mathrm{hd}(a)})$$

Since we are interested only in representations of the bound quiver (Q, R), we work not with $\mathbb{A}^{nr}_{\Bbbk}$, but with the subscheme $Z \subset \mathbb{A}^{nr}$ defined by the ideal

$$I = \big\langle z_j^{\rho\rho_i} z_i^\rho - z_i^{\rho\rho_j} z_j^\rho : \rho \in \mathrm{Irr}(G), 1 \leq i, j \leq n \big\rangle.$$

This restriction is equivalent to considering only representations of the bound quiver (Q, R). The algebraic torus $T_Q := \big(\prod_{i \in Q_0} \mathrm{GL}(1)\big)/\mathbb{C}^*$ of dimesion $r-1$ acts faithfully on the subscheme $Z \subset \mathbb{A}^{nr}_{\Bbbk}$ by change of basis, and T_Q-orbits correspond to isomorphism classes of representations of the quiver Q. To form an orbit space with nice properties we turn to Geometric Invariant Theory (GIT), and to do this we must choose a character of the group T_Q that acts. The key link that relates the GIT quotient construction with the discussion of G-constellations from above is that group of characters of T_Q is

$$T_Q^* = \big\{\theta \in \mathbb{Z}^{Q_0} : \textstyle\sum_{i \in Q_0} \theta_i = 0\big\} \cong \big\{\mathrm{Hom}(\bigoplus_{\rho \in \mathrm{Irr}(G)} \mathbb{Z}\rho, \mathbb{Z}) : \theta(R) = 0\big\}$$

hence $T_Q \otimes_{\mathbb{Z}} \mathbb{Q} = \Theta$ (!). Applying the main result of King [19] to this situation shows that a point of Z is θ-(semi)stable in the sense of GIT if and only if it the corresponding G-constellation is θ-(semi)stable in the sense defined above, and we write

$$\overline{\mathcal{M}_\theta}(Q, R) := Z_\theta^{\mathrm{ss}}/T_Q \quad \text{and} \quad \mathcal{M}_\theta(Q, R) := Z_\theta^{\mathrm{s}}/T_Q$$

for the categorical quotient and geometric quotient of the open subschemes of Z parametrising θ-semistable G-constellations and θ-stable G-constellations respectively. A parameter $\theta \in \Theta$ is *generic* if every point of Z that is θ-semistable is in fact θ-stable, in which case $\overline{\mathcal{M}_\theta}(Q, R) = \mathcal{M}_\theta(Q, R)$. The subset of generic parameters decomposes into finitely many open cones separated by *walls*, where $\mathcal{M}_\theta(Q, R)$ remains unchanged as θ varies in a chamber (though its polarising line bundle varies).

Definition 6.2. For any generic parameter $\theta \in \Theta$, the scheme $\mathcal{M}_\theta(Q, R)$ is the *fine moduli space of θ-stable G-constellations*. In addition, the moduli space comes armed with a *universal bundle* $\mathscr{U}_\theta$ on the product $\mathcal{M}_\theta(Q,R)\times\mathbb{C}^n$ whose fibre over any point of $\mathcal{M}_\theta(Q,R)$ is the corresponding G-equivariant coherent sheaf on $\mathbb{C}^n$. The push-forward via the first projection gives the tautological bundle $\mathscr{R}_\theta$ that decomposes as the regular representation

$$\mathscr{R}_\theta = \bigoplus_{\rho\in\mathrm{Irr}(G)} (\mathscr{R}_\theta)_\rho$$

giving the tautological line bundles $\mathscr{R}_\rho := (\mathscr{R}_\theta)_\rho$ on the moduli space $\mathcal{M}_\theta(Q,R)$. Without loss of generality, we normalise so that $\mathscr{R}_{\rho_0}$ for the trivial representation ρ_0 is the trivial bundle on $\mathcal{M}_\theta(Q,R)$.

6.3. Physics interpretation. The moduli spaces $\mathcal{M}_\theta(Q,R)$ appear in the physics literature as moduli of $D0$-branes on the orbifold $\mathbb{C}^n/G$, where θ is a Fayet-Iliopoulos term for U(1) gauge multiplets present in the world-volume theory (see Douglas–Greene–Morrison [14]). In this case, the ideal of relations I arises from the F-terms obtained from the partial derivatives of the superpotential of the quiver gauge theory, while the action of T_Q on Z arises from the D-term (which is often described in the physics literature via a moment map). The link between the physics and mathematics literature is madetransparent in the construction of the coherent component by Craw–Maclagan–Thomas [12]).

