Homework Solutions for Math 4800

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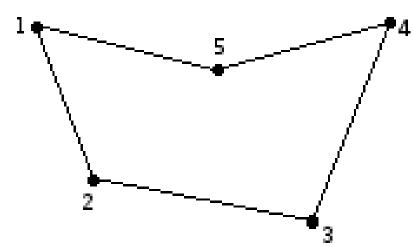
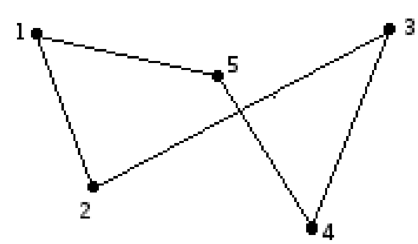
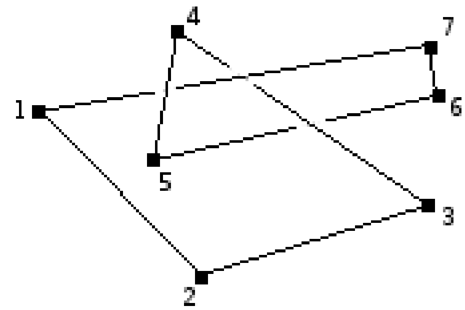
Chapter 2

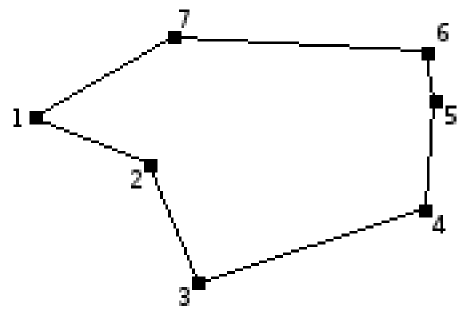
2.2.1 (Miles) The ordering of the vertices of a knot is very important. If the the order of the vertices is changed you might not only change the knot, but you might not even get a knot. Take for example the following unknot:





The first picture shows an unknot but in the second (where the order of the vertices is changed) we no longer have a knot (because it isn't simple).
Another simple example shows how that trefoil knot can be turned into the unknot if the ordering of the vertices is changed. Take the following trefoil:





The first picture shows the trefoil, the second has the exact same vertices, but the order is again changed. Notice in the second picture the unknot is the result of changing the order of the vertices.
Thus if the order of the vertices is changed you may get a different knot, or you may not even get a knot at all.

2.2 (Charlotte) Prove that the vertices of a knot form a well-defined set.

A set of points $\{p_i\}$ in $\mathbb{R}^3$ is a *defining set* for a knot K if the knot given by $\{p_i\}$ is K.

A *set of vertices* for a knot K is a defining set $\{p_i\}$ for K such that no proper subset of $\{p_i\}$ is a defining set for K.

A *corner* of a knot K is a point p on K so that the points on K near p do not lie in a single line.

The set of corners of a knot is well-defined.

We want to show that a point p of the knot K is a corner if and only if p is in every set of vertices for K.

Suppose a point p of the knot K is a corner. Then p must be in every defining set for K, and so it must also be in any set of vertices for K.

Next, we show that if p is in every set of vertices, then p must be a corner by showing that if p is not a corner then p is not in the set of vertices.

For a knot K, defined by the points $\{p_1, p_2, ..., p_n\}$, if p_1 is a point that is not a corner, then K can also be defined by the subset $\{p_2, ..., p_n\}$, because a point that is not a corner is a point lying on a straight line, so removing such a point does not affect K. So p_1 is not in the set of vertices, because without p_1, the subset $\{p_2, ..., p_n\}$ is still a defining set for K.

So any set of vertices is exactly the set of corners.

3.1 (Alex)

3.2 (Alex)

3.3 (Alex) **Problem 2.3.3: Show there is only one planar knot.**

This is equivalent to taking any planar knot $K = (v_1, v_2, ...v_n)$ and deforming it into the unknot. For any vertice v_i, define $\theta(v_i)$, the angle of the vertice, as the angle between the line segments $[v_{i-1}, v_i]$ and $[v_i, v_{i+1}]$. Notice that $\sigma\theta(v_i) = 360$, but $\sigma|\theta(v_i)| = 360$ iff K is convex.

We've already shown in **2.3.1** that all convex planar knots are equivalent to the unknot, so these knots aren't interesting. Instead, let's look at knots that are not convex. Note that it suffices to show that any non-convex knot of n vertices can be reduced to a non-convex knot of $n - 1$ vertices, and the desired result follows from induction.

Knots that are not convex are knots K where $\sigma|\theta(v_i)| > 360$. Since $\sigma\theta(v_i) = 360$, $\exists\, v_i$ where $\theta(v_i)$ has opposite sign from $\theta(v_{i-1})$. Let all such v_i be called switches. Using any switch v_i we will be able to find a point we can remove with an elementary deformation. The reason is that, to prevent v_i itself from being removed, we require another switch, v_j, somewhere in the knot. (See Figure A) With one exception, this v_j will also require another switch to prevent itself from being removed. Since any knot has only a finite number of vertices, there can only be a finite number of switches, so at least one switch can be removed via an elementary deformation, reducing K to a knot with n vertices.

There is one exception to this rule. It is possible that the switch v_j cannot be removed because it is blocked by the switch v_i. But then the only way to block v_i will be with a spiral. (See Figure B) To escape from the spiral will require another switch v_k, which in turn will require a switch to keep itself from being removed. By the same logic as before, then, there is at least one switch that can be removed, simplyfying K into an $n - 1$ knot.

3.4 (Brian) $\widehat{\text{T}}$hrm: Any knot formed by 4 vertices is the unknot. We have three basic cases:
Case 1: A square is formed.
Case 2: A overcrossing as seen going from P1 to P2.
Case 3: A undercrossing as seen going from P1 to P2.

For case A we have a simple deformation that takes point 3 to the center of the square. This move is seen as taking point 3 to the dotted line. This automatically gives us the unknot.

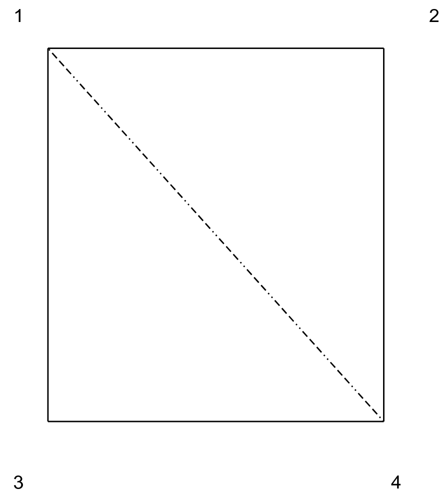


Figure 1: Start as a square box and then take either point 2 or 4 and bring it to the midpoint and you will form a triangle. This is just the unknot

For case B and C look at the following page for the figures.

Since this contains all possible cases we have shown that all 4 vertice knots are the unknot.

3.5 (Tim) Let K be a knot determined by points $(p_1, p_2, ...p_n)$. Show that there is a number z such that if the distance from p1 to p1' is less than z, then K is equivalent to the knot determined by $(p_1', p_2, ...p_n)$. Similarly, show there is a z such that every vertex can be moved a distance z without changing the equivalence class of the knot.

for the region around p1 consider the two closest connected line segments $[p_1, p_2][p_n, p_1]$ Because the knot is a simple polygonal curve there must exist an open tubular region centered on each of these line segments of radii a and b with $a < b$ such that no line segment other than $[p_1, p_2]$ $[p_2, p_3]$ $[p_n, p_{n-1}]$ $[p_1, p_n]$ pass through these regions. Because p_1 p_2 and p_n are centered in their tubular regions if we move p_1 by an amount $z < a$ Then we will have moved the point by less than the radius of the tube. Therefore the resulting line segment will also still be in the open tubular region and so the knot could not have possibly passed through itself. Since we said before that no other segments passed through this region. Thus there always exists a z by which we may shift p_1 to p_1' without changing the knot.

if instead of choosing z such that $z < a$ if for each point we choose z¡ $\frac{a}{2}$ then even if both end points are moved simultaneously the resulting line segment can move less than half the distance towards the closest line and that line can move less than half the distance also which means that no line segments can cross and so there also must exist a z which preserves the knot when every point is moved simultaneously.