6.4. As an example, consider the cyclic group $G\cong\mathbb{Z}/3\subset\mathrm{SL}(3,\mathbb{C})$ acting with weights $(1,1,1)$, the McKay quiver has three vertices $Q_0=\{\rho_0,\rho_1,\rho_2\}$, with three arrows $\rho_{i+1}\to\rho_i$ for $i=0,1,2$ (where addition is modulo 3). The space

$$\Theta=\{(\theta_0,\theta_1,\theta_2)\in\mathbb{Q}^3:\theta_0+\theta_1+\theta_2=0\}\cong\mathbb{Q}^2$$

decomposes into three GIT chambers given by

$$\begin{aligned}C_0 &= \{\theta\in\Theta:\theta_2>0,\ \theta_1+\theta_2>0\},\\ C_1 &= \{\theta\in\Theta:\theta_1<0,\ \theta_1+\theta_2<0\},\\ C_2 &= \{\theta\in\Theta:\theta_1>0,\ \theta_2<0\}.\end{aligned}$$

Since C_0 contains parameters of the form $\{\theta_+=(\theta_0,\theta_1,\theta_2)\in\mathbb{Q}^3:\theta_1>0,\theta_2>0\}$, we deduce from above that $\mathcal{M}_\theta(Q,R)=G$-Hilb for all $\theta\in C_0$. It is easy to show that G-Hilb is a smooth toric variety that can be obtained as the unique crepant resolution $\tau\colon Y\to\mathbb{C}^3/G$ contracting a divisor $E\cong\mathbb{P}^2$ to the singular point. This resolution is isomorphic to the total space of the bundle $\mathcal{O}_{\mathbb{P}^2}(-3)$.

In fact, the moduli space $\mathcal{M}_\theta$ is isomorphic to Y for any generic $\theta\in\Theta$; nevertheless, the moduli spaces are different for parameters lying in different chambers since the rank 3 tautological bundle $\mathcal{R}_\theta$ on $\mathcal{M}_\theta$ changes as θ varies between the chambers. To emphasise this point we list in Table 2 the restriction of the tautological bundles to the exceptional divisor $E\subset\mathcal{M}_\theta$ for parameters in all three chambers. For example, parameters $\theta\in C_0$ such that $\mathcal{M}_\theta=G$-Hilb give $\mathcal{R}_{\rho_1}|_E\cong\mathscr{O}_E(1)$ since

	$\theta \in C_0$	$\theta \in C_1$	$\theta \in C_2$
$\mathcal{R}_{\rho_0}\vert_E$	$\mathscr{O}_E$	$\mathscr{O}_E$	$\mathscr{O}_E$
$\mathcal{R}_{\rho_1}\vert_E$	$\mathscr{O}_E(1)$	$\mathscr{O}_E(-2)$	$\mathscr{O}_E(1)$
$\mathcal{R}_{\rho_2}\vert_E$	$\mathscr{O}_E(2)$	$\mathscr{O}_E(-1)$	$\mathscr{O}_E(-1)$

TABLE 2. Tautological bundles on $\mathcal{M}_\theta(Q,R)$ for $\mathbb{Z}/3 \subset \mathrm{SL}(3,\mathbb{C})$

$\mathcal{R}_{\rho_1}$ has degree one on the class of a line in E, and $\mathcal{R}_{\rho_2}|_E \cong \mathscr{O}_E(2)$ follows since $\mathcal{R}_{\rho_2} = \mathcal{R}_{\rho_1} \otimes \mathcal{R}_{\rho_1}$.

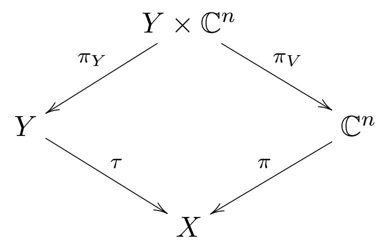
Exercise 6.3. Repeat this above example for the A_2 singularity. Do you recognise the GIT chamber decomposition? [Hint: you're working with the A_2 singularity!]

6.5. The McKay correspondence via Fourier–Mukai transform. Assume that $\theta \in \Theta$ is generic, so that $\mathcal{M}_\theta(Q,R)$ is the fine moduli space of θ-stable G-constellations. There is a projective morphism

$$\tau \colon \mathcal{M}_\theta(Q,R) \to \mathbb{C}^n/G$$

sending any point of $\mathcal{M}_\theta(Q,R)$ to the G-orbit that supports the corresponding G-constellation[3]. As before, there is an irreducible component of the moduli space containing the G-constellations arising from the structure sheaves of the free G-orbits; this is the *coherent component*. Write $\pi \colon \mathbb{C}^n \to X = \mathbb{C}^n/G$ for the quotient morphism and set $Y := \mathcal{M}_\theta(Q,R)$. There is a commutative diagram

(6.1)



where π_Y and π_V are the projections to the first and second factors, and where G acts trivially on both Y and X. The fine moduli construction gives the universal sheaf $\mathscr{U} := \mathscr{U}_\theta$ on the product $Y \times \mathbb{C}^n$, and we define a functor $\Phi^{\mathscr{U}} : \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(S * G))$ via

$$\Phi(-) := \Phi^{\mathscr{U}}(-) = \mathbf{R}(\pi_V)_*\Big(\mathscr{U} \overset{\mathbf{L}}{\otimes} (\pi_Y)^*(- \otimes \rho_0)\Big). \tag{6.2}$$

The method of Bridgeland, King and Reid [6] generalises from the fine moduli space of G-clusters to the fine moduli space of θ-stable G-constellations for any generic parameter $\theta \in \Theta$ as follows:

Theorem 6.4. *Let $G \subset \mathrm{SL}(n,\mathbb{C})$ be a finite subgroup and let $\theta \in \Theta$ be generic. If the coherent component $Y \subseteq \mathcal{M}_\theta(Q,R)$ satisfies $\dim\big(Y \times_X Y\big) \leq n+1$, then:*

[3]The orbifold $\mathbb{C}^n/G$ is isomorphic to an irreducible component of the scheme $\overline{\mathcal{M}_0}(Q,R)$ of 0-semistable G-constellations

(1) *the morphism* $\tau\colon Y \to X$ *is a crepant resolution; and*
(2) *the functor* $\Phi^{\mathscr{U}}$ *with kernel the universal bundle for* $\mathcal{M}_\theta(Q,R)$ *is an equivalence of derived categories that sends* $\mathscr{R}_\rho^\vee$ *to the* G*-equivariant coherent sheaf* $\mathscr{O}_{\mathbb{C}^n}\otimes\rho$ *for all* $\rho\in\operatorname{Irr}(G)$.

Proof. In the course of the proof that was sketched in lecture 5, we used the morphism $\tau\colon G\text{-Hilb}\to X$, the universal G-equivariant coherent sheaf on $\mathbb{C}^n$, and the fact that

$$G\text{-Hom}_S(S/I,S/I') = \begin{cases} \mathbb{C} & \text{if } I=I';\\ 0 & \text{otherwise}\end{cases}$$

for any two G-clusters $S/I, S/I'$. The analogous map and sheaf have been constructed, and the G-Hom result holds for any pair of θ-stable G-constellations, since θ-stable G-constellations are simple objects in the full category of θ-semistable $S*G$-modules.

It remains to show that $\Phi(\mathscr{R}_\rho^\vee)=\mathscr{O}_{\mathbb{C}^n}\otimes\rho$ for all $\rho\in\operatorname{Irr}(G)$. As before, the quasi-inverse of Φ is its the left adjoint, namely, $\Psi\colon\mathcal{D}(\operatorname{mod}(S*G))\to\mathcal{D}^b(\operatorname{coh}(Y)$ given by

$$\Psi(-)=\big[\mathbf{R}(\pi_Y)_*(\mathscr{U}^\vee[n]\overset{\mathbf{L}}{\otimes}(\pi_V^*(-))\big]^G,$$

where $\mathcal{U}^\vee=\mathbf{R}\mathcal{H}om_{\mathscr{O}_{Y\times\mathbb{C}^3}}(\mathscr{U},\mathscr{O}_{Y\times\mathbb{C}^3})$. We have

$$\begin{aligned}
\Psi(\mathscr{O}_{\mathbb{C}^n}\otimes\rho) &= \big[\mathbf{R}(\pi_Y)_*(\mathscr{U}^\vee[n]\otimes(\mathscr{O}_{Y\times\mathbb{C}^n}\otimes\rho))\big]^G\\
&= \big[\mathbf{R}(\pi_Y)_*(\mathbf{R}\mathcal{H}om(\mathscr{U},\mathscr{O}_{Y\times\mathbb{C}^n}[n]))\otimes\rho)\big]^G\\
&= \big[\mathbf{R}\mathcal{H}om(\mathbf{R}(\pi_Y)_*(\mathscr{U}),\mathscr{O}_Y)\otimes\rho\big]^G\\
&= \big[\mathbf{R}\mathcal{H}om(\mathscr{R},\mathscr{O}_Y)\otimes\rho\big]^G\\
&= \mathscr{R}_\rho^\vee,
\end{aligned}$$

where the key step from line two to three invokes Grothendieck duality. Thus $\Phi(\mathscr{R}_\rho^\vee)=\mathscr{O}_{\mathbb{C}^n}\otimes\rho$. □

The restriction of $\Phi^{\mathscr{U}_\theta}$ to the full subcategory of $\mathcal{D}^b(\operatorname{coh}(Y))$ consisting of objects whose cohomology sheaves are supported on the subscheme $\tau^{-1}(\pi(0))$ is a derived equivalence

$$\Phi_0^{\mathscr{U}_\theta}\colon\mathcal{D}_0^b(\operatorname{coh}(Y))\longrightarrow\mathcal{D}_0^b(\operatorname{mod}(S*G))$$

with image the full subcategory $\mathcal{D}_0^b(\operatorname{mod}(S*G))\subset\mathcal{D}^b(\operatorname{mod}(S*G))$ consisting of objects with nilpotent cohomology modules (or equivalently, whose cohomology sheaves are supported at the origin in $\mathbb{C}^n$). There is a commutatve square consisting offunctors between these derived categories, and this in turn induces a commutative square of maps between the Grothendieck groups of these categories.