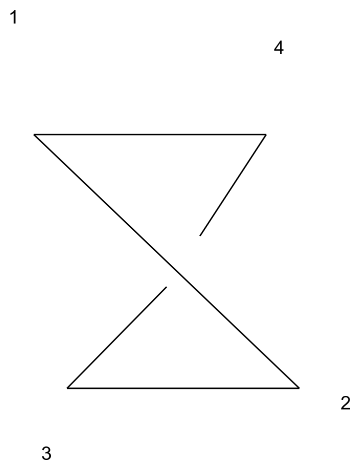


Figure 2: Here we start with an overcrossing of our knot.

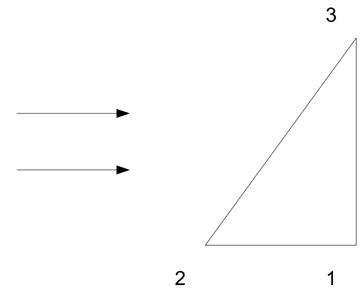


Figure 3: Here we have taken point P1 and taken it to the midpoint of [P3,P1]. After performing this elementary deformation we have made the knot into a triangle. This is again the unknot.

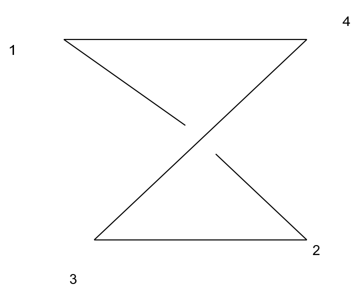


Figure 4: Here we start with an undercrossing.

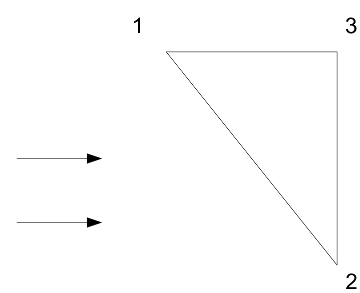


Figure 5: After an elementary deformation we have a triangle. In this step we have taken point P3 to the midpoint of [P2,P3] to get rid of the point This shows we get the unknot as expected.

3.6 (Jason)

Theorem 1. *Let K be a knot determined by $(p_1, p_2, ..., p_n)$. Then there is a number z such that if the distance from p_1 to p_1' is less than z then K is equivalent to the knot K' determined by $(p_1', p_2, ..., p_n)$.*

Let K be a knot defined by $(p_1, p_2, ..., p_n)$ then by the stated theorem there exists an equivalent knot K' and this knot can be created using simple deformations to move p_1 to the position p_1'.

By definition of a knot the cyclic permutation of points $(p_1, ..., p_n)$ define equivalent knots. We then apply a cyclic permutation of points to the knot K' to get the knot defined by the points $(p_i, p_{i+1}, ..., p_1', ..., p_j)$ and then transform it into an equivalent knot K'' of points $(p_i', p_{i+1}, ..., p_1', ..., p_j)$ This process continues an arbitrary number of times for all knots equivalent to our general knot K.

3.7 (Tim) Generalize the definition of elementary deformation and equivalence, to apply to links. (your definition should not permit one component to pass through another.)

A link L and a link L' consisting of sets of non intersecting knots are elementary deformations of each other when: 1) L' contains only one knot K' whose points differ from the points of the knots in L and K is an elementary deformation of a knot K in L and 2) The triangle spanned by the difference of the knot in L and L' does not intersect any other knot.

4.1 (Onye) Suppose K is a knot defined by $(p_1, p_2, \ldots, p_n)$, and J is a knot defined by $(q_1, q_2, \ldots, q_n)$. If K and J have regular projections, then the number of crossings for both of them in $\mathbb{R}^2$ is the same. Since, K and J have the same number of crossings and vertices, and their diagrams are identical, we can easily undertake an elementary deformation such that $(p_1, p_2, \ldots, p_n)$ is equivalent to $(q_1, q_2, \ldots, q_n)$.

2.4.2 (Miles)

Problem: Sketch a proof of theorem 1.

Theorem 1: Let K be a knot determined by the ordered set of points (p_1, p_2, ... , p_n). For ever number $t > 0$ there is a knot K' determined by an ordered set (q_1, q_2, ... , q_n) such that the distance from q_i to p_i is less than t for all i, K' is equivalent to K, and the projection of K' is regular.

A regular projection satisfies the following conditions:
1) No line joining two vertices is parallel to the vertical axis.
2) No vertices span a plane containing a line parallel to the vertical axis.
3) There are no triple points in the projection

First label each arc $a_1, a_2, ..., a_n$ where a_1 connects p_1 to p_2, a_2 connects p_2

to p_3, and so on, ending with a_n connecting p_n to p_1. Then, for any $t > 0$ construct an open ball at every vertex with radius t. If a point p_i is moved to any other point q_i in the open ball around it the distance between p_i and q_i will be less than t.

Fix p_1 (that is let $q_1 = p_1$). Now, if a_1 causes part of the projection to not be regular it can be moved (via moving p_2) so that the irregularities are gone, and the knot is still equivalent. There is a single line running through p_1 which is parallel to the vertical axis. If this line runs through the open ball around p_2 then remove it from the open ball. Also, there will be a finite number crossings in the diagram. At each of these crossings create a plane that is parallel to the vertical axis, and runs through the projection of p_1 and the crossing point. If any of these planes intersect the ball around p_2 then remove that region from the ball. We know the deleted ball will be nonempty because a ball with a finite number of slices taken out will always be nonempty. Now if p_2 is moved to any point in the ball that we have created it will satisfy the conditions of making the projection regular, and will be within distane t of its old position. However, we may still need to limit further the places p_2 can be moved to ensure that K' is equivalent to K. Do not let a_1 be moved anywhere that would cause it to cross another arc in the knot. We know we can do this from theorem two. This still leaves infinitely many places in the ball around p_2 that we can move p_2. Pick one and label it q_2.

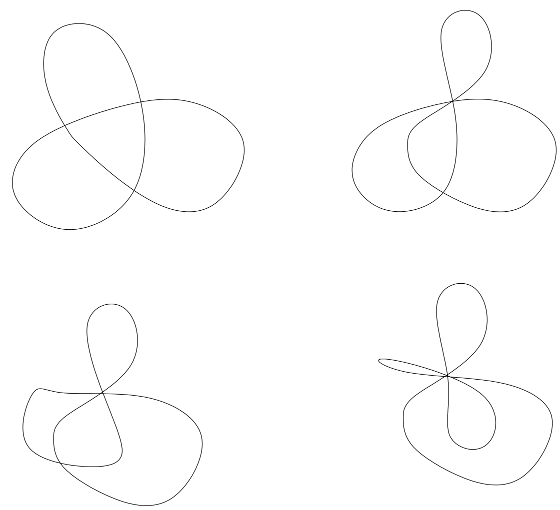
Follow this same procedure for a_2, only this time we need to conscider condition 2 (above) for a regular projection. If the plane that contains a_1 and is parallel to the vertical axis runs through the ball around p_3, then remove that portion of the ball also. Continue doing this with the rest of the vertices. When you are finished there will be a new knot K' that is equivalent to K (made up of vertices $q_1, q_2, ..., q_n$), has a regular projection, and every q_i is within distance t of p_i.

4.3 (Onye) We are given that K is determined by the sequence $(p_1, p_2, \ldots, p_n)$ and has a regular projection. This means that when K is projected from $\mathbb{R}^3$ to $\mathbb{R}^2$, no three vertices of K are collinear. It also means that no singular point p_i of K is on the same point as any other point p_l of K. We are also given that K' has equal number of vertices as K such that $t > 0$, $qi - pi < t$ for all i.

Then when K' is projected from $\mathbb{R}^3$ to $\mathbb{R}^2$, it is going to be a regular projection. Having equal number of vertices and crossings as K, it is therefore equivalent to K because we can undertake some elementary deformations to transform K' into K.

4.4 (Charlotte)

2.4.4 Show that the trefoil knot can be deformed so that its nonregular projection has exactly one multiple point.



5.1 (Jason) HW. 5.1 Definition of an oriented link. Two definitions came to mind when considering this what an oriented knot is. The first is the simplest an oriented link is the union of oriented knots. But this seems to lack a feel for the word oriented and more restrictive definition might be interesting for example a link is oriented if at every crossing of knots $k_1, .., k_n$ in our diagramed link the orientations of the overlapping strands is opposite i.e.