Example 6.5. To illustrate the functor $\Phi_0^{\mathscr{U}_\theta}$, consider the group $G \cong \mathbb{Z}/3 \subset \mathrm{SL}(3,\mathbb{C})$ from the previous example. It is enough to calculate the images under $\Psi_\theta := (\Phi^{\mathscr{U}_\theta})^{-1}$ of the objects $\mathscr{O}_0 \otimes \rho_i$ that generate $\mathcal{D}_0^b(\mathrm{mod}(S * G))$. The results are presented in Table 3, where we write $E := \tau_\theta^{-1}(\pi(0)) \cong \mathbb{P}^2$ for the exceptional divisor of the crepant resolution $\tau_\theta \colon \mathcal{M}_\theta \to \mathbb{C}^3/G$. These results may be simplified

	$\theta \in C_0$	$\theta \in C_1$	$\theta \in C_2$
$\Psi_\theta(\rho_0 \otimes \mathscr{O}_0)$	$\mathscr{O}_E(-3)[2]$	$\Omega_E^2(3)$	$\Omega_E^1[1]$
$\Psi_\theta(\rho_1 \otimes \mathscr{O}_0)$	$\Omega_E^1(-1)[1]$	$\mathscr{O}_E(-1)[2]$	$\mathscr{O}_E(-1)$
$\Psi_\theta(\rho_2 \otimes \mathscr{O}_0)$	$\Omega_E^2(1)$	$\Omega_E^1(1)[1]$	$\Omega_E^2(1)[2]$

TABLE 3. Fourier–Mukai transforms on $\mathcal{M}_\theta(Q,R)$ for $\mathbb{Z}/3 \subset \mathrm{SL}(3,\mathbb{C})$

via the isomorphism $\Omega_E^2 \cong \mathscr{O}_E(-3)$, but the pattern in each column is clearer in the present form. The three entries in any one of these columns generated the derived category $\mathcal{D}_0^b(\mathrm{coh}(\mathcal{M}_\theta(Q,R)))$ for the appropriate $\theta \in \Theta$. The autoequivalences of $\mathcal{D}_0^b(\mathrm{coh}(\mathcal{M}_\theta(Q,R)))$ are induced by moving from one chamber to another.

To illustrate the method we present two calculations in full. To perform the calculations in the example below, we repeatedly use the formula

$$\pi_*\Phi_\theta^i(-) \cong R^i\tau_*(- \otimes \mathcal{R}_\rho) = \bigoplus_{\rho \in \mathrm{Irr}(G)} H^i(- \otimes \mathcal{R}_\rho) \otimes \rho, \tag{6.3}$$

where $\mathcal{R}_\rho$ denote the tautological bundles on $\mathcal{M}_\theta$ (we often omit π_* from the left hand side). To begin, fix $\theta \in C_0$, hence $\mathcal{M}_\theta = G$-Hilb. Using (6.3) and the first column of Table 2 we calculate

$$\Phi_\theta^i(\mathscr{O}_E(-3)) = H^i(\mathscr{O}_E(-3)) \otimes \rho_0 \oplus H^i(\mathscr{O}_E(-2)) \otimes \rho_1 \oplus H^i(\mathscr{O}_E(-1)) \otimes \rho_2.$$

Since $E \cong \mathbb{P}^2$, the only nonzero vector space in this expansion is $H^2(\mathscr{O}_E(-3)) \cong \mathbb{C}$. Therefore $\Phi_\theta(\mathscr{O}_E(-3)[2]) = \Phi_\theta^2(\mathscr{O}_E(-3)) = H^2(\mathscr{O}_E(-3)) \otimes \rho_0 \cong \mathbb{C} \otimes \rho_0$. This can be written as $\Phi_\theta(\mathscr{O}_E(-3)[2]) = \mathscr{O}_0 \otimes \rho_0$ or, equivalently using the inverse transform Ψ_θ, as

$$\Psi_\theta(\mathscr{O}_0 \otimes \rho_0) = \mathscr{O}_E(-3)[2].$$

Similarly, fix $\theta' \in C_1$ and use (6.3) with column two of Table 2 to give

$$\Phi_{\theta'}^i(\Omega_E^1(1)) = H^i(\Omega_E^1(1)) \otimes \rho_0 \oplus H^i(\Omega_E^1(-1)) \otimes \rho_1 \oplus H^i(\Omega_E^1) \otimes \rho_2.$$

Here, only $H^1(\Omega_E^1) \cong \mathbb{C}$ is nonzero, hence $\Phi_{\theta'}^0(\Omega_E^1(1)[1]) = \Phi_{\theta'}^1(\Omega_E^1(1)) \cong \mathbb{C} \otimes \rho_2$. Write this as $\Phi_{\theta'}(\Omega_E^1(1)[1]) = \mathscr{O}_0 \otimes \rho_2$ or, equivalently, as

$$\Psi_{\theta'}(\mathscr{O}_0 \otimes \rho_2) = \Omega_E^1(1)[1].$$

7. Lecture 7: The derived McKay correspondence beyond G-Hilb

7.1. The Derived McKay Correspondence Conjecture as formulated by Reid proposes that for a finite subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$, the bounded derived category of coherent sheaves of every crepant resolution Y of $\mathbb{C}^n/G$ should be equivalent to the bounded derived category of finitely generated modules over the skew group algebra. Theorem 4.6 establishes the $n = 2$ case, and Theorem 5.1 establishes the case $n = 3$ if the distinguished crepant resolution $Y = G$-Hilb is chosen. What if $\mathbb{C}^3/G$ admits more than one crepant resolution? In fact this question was posed independently by Nakajima, Reid and Douglas in the following form[4]:

Question 7.1. Let $G \subset \mathrm{SL}(3, \mathbb{C})$ be a finite subgroup, and let $\tau \colon Y \to \mathbb{C}^3/G$ be a (projective) crepant resolution. Does there exist a generic parameter $\theta \in \Theta$ such that $Y \cong \mathcal{M}_\theta(Q, R)$?

If the question has an affirmative answer then Theorem 6.4 immediately establishes a derived equivalence $\Phi^{\mathscr{U}_\theta} \colon \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{mod}(S * G))$. Moreover, if Y, Y' are two such resolutions that satisfy $Y \cong \mathcal{M}_\theta(Q, R)$ and $Y' \cong \mathcal{M}_{\theta'}(Q, R)$ for generic $\theta, \theta' \in \Theta$, then the composition

$$(\Phi^{\mathscr{U}_{\theta'}})^{-1} \circ \Phi^{\mathscr{U}_\theta} \colon \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{coh}(Y'))$$

shows that different crepant resolutions have equivalent derived categories.

7.2. In the abelian case we can say the following:

Theorem 7.2 (Craw–Ishii [11])**.** *Let $G \subset \mathrm{SL}(n, \mathbb{C})$ be a finite abelian subgroup with $n \leq 3$. For every projective crepant resolution $\tau \colon Y \to \mathbb{C}^n/G$ we have $Y = \mathcal{M}_\theta(Q, R)$ for some generic $\theta \in \Theta$. Thus, the Derived McKay correspondence holds in this case.*

Remark 7.3. Inevitably, Tom Bridgeland [5] beat us to the punch with his stunning construction of a derived equivalence between the bounded derived categories of coherent sheaves on any two crepant resolutions Y, Y' of $\mathbb{C}^3/G$, This establishes the Derived McKay Correspondence Conjecture in dimension $n = 3$. Principle 4.5 has also been established as an equivalence of derived categories for finite subgroups $G \subset \mathrm{Sp}(n, \mathbb{C})$ by Kaledin–Bezrukavnikov [3] and for finite abelian subgroups $G \subset \mathrm{SL}(n, \mathbb{C})$ by Kawamata [18].

Before sketching the proof of the above result we emphasise the importance of the ample bundle on $\mathcal{M}_\theta(Q, R)$ inherited from the GIT construction for generic $\theta \in \Theta$. Let $\mathscr{R}_\rho$ for $\rho \in \mathrm{Irr}(G)$ denote the tautological bundles on the fine moduli space $\mathcal{M}_\theta(Q, R)$, and let $C \subset \Theta$ denote the open GIT chamber containing θ. It

[4] OK, so Mike didn't put it quite this way. He asked whether every 'geometric phase' could be realised in open string theory.

follows tautologically that the map

$$L_C\colon \Theta \longrightarrow \operatorname{Pic}\big(\mathcal{M}_\theta(Q,R)\big) \;:\; \theta \mapsto \bigotimes_{\rho\in\operatorname{Irr}(G)} \mathscr{R}_\rho^{\theta_\rho}$$

sends any parameter in the chamber C to an ample line bundle on $\mathcal{M}_\theta(Q,R)$. The Picard groups of each $\mathcal{M}_\theta(Q,R)$ are isomorphic since any two are related by a finite sequence of flops, so may knit these maps together, one piece for each chamber $C \subset \Theta$, to obtain a piecewise-linear map.

$$\textstyle\coprod_{C\subset\Theta} L_C\colon \Theta \longrightarrow \operatorname{Pic}\big(\mathcal{M}_\theta(Q,R)\big).$$