$\leftarrow \rightarrow$ or $\uparrow \downarrow$.

5.2 (Brian) What is the largest possible number of distinct oriented n component links which can determine the same unoriented link, up to equivalence?
b)Show that any two oriented links which determine the unlink as an unoriented link are oriented equivalent.

Part a)
Each link can be oriented two ways which we will call (F,B). Given two links we can have FF,FB,BF,BB. This is just the binomial theorem of probability. For 2 choices and N links, we have 2^N possibilities that gives a set unoriented link as a maximum. One possibility of reaching this upper bound would be to find n non-reverable(given that there are an infinite number of distinct non-reversable knots) knots and link them up into a long chain. This would give us our upper bound. Part b)By looking at the bottom of the following figure, we see that the unlink is just the union of two unknots. By the upper four pictures it is shown that each unknot is oriented equivalent. So the Union of Unknots is oriented equivalent as well.

5.3 (Miles) Explain why if an oriented knot is reversible then any choice of orientation is reversible.

Let K be an oriented knot. Then it can be be determined by an ordered set of vertices $(p_1, ..., p_n)$. The reverse of this knot is the knot K^r with the same vertices, but their order reversed. A knot is reversible if K and K^r are oriented equivalent. If you chose an orientation of a reversible knot, then K is orientation equivalent to K^r. However, since there are only two ways to orient a knot, and

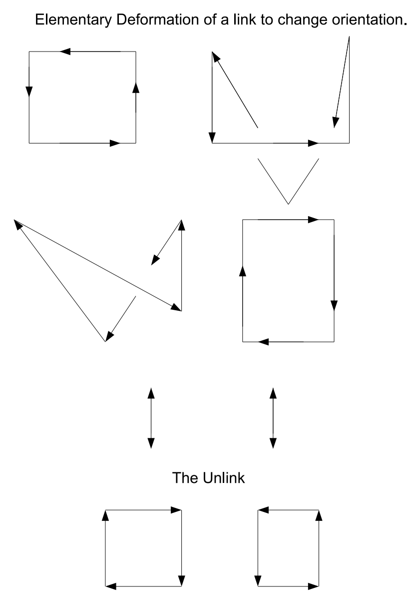


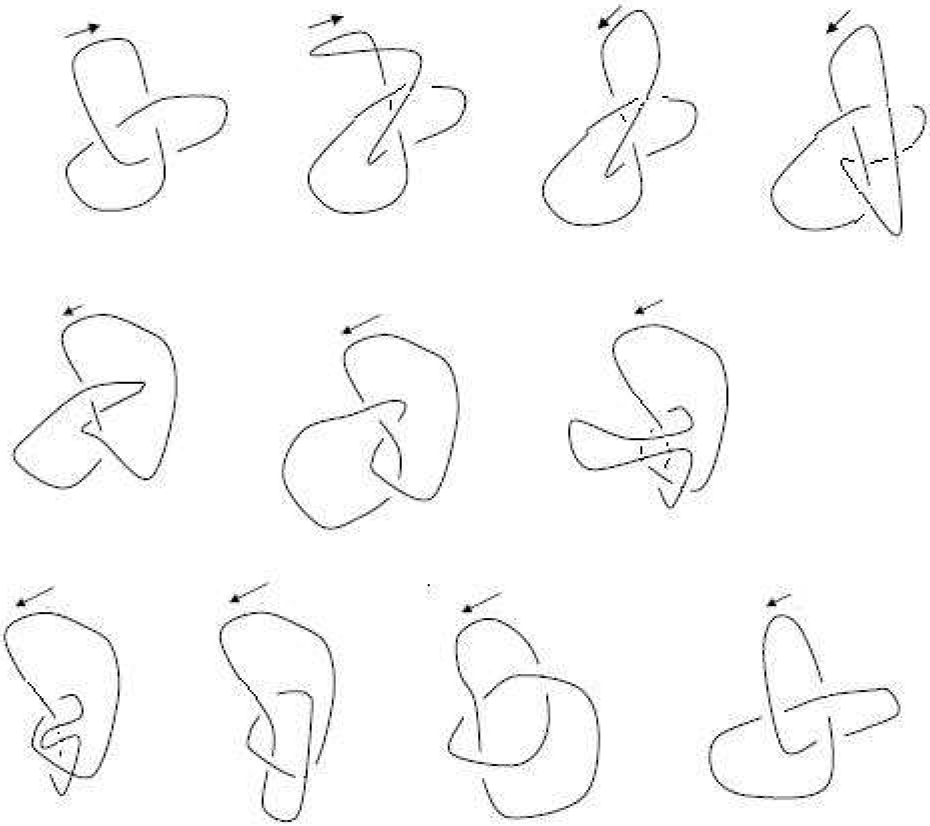
Figure 6: The top four figures show that the unknot is orientable equivalent. The bottom two show that the unlink is just two unknots.

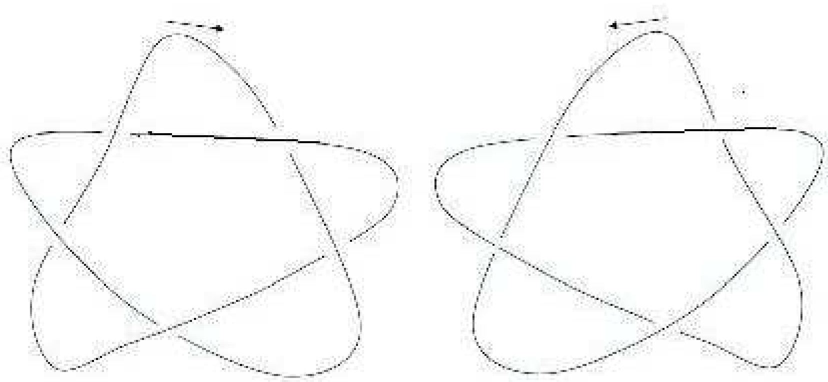
if you chose the other orientation it is the knot K^r, it follows that if a knot is reversible then any choice of orientation is reversible.

5.4 (Tim) Show the (p,p,q) pretzel knot is reversible.

The (p,p,q) pretzel knot is equivalent to the (p, q, p) pretzel knot. If you consider the symmetric diagrams of the (p,p,q) pretzel knot it is clear that a rotation by 180 degrees changes the orientation of the diagram but not the diagram itself. Thus the (p, p, q) pretzel knot is reversible.

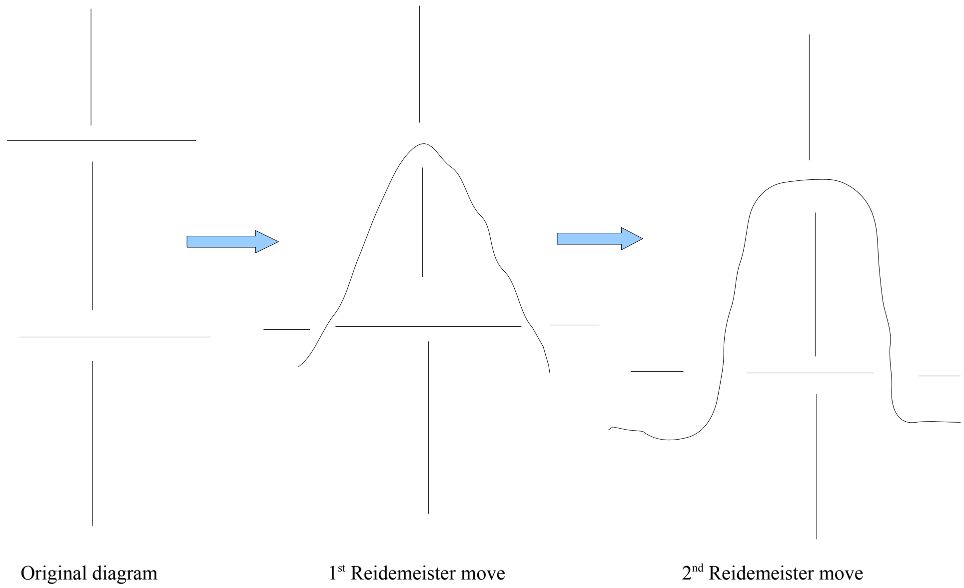
5.5 The knot 8_{17} is the first knot in the appendix that is not reversible. Below is a series of diagrams to show that the trefoil knot is reversible by Reidemeister moves. The second set of diagrams shows knot 5_1, and then how it appears if you flip the knot in 3-dimensional space. Because it looks the same but oriented in the opposite direction, this knot is reversible. With great difficulty, this could also be shown by Reidemeister moves.