Consider the target of this map. Once a line bundle moves to the edge of the ample cone and passes through the boundary of the Kahler cone, one induces a map on $\mathcal{M}_\theta(Q,R)$ that is a well-understood birational transformation; it is either a divisorial contraction, a flip, a flop, or a Mori Fibre Space, and arises naturally from the Mori theoretic point-of-view[5]. The important point is this: *the geometric implications for $\mathcal{M}_\theta(Q,R)$ as θ moves to the boundary of a given chamber in Θ are tracked by how the image of θ under the piecewise-linear map $\coprod L_C$ varies.*

Example 7.4. Consider what happens to the running example $\frac{1}{3}(1,2)$ in light of the above. As the parameter θ moves continuously around the plane on, say, a circle centred at the origin, the image of θ under the piecewise-linear map $\coprod L_C$ moves back and forth between the boundary walls of the nef cone of Y, the crepant resolution of the A_2-singularity.

Example 7.5. Consider the action of the group $G := \mathbb{Z}/2\times\mathbb{Z}/2$ in $\mathrm{SL}(2,\mathbb{C})$, where the three nontrivial group elements act as diagonal matrices, each having two entries -1 and a single entry $+1$. The G-Hilbert scheme is one crepant resolution, but there are three others, each of which can be obtained as a moduli space $\mathcal{M}_\theta(Q,R)$ by varying the parameter θ.

7.3. We now summarise the proof of Theorem 7.2. The G-Hilbert scheme is a crepant resolution, and every crepant resolution of $\mathbb{C}^3/G$ is obtained from G-Hilb by a finite sequence of flops. Moreover, $G\text{-Hilb} = \mathcal{M}_\theta(Q,R)$ for the special parameter $\theta = \theta_+$. Thus, it is enough to prove that if $Y = \mathcal{M}_\theta(Q,R)$ is a given crepant resolution for $\theta \in C$ for some chamber C, and if Y' is another crepant resolution that is obtained from Y by the flop of a single curve, then $Y' = \mathcal{M}_{\theta'}(Q,R)$ for θ' in some other chamber C'.

Since $Y = \mathcal{M}_\theta(Q,R)$ for $\theta \in C$, the derived equivalence

$$\Phi_C := \Phi_0^{\mathscr{U}_\theta}\colon \mathcal{D}^b_0(\operatorname{coh}(Y)) \longrightarrow \mathcal{D}^b_0(\operatorname{mod}(S*G))$$

induces a $\mathbb{Z}$-linear isomorphism

$$\varphi_C\colon K_0(Y) \longrightarrow K_0(\operatorname{mod}(S*G)) \cong \textstyle\bigoplus_{\rho\in\operatorname{Irr}(G)} \mathbb{Z}\rho.$$

[5]In the physics literature, one can even make sense of what happens to the corresponding gauged linear sigma model beyond the Kahler cone giving 'hybrid sectors'.

There is a perfect pairing $K(\mathrm{mod}(S*G)) \times K_0(\mathrm{mod}(S*G)) \to \mathbb{Z}$. Thus, when we dualise and tensor over $\mathbb{Q}$, the GIT parameter space Θ is a codimension-one subspace of the space that is dual to the target of the map φ_C. This observation enables us to provide a geometric interpretation of the walls of any given chamber:

Proposition 7.6. *Let $C \subset \Theta$ be a chamber. Then $\theta \in C$ if and only if*

- *for every exceptional (e.g., flopping) curve ℓ in Y we have $\theta(\varphi_C(\mathscr{O}_\ell)) > 0$.*
- *for every compact reduced divisor D in Y and $\rho \in \mathrm{Irr}(G)$ we have*

$$\theta(\varphi_C(\mathscr{R}_\rho^{-1} \otimes \omega_D)) < 0 \quad \textit{and} \quad \theta(\varphi_C(\mathscr{R}_\rho^{-1}|_D)) > 0.$$

The inequalities of the form $\theta(\varphi_C(\mathscr{O}_\ell)) > 0$ are 'good' in the sense that they lift via the map L_C from inequalities defining the walls of the ample cone of Y in $\mathrm{Pic}(Y)$. The latter ones, however, are problematic a priori since they are not present in $\mathrm{Pic}(Y)$. So, what happens to the variety (and it's derived category!) as θ passes through a wall of the latter kind?

7.4. Recall that $\omega_Y \cong \mathscr{O}_Y$ since $Y \to \mathbb{C}^3/G$ is crepant. An object $E \in \mathcal{D}^b_0(\mathrm{coh}(Y))$ is said to be *spherical* if $\mathrm{Hom}_{\mathcal{D}^b(\mathrm{coh}(Y))}(E, E[k]) = 0$ unless k is 0 or n, in which case it is $\mathbb{C}$. Notice then that the self-Hom groups of a spherical object coincide precisely with the homology groups of a sphere, hence the name. The *twist* along a spherical object E is defined via the distinguished triangle

$$\mathbf{R}\mathrm{Hom}_{\mathscr{O}_Y}(E, F) \overset{\mathbf{L}}{\otimes}_{\mathbb{C}} E \xrightarrow{ev} F \longrightarrow T_E(F)$$

for any $F \in D(Y)$, where ev is the evaluation morphism. The *twist functor* $T_E \colon \mathcal{D}^b(\mathrm{coh}(Y)) \longrightarrow \mathcal{D}^b(\mathrm{coh}(Y))$ is an exact autoequivalence.

Lemma 7.7. *When a parameter θ passes through a wall of the latter type (called a wall of* type 0*), then $\mathcal{M}_\theta(Q,R)$ is isomorphic to $\mathcal{M}_{\theta'}(Q,R)$, and the composition*

$$\Phi_{C'}^{-1} \circ \Phi_C \colon \mathcal{D}^b(\mathrm{coh}(\mathcal{M}_\theta(Q,R))) \longrightarrow \mathcal{D}^b(\mathrm{coh}(\mathcal{M}_{\theta'}(Q,R)))$$

is a twist functor (up to tensoring by a line bundle)

If C does not have the appropriate wall that defines the desired flop, we pass into an adjacent chamber C_1 separated from C by a wall of type 0. A property of twists reveals that, roughly speaking, the images $L_C(C)$ and $L_{C_1}(C_1)$ are adjacent cones in the Picard group. We proceed in this way towards the desired wall of the ample cone, passing through at most finitely many such type 0 walls. By crossing the final wall we induce the desired flop $Y = \mathcal{M}_\theta(Q,R) \dashrightarrow \mathcal{M}_{\theta'}(Q,R) = Y'$.

7.5. Towards McKay in higher dimensions. It is natural to ask whether new examples of the McKay correspondence can be constructed using the moduli $\mathcal{M}_\theta(Q,R)$ in dimension $n \geq 4$, when the singularity $X = \mathbb{C}^n/G$ may admit crepant resolutions even though G-Hilb is singular or discrepant.

One such example is the quotient of $\mathbb{C}^4$ by the maximal diagonal subgroup $G = (\mathbb{Z}/2)^{\oplus 3} \subset \mathrm{SL}(4,\mathbb{C})$ of exponent two. The resolution $Y = G\text{-Hilb} \to X$ has one exceptional divisor $E \cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ of discrepancy one. The divisor $E \subset G\text{-Hilb}$ can be blown down to $\mathbb{P}^1 \times \mathbb{P}^1$ in three different ways, giving rise to crepant resolutions $Y_i \to X$ with exceptional loci $E_i \cong \mathbb{P}^1 \times \mathbb{P}^1$ for $i = 1, 2, 3$. All four resolutions of X are toric morphisms, and the 3-dimensional cross-sections of the 4-dimensional fans defining these resolutions are shown in Figure 7.5:

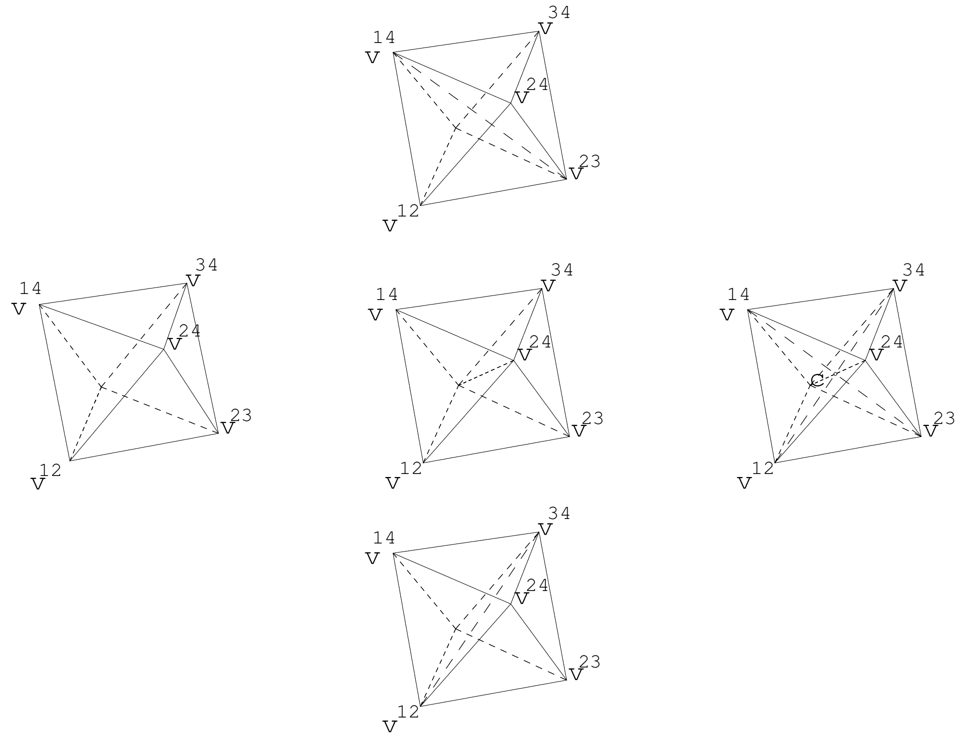


FIGURE 3. Toric picture as drawn beautifully by Chiang–Roan [9, Figure 3]

We now show that each crepant resolution Y_i is a moduli space $\mathcal{M}_\theta(Q,)$ of θ-stable G-constellations for some generic $\theta \in \Theta$ (these are not the only ones; there are another 189 distinct crepant resolutions!).