Chapter 3

1.1 (Onye)



The diagrams represented above show how to accomplish the solution through two Reidemeister moves.

1.2 (Brian) Show that the given knot is equivalent to the unknot by performing a series of unknots.

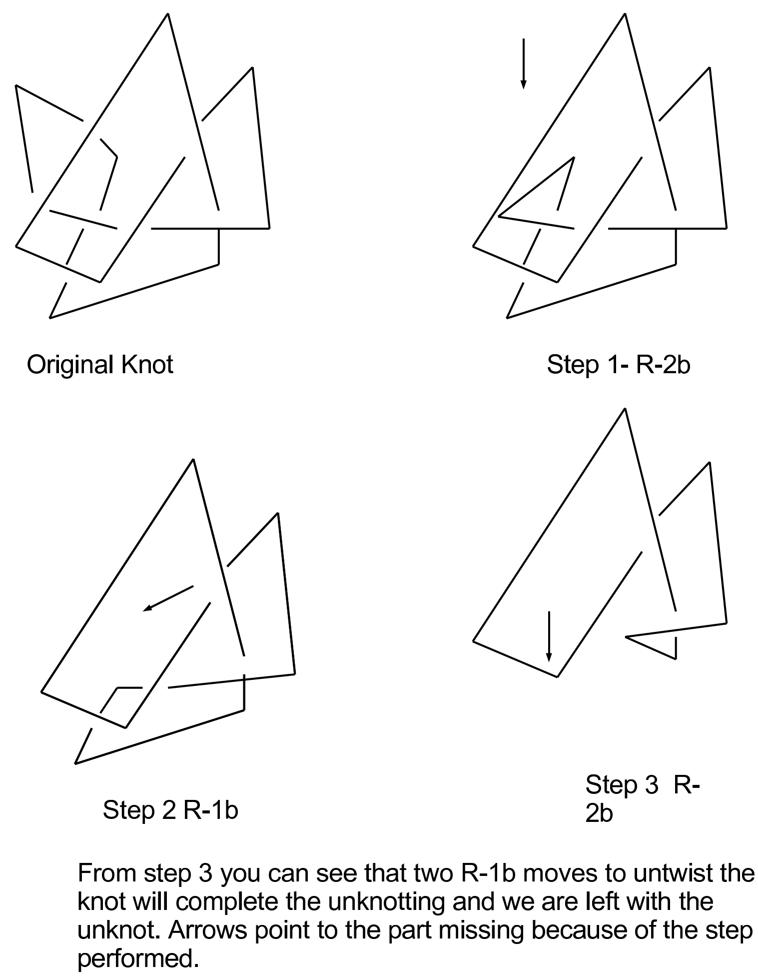
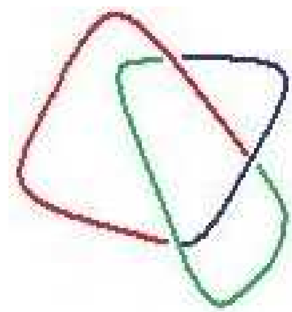


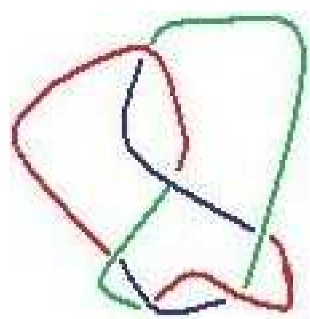
Figure 7: Here we perform steps of Reidemeister moves to show that this is equivalent to the unknot.

2.1 (Miles) I counted a total of four knots with with seven or fewer crossings that were colorable. They were the $3_1, 6_1, 7_4$, and the 7_7 knots. Here are the coloring I found of them.

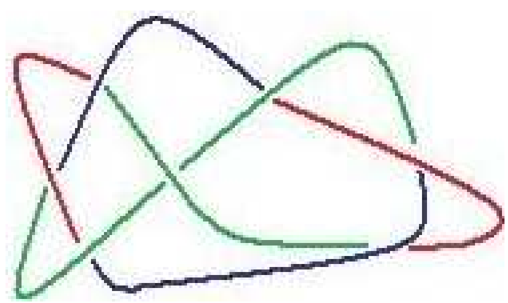
3_1:



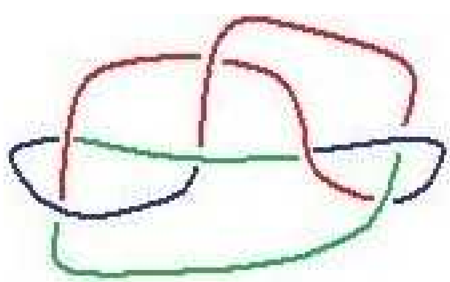
6_1:



7_4:



7_7:



2.2 (Tim) For which integers n is the (2, n)-torus knot colorable? For which values of n is the n twisted double of the unknot colorable? (2n is the number of crossings in the vertical band of the n-twisted double of the unknot)

if the diagram is colorable then the left strand of the torus knot must be a different color from the right one since if they were both the same color then the entire knot would be forced to be the same color. If we let the left strand be red and the right strand be green then the first crossing will leave us with a green strand and then a blue strand from left to right then a blue strand and a red strand and then a red strand and a green strand again. Clearly after this point the pattern repeats. We had to go through 3 crossings to get to a point with red on the left and green on the right which is the coloring we started with so we have found a colorable knot and if we add three more crossings then we will also have a colorable knot. therefore any (2, 3n)-torus knot is colorable.

Again the n twisted double of the unknot must start with two different color strands on top or the whole thing would be a single color. Let the left strand starting in to the twists be red and the right strand be green. In order for a valid coloring to be found the vertical band of twists would need to end with blue on the left and red on the right. the color changes as before leaving us with a sequence rg gb br

3.2.3 (Onye) I shall attempt to explain the colorability of a (p,q,r) pretzel knot without reference to the determinant or modularity of the knot since these ideas had not been introduced in section 2 of the chapter at the time of this exercise.

My trial of all the possible combinations of colorings did not seem to work. In general, a (p,q,r) pretzel knot where p,q, and r are different, produce a pretzel knot that is always not colorable. However, when p is odd and is set to be equal to r, and q is an odd number less than p and r, the system becomes colorable. Some examples of such a combination are the (5, -3, 5), (7, -5, 7), and (9, -7, 9) pretzel knots. This concludes the discussion.

3.2.5 (Brian) Part a) For the Reidemeister move 1 we obviously don't change linking numbers since this is just an untwisting. Even if the twisting crosses over another link, by performing a R2 move to seperate them we get a link with no crossing and we have the same linking number if R2 works. So we must look at R2 and R3. These are performed on the next page as Fig 2.diagram.

Part b) Show that the Whitehead Link has linking number 0. See figure on following page - Fig. 3.

Part c) Two examples that have linking numbers of 3 and 4. See last page(Fig. 4)

2.6 (Charlotte) The sum used to compute linking numbers can be split into the sum of the signs of the crossings where K passes over J, which we will write as *cr(K,J)* and the sum of the signs of the crossings where J passes over K,

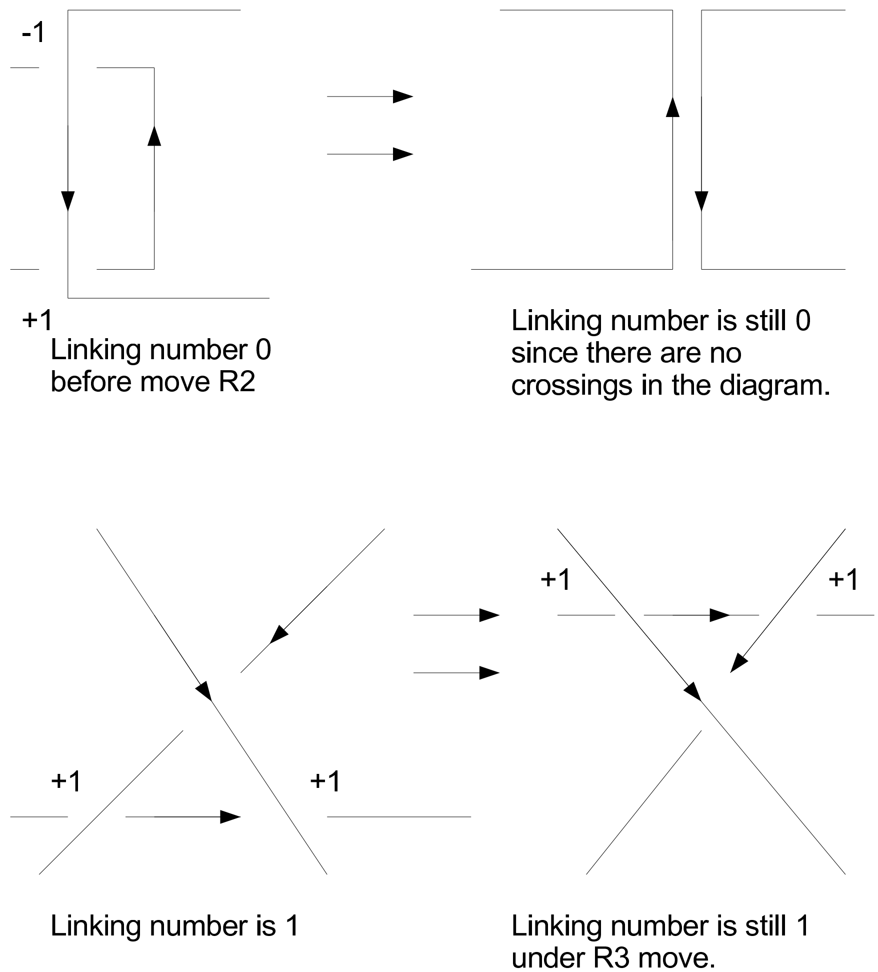


Figure 8: Reidemeister moves 2 and 3 to show that the linking number is invarient to any R move that we can make on any 2 links. Obviously this also applies to more than 2 links by making sure each triangle only passes through 1 link at a time.

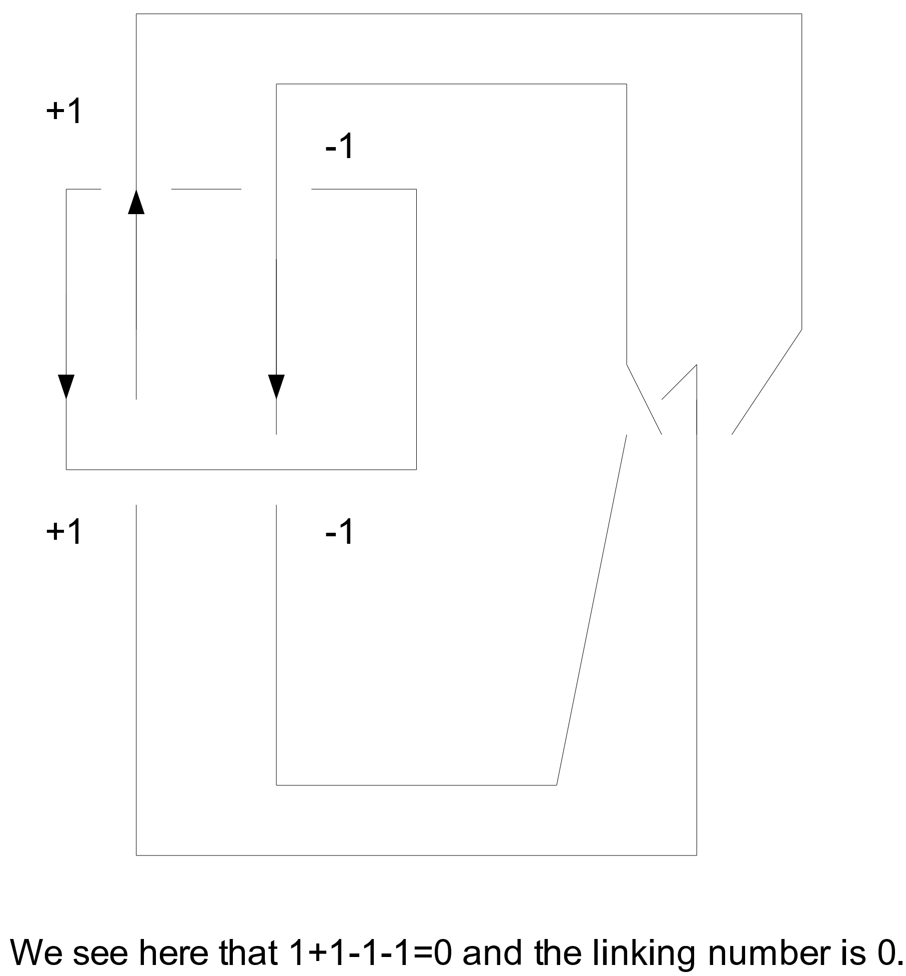


Figure 9: Simply by putting the appropriate linking number of 1 we show that it is 0.

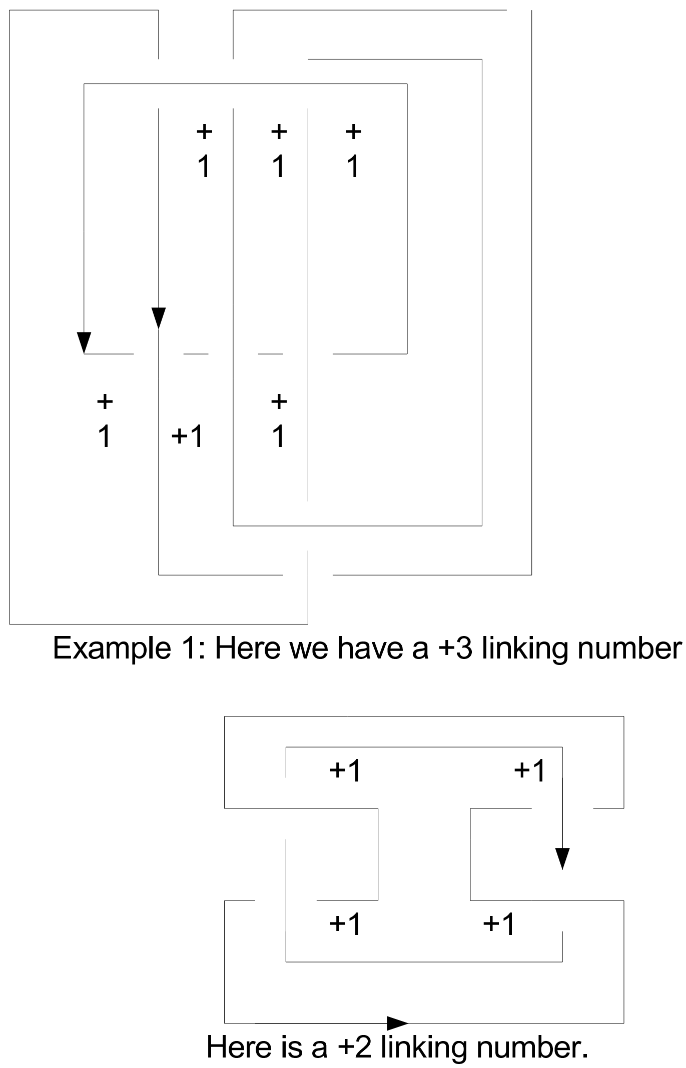
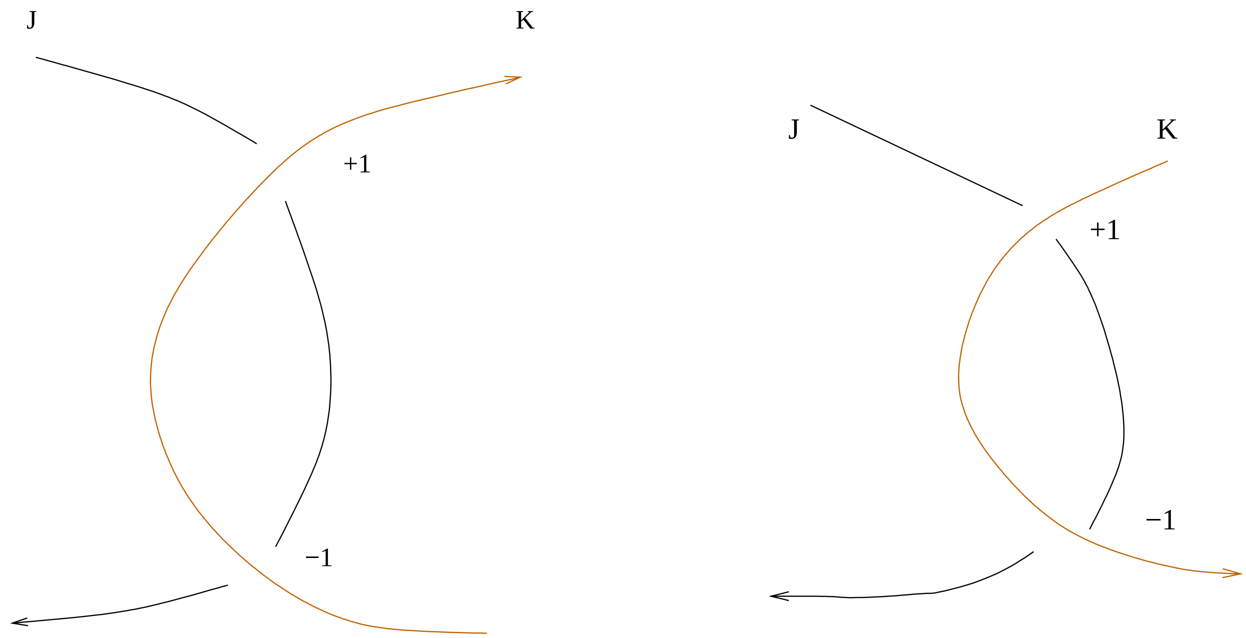