The chamber containing the weights defining $\mathcal{M}_\theta = G\text{-Hilb}$ is $C_0 = \{\theta \in \Theta; \theta(\rho) > 0 \text{ if } \rho \neq \rho_0\}$. Let x, y, z, w denote the coordinates of $\mathbb{C}^4$. These lie in distinct eigenspaces of the G-action denoted $(1,0,0), (0,1,0), (0,0,1), (1,1,1) \in (\mathbb{Z}/2)^{\oplus 3} \cong G^*$. Write $\rho_1 := (0,1,1), \rho_2 := (1,0,1), \rho_3 := (1,1,0)$ for the complementary nontrivial irreducible representations, and write $W_i := \{\theta \in \Theta : \theta \in \overline{C_0}, \theta(\rho_i) = 0\}$ for the corresponding wall of C_0 for $i = 1, 2, 3$. The claim is that the unique chamber C_i lying adjacent to C_0 satisfying $W_i = \overline{C_i} \cap \overline{C_0}$ defines the crepant resolution Y_i; in particular, we claim that $Y_i = \mathcal{M}_\theta$ for all $\theta \in C_i$.

To prove the claim we follow the behaviour of the G-constellations defined by the twelve torus-invariant points of G-Hilb as θ passes through the wall W_i. In each case, the four G-constellations (in fact G-clusters) defining the torus-invariant points corresponding to the outermost simplices in the fan Σ_0 in Figure 7.5 (containing the vectors e_1, e_2, e_3, e_4 as vertices) remain θ-stable with respect to parameters $\theta \in C_i$. On the other hand, all of the remaining eight torus-invariant G-clusters become unstable. In addition, for each $i = 1, 2, 3$ there are four new torus-invariant G-constellations that are θ-stable for $\theta \in C_i$ (these are not G-clusters so were not present on G-Hilb). An explicit deformation calculation shows that for each $i = 1, 2, 3$, the four new G-constellations define the four torus-invariant points of Y_i corresponding to the four innermost simplices in the fans Σ_i in Figure 7.5. This shows that for each $i = 1, 2, 3$, the birational map $G\text{-Hilb} \to \mathcal{M}_\theta$ induced by moving the GIT parameter through the wall W_i coincides with the contraction from G-Hilb to the crepant resolution Y_i.

Applying Theorem 6.4 carefully leads to the following result:

Theorem 7.8. *For $i = 1, 2, 3$, there is a chamber $C_i \subset \Theta$ such that $Y_i = \mathcal{M}_\theta(Q, R)$ for $\theta \in C_i$. Moreover, the functor $\Phi^{\mathscr{U}_\theta}$ defined by the universal sheaf is an equivalence of categories $\mathcal{D}^b(\mathrm{coh}(Y_i)) \cong \mathcal{D}^b(\mathrm{mod}(S * G))$.*

Proof. It remains to prove the second statement. Write $\tau_i \colon \mathcal{M}_{\theta_i} \to X$ for the crepant resolution with $\theta_i \in C_i$ for $i = 1, 2, 3$. The fibre $\tau_i^{-1}(\pi(0))$ over the point $\pi(0) \in \mathbb{C}^4/G$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, so it satisfies the dimension condition required to apply Theorem 6.4. This is not yet enough since the images under $\pi \colon \mathbb{C}^4 \to \mathbb{C}^4/G$ of all four coordinate axes are also singular and hence have als been resolved in creating the resolution. Nevertheless, these singularities arise along a subvariety of dimension one, so the dimension of the fibre product over any such point $\pi(x) \in \mathbb{C}^4/G$ is 2(3)-1=5. This equals $n + 1$ in this case, so Theorem 6.4 applies and the equivalence of categories follows. □

This is the first known example in dimension $n \geq 4$ of a nonsymplectic group action for which G-Hilb is not a crepant resolution and yet the McKay correspondence has been established as an equivalence of derived categories. It would be interesting to discover the extent to which crepant resolutions of $\mathbb{C}^n/G$ can be constructed as moduli $\mathcal{M}_\theta(Q, R)$ in general. The challenge is then to show that the integral functor $\Phi^{\mathcal{U}_\theta}$ is a Fourier–Mukai transform.

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