Figure 10: Here are two examples with different linking numbers.



cr(J,K), where each *right-handed* crossing is assigned a $+1$ and each *left-handed* crossing is assigned a -1.

a) Use Reidemeister moves to show that each sum is unchanged by a deformation.

The first type of move does not affect *cr(K,J)* or *cr(J,K)*, because it only involves one of the knots, either K or J, passing over itself.

The second type of move will always involve one right-handed (+1) and one left-handed (-1) crossing. The sum of these is zero, so this move will not affect *cr(K,J)* or *cr(J,K)*.

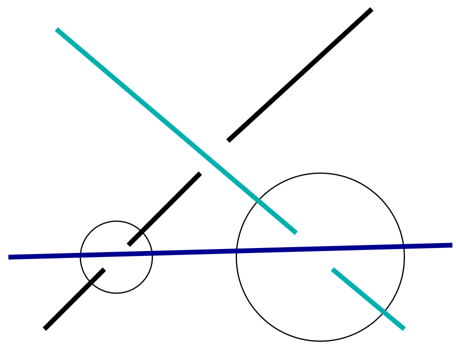
The third type of move involves three strands and three crossings. Regardless of orientation and to which knot each strand belongs, this move will not affect the linking number because the three crossings that exist before the move is performed are the same as the three crossings which result; they simply occur at a different place in the diagram. Therefore *cr(K,J)* and *cr(J,K)* are unaltered.

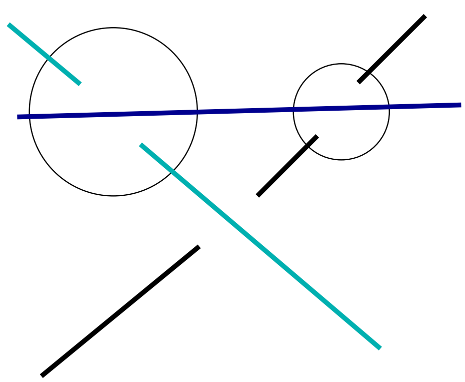
b) The value of $cr(K,J) - cr(J,K)$, is unchanged if a crossing is changed in a diagram.

At any crossing, either K passes over J or J passes over K. There are also two possible orientations.

In the first case, if K passes over J, and J passes under from the right, we get $cr(K,J) - cr(J,K) = (+1) - 0 = +1$. If switched so that J passes over K, then K will pass under J from the left, giving $0 - (-1) = +1$.

In the second case, if K passes over J, and J passes under from the left, we get $cr(K,J) - cr(J,K) = (-1) - 0 = -1$. If switched so that J passes over K, then K will pass under J from the right, giving $0 - (+1) = -1$.





So switching the crossing does not affect $cr(K,J) - cr(J,K)$.

c) If the crossings are changed so that K always passes over J, then $cr(K,J) - cr(J,K)$ is zero.

Since deforming the diagram with Reidemeister moves does not change either $cr(K,J)$ or $cr(J,K)$, and switching crossings so that K passes over J or J passes over K does not change the difference of the sums $cr(K,J) - cr(J,K)$, the diagram of a link can be deformed by these two methods until the projections are disjoint. At this point it is clear that $cr(K,J) = cr(J,K) = cr(K,J) - cr(J,K) = 0$.

d) The linking number is always an integer, given by either of the two sums, $cr(K,J)$ or $cr(J,K)$.

The text defined the linking number to be an integer given by $(cr(K,J) + cr(J,K)) \div 2$. Since we have shown that $cr(K,J) - cr(J,K) = 0$ if all the crossings are changed, and changing a crossing does not alter the value of $cr(K,J) - cr(J,K)$, then $cr(K,J) - cr(J,K)$ must always be zero. Thus, $cr(K,J)$ and $cr(J,K)$ must be equal and, by their definition, integers. So instead of adding them together and dividing by 2, we can take the linking number to be either value.

2.7 (Alex)

Problem 3.2.7: Show that (a) the presence of two colors on a colorable knot diagram forces the presence of a third color, but (b) there is a diagram of the unlink of two components that can be colored with two colors, and one with three. Why is this the case? (c) Prove Reidemeister moves don't change the colorability of links.

a) This follows from the observation that if a knot has two colors, then there must be a vertex where both colors are present. To color the knot, this vertex would require the presence of a third color as well.

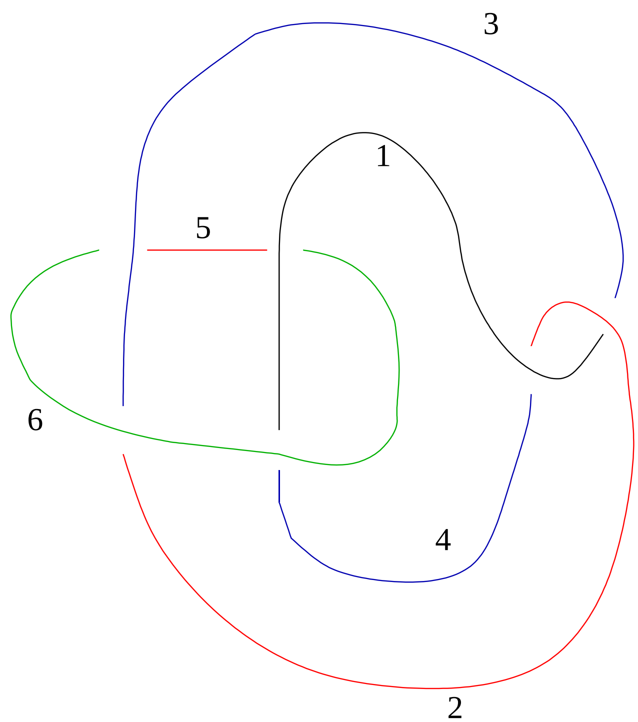
b) Figure C.a can be colored with two colors, and Figure C.b requires three.

c) The proof that works for knot diagrams works for diagrams of links as well. The reason for this is that nowhere in the proof is it assumed that the strands affected by any reidemeister move ever join together.

2.8 (Charlotte) 3.2.8 The Whitehead link is non-trivial.

The unlink of two components is trivial and has a diagram which is colorable with three colors. Since colorability is preserved under Reidemeister moves, every diagram of the unlink is colorable.

To check for colorability, color arc 1 black. Choosing to make arcs 2 and 3 black would force the whole diagram to be black which is not a valid coloring. Coloring arc 2 red forces arcs 3 and 4 to be blue. Arcs 3 and 1 force arc 5 to be red. Arc 6 would then have to be both blue and black, which is also not valid. The Whitehead link is not colorable, therefore it is not trivial.



2.9 (Tim) If a knot is colorable there are many different ways to color it. For instance arcs that were colored red can be changed to yellow, yellow arcs changed to blue, and blue arcs to red. The requirements of the definition of colorability will still hold. There are six permutations of the set of three colors, so any coloring yields a total of six colorings. For some knots there are more possibilities. (a) Show that the standard diagram for the trefoil knot has exactly six colorings

Since the standard diagram of the trefoil knot has only three arcs since all three arcs are the same we can pick one of them at random. We can choose any of the 3 colors to color this arc and we can pick two ways to color the second two arcs. Therefore we have 3*2 choices of possible colorings which gives us the 6 colorings we get from permuting the colors.

(b)How many colorings does the square knot have?

Since the square knot is the connected sum of two colorable knots clearly we can either make one of the two knots all the same color, the color of the strand connecting it to the other knot or we can color it in exactly the way we would if the strands connecting it to the other knot were simply linked together and we just had a single knot. Since the first trefoil has 6 different colorings and for each coloring the strands linking it to the other trefoil are the same color, the color of the linking strand makes the other knot be colorable in only two ways that aren't monochromatic. The total number of colorings is then 18.

(c) The number of colorings of a knot projection depends only on the knot; that is, all diagrams of a knot will have the same number of colorings. Outline a proof of this.

two knot diagrams of the same knot differ only by Reidemeister moves and because the coloring of all of the diagrams of the Reidemeister moves are completely determined by the color of the incoming strands they cannot possibly change the number of colorings of the knot as a whole.

(d)Use the connected sum of n trefoils to show that there are an infinite number of distinct knots.

The first knot in the chain can be colored freely giving 6 colorings for it and each connected knot will have 3 possible colorings that will be consistent with the color of the incoming strands. Thus for a connected sum of n trefoil knots there will be 18(n-1) total colorings for n greater than 1. Since the number of colorings is different for each n then each n connected sum is a different knot. Therefore we have found an infinite class of different knots.

3.2 (Brian) For what values of p does the trefoil knot have a mod p solution? Here is a labeling of a trefoil knot.

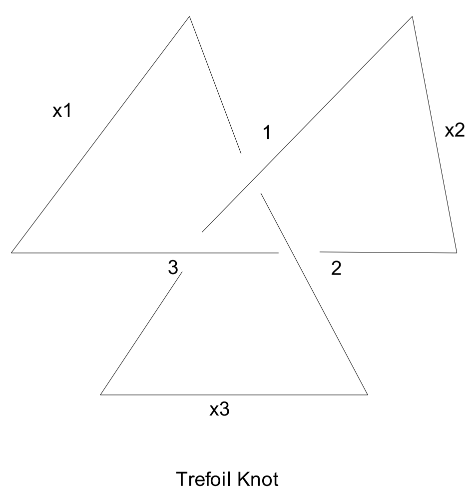


Figure 11: Labeling for a trefoil knot.

To show which mod p solutions exist for the trefoil knot, we must calculate the determinate of the matrix given below minus one row and column.

$$\begin{vmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \\ 2 & -1 & -1 \end{vmatrix}$$

$$Det(A) = \begin{vmatrix} -1 & 2 \\ -1 & -1 \end{vmatrix} = 1 - (-2) = 3.$$

The only mod p solution of this matrix is 3. So the trefoil knot can be labeled mod 3.

3.4 (Onye) Question 3.3.4

Suppose a knot is labelled mod(3), then the following relationship is correct.

2x-y-z=0 mod(3) for any crossing point where x is the over crossing and y and z are the undercrossings. We had shown in a previous proof that this relationship holds true for all colorable knots.

If the labelling is therefore multiplied by 5, then the new relationship becomes 10x-5y-5z=0 mod(15). This new relationship holds true for whatever values we choose for x, y, and z.

Suppose x=0, y=1, z=2. Then the original relationship implies that 0-1-2=0 mod(3). This is a true statement. On the other hand, the new relationship will imply that 0-5-10=0 mod(15). This is also a true statement.

We may therefore conclude that a knot labelled mod(3) could be multiplied by any prime number which will still preserve the colorability and labelling of the knot. This concludes the proof.

3.5 (Miles) Problem: If p is 2, other difficulties come up. Explain why no knot can be labeled mod 2.

If we think of trying to color a knot it makes sense that we cannot color it mod two, as it would take at least three colors to color. However, if we look at the crossing relationship that $2x - y - z = 0$ we see that it also doesn't make sense. Since there are three variables and only two "colors" it must be that either $x = y$ or $y = z$. If $x = y$ then we have $2x - x - z = 0 \Rightarrow x - z = 0 \Rightarrow x = z$, but then every arc would be labeled the same, and the condition that there needs to be at least two distinct labels wouldn't be satisfied. Also, if $y = z$ then we have $2x - y - z = 0 \Rightarrow 2x - 2z \Rightarrow x = z$. Again the condition of at least two labels being distinct doesn't hold. Thus no knot can be labeled mod 2.

4.1 (Tim) For each knot with 6 or fewer crossings find the associated matrix, and its determinant. In each case, for what p is there a mod p labeling.

using the notation of the table in Appendix 1.

3_1 has an $n \times n$ matrix.

$$\begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

the determinant of 3_1 is 3 which implies a mod 3 labeling.

4_1 has 4×4 matrix

$$\begin{pmatrix} 2 & 0 & -1 & -1 \\ -1 & -1 & 2 & 0 \\ -1 & 2 & 0 & -1 \\ 0 & -1 & -1 & 2 \end{pmatrix}$$

the determinant of 4_1 is 5 which implies a mod 5 labeling.

5_1 has a 5×5 matrix

$$\begin{pmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{pmatrix}$$

the determinant of 5_1 is 5 which implies a mod 5 labeling.

5_2 has a 5×5 matrix

$$\begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 2 & 0 & -1 & -1 \\ -1 & 0 & 2 & 0 & -1 \\ -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

the determinant of 5_2 is 7 which implies a mod 7 labeling.

6_1 has a 6×6 matrix

$$\begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 0 & 2 \end{pmatrix}$$

the determinant of 6_1 is 9 which implies a mod 3 labeling.

6_2 has a 6×6 matrix.

$$\begin{pmatrix} 2 & 0 & 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & -1 & -1 \\ -1 & -1 & 0 & 2 & 0 & 0 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{pmatrix}$$

the Determinant of 6_2 is 11 which implies a mod 11 labeling.

6_3 has a 6×6 matrix.

$$\begin{pmatrix} 2 & 0 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ -1 & 0 & 0 & 2 & 0 & -1 \\ 0 & -1 & -1 & 0 & 2 & 0 \\ 0 & 0 & -1 & -1 & 0 & 2 \end{pmatrix}$$

The determinant of 6_3 is 13 which implies a mod 13 labeling.

4.3 (Brian) Complete the first part of the proof that the determinate is the same regardless of which row and column you remove.

Since we are free to choose our labeling, we can set the last arc $a_n = 0$. This gives us a matrix where the last column does not matter so we can make all the last values 0. When a column becomes 0, it gives us a nullity of 1. Then when a matrix has nullity 1 is row reduced, we get a row on the bottom that contains only 0's. So we can remove both the row and column that are filled with 0's. Because the labeling is arbitrary, we could have picked any arc to equal 0. An arc is just a column in our matrix and so it does not matter which column gets removed. Likewise, because we can do a row swap, it doesn't matter which row gets removed. All we need to do before row reducing is swap one row for the last row and it would make the new last row have 0's.

4.4 (Charlotte) 3.4.4 The diagram of the unknot which has no crossings and the diagram of the unknot which has only one crossing do not have typical corresponding matrices. An $(n) \times (n)$ matrix corresponds to a diagram with n arcs, and the dete rminant of a knot is the absolute value of the associated $(n-1) \times (n-1)$ matrix. The problem with these two diagrams of the unknot is that, unlike any others, they have only one arc each, so the corresponding matrix for either diagram is a 1×1 matrix, and the matrix from which we are supposed to find the determinant is a 0×0 matrix. We manage this problem by defining the determinant and nullity of a 0×0 matrix to be 1. A nullity of 1 means there is one solution. Since the determinant is not 0, the only existing solution for the system of equations is trivial.

4.5 (Miles) Problem: Prove that the determinant of a knot is always odd.

We know from that if a knot has a determinant that is divisible by p then a mod p solution exists that satisfies the crossing equation at each crossing. That is to say, if the determinant of a knot is divisible by p then the knot can be labeled mod p.
Assume the determinant of a knot is l, where l is an even integer. That means that $l = 2k$ for some integer k. Since the determinant of the knot is $2k$ we know that a mod 2 solution exists (because $2k$ is definitely divisible by 2). However, we know this can't happen (from exercise 3.3.5). Thus we have a contradiction, and it must be that the determinant of a knot can't be even.

4.6 (Alex) **Problem 3.4.6: Show that if a knot has mod p rank n, then the number of mod p labelings is $p(p^n - 1)$.**

This follows from a basic understanding of what the rank of a knot signifies. Let us look at any $n-1 \times n-1$ matrix of a knot with mod p rank n. Then, since the rank of the knot is n, this matrix has nullity n, meaning that its kernel has rank n. This means that the space of vectors we can multiply by this matrix that return 0 have dimension n.

But this is the space we're interested in, because all labelings of this knot, when written as a vector and multiplied by this matrix, should return 0. Notice then that the amount of elements in an n-dimensional mod p space is p^n. We toss one of these elements out, namely the trivial solution of the 0 vector, because although this vector multiplied by our matrix returns 0, it corresponds to a labeling of the knot using only either one or two colors, and so is an illegitimate solution. So there are $(p^n - 1)$ solutions of the $n - 1 \times n - 1$ matrix.

Now notice that to get down to the $n - 1 \times n - 1$ matrix, we first choose an arc and fix a labeling for it. As there are p labels, there are p ways to go from the $n \times n$ matrix to the $n - 1 \times n - 1$ matrix.

But if there are p ways to choose an $n - 1 \times n - 1$ matrix, and each of these matrices has $(p^n - 1)$ solutions, then the total number of solutions of the matrix, or, equivalently, the total number of labeling of the knot, is $p(p^n - 1)$.

5.2 (Charlotte) 3.5.2 Relate the value of the Alexander polynomial of a knot evaluated at -1 to the determinant of the knot, defined in section 3.4.

The first method we learned for labeling a knot, making a matrix and finding the determinant was in section 3.4. In this matrix, as in the Alexander matrix, each row represents a crossing and each column represents an arc.

In the first method, we put a 2 in every row, in the column of the over-crossing arc. In the Alexander matrix we put $(1-t)$ in for every over-crossing. Evaluating this at $t = -1$ we get $1 - (-1) = 2$.

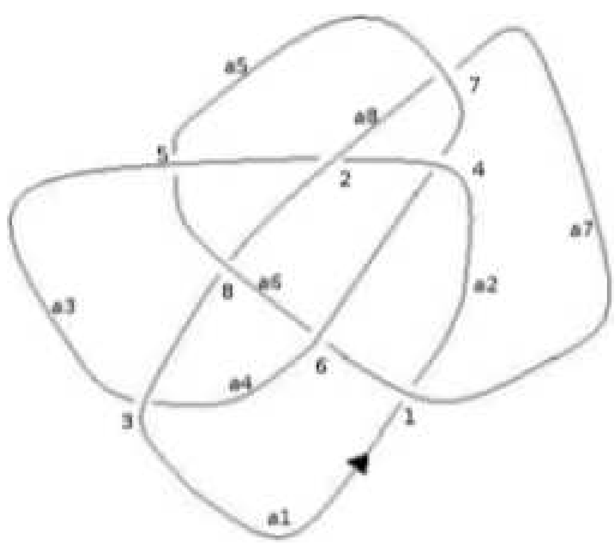
In the first method, we labeled the two under-crossing arcs that approach and leave each crossing with the value -1. In the Alexander matrix, we label the under-crossing arc that approaches the crossing with a -1, and the under-crossing arc that leaves the crossing with t. Evaluating this at $t = -1$ we get -1 for both under-crossing arcs, as we did in the first method.

Finally, in both methods, we put 0 in the column of every arc that is not involved in the crossing represented by any given row. Since all entries of the matrix are the same by either method when we evaluate the entries of the Alexander matrix at $t = -1$, the value of the determinant will be the same by either method. Therefore, since the Alexander polynomial of a knot K is the determinant of the Alexander matrix for K, the value of the Alexander polynomial, when evaluated at $t = -1$ will be the same as the determinant we find by the method in section 3.4.

5.4 (Miles)

3.5.4 (Miles) Problem: Compute the polynomials of 8_{18} and 9_{24} to check they are identical.

Using the following labelings and orientation for the 8_{18} knot:

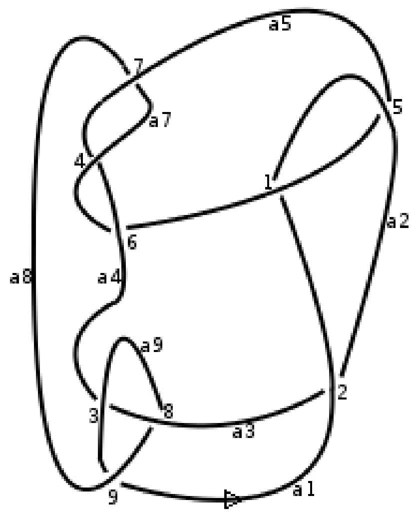


If we set up the matrix for the knot using the diagram shown, and remove the last row and column we get the determinant:

$$\begin{vmatrix}
-1 & t & 0 & 0 & 0 & 0 & 1-t \\
0 & t & -1 & 0 & 0 & 0 & 0 \\
1-t & 0 & -1 & t & 0 & 0 & 0 \\
0 & 1-t & 0 & t & -1 & 0 & 0 \\
0 & 0 & 1-t & 0 & -1 & t & 0 \\
0 & 0 & 0 & 1-t & 0 & t & -1 \\
0 & 0 & 0 & 0 & 1-t & 0 & -1
\end{vmatrix}$$

This determinant equals (via TI-89): $-t^8 + 5t^7 - 10t^6 + 13t^5 - 10t^4 + 5t^3 - t^2$

Now use the following labelings and orientation for the 9_{24} knot:



If we set up the matrix for the knot using the diagram shown, and remove the last row and column we get the determinant:

$$\begin{vmatrix} t & -1 & 0 & 0 & 0 & 1-t & 0 & 0 \\ 1-t & -1 & t & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & t & 0 & 1-t & 0 \\ 0 & 1-t & 0 & 0 & t & -1 & 0 & 0 \\ 0 & 0 & 0 & 1-t & 0 & -1 & t & 0 \\ 0 & 0 & 0 & 0 & 1-t & 0 & -1 & t \\ 0 & 0 & 1-t & 0 & 0 & 0 & 0 & t \end{vmatrix}$$

Calculating this determinant we get: $-t^8 + 5t^7 - 10t^6 + 13t^5 - 10t^4 + 5t^3 - t^2$ Which is the exact same polynomial that we got for the 8_{18} knot. Thus the Alexander polynomial fails to distinguish the 9_{24} knot from the 8_{18} knot.

5.6 (Tim) Prove that a knot and its mirror image have the same Alexander polynomial.

If the Knot has left and right reflected then the right handed crossings will become left handed and vice versa but if we also reverse the orientation of the knot then the handedness of the crossings remains unchanged. If we label the knot arcs in the same way as before then we will get precisely the same matrix and so we must get the same determinant.

5.7 (Alex) **Problem 3.5.7: Show the Alexander Polynomial of a knot K with its orientation reversed is obtained from the polynomial of K by**

substituting t^{-1} for t, multiplying by the appropriate power of t, and perhaps changing sign.

This is equivalent to asking to show that the Alexander Polynomial is invariant under changes of orientation. This can be seen in the following way. The effect of a change of orientation on the matrix from which the Alexander Polynomial is computed is to change all -1s to ts and vice versa. It is merely necessary, then, to show that this new matrix will have the same determinant as the old, up to changes of sign and powers of t.

Consider the columns of this matrix. They will either: i) Have some amount of -1, t, and $1-t$ terms, iia) Will only have -1 or iib) will only have t terms, or iii) will have some -1 and t terms, but no $1-t$ terms. Considering what happens when finding the determinant by expanding down the column in each of these cases, along with some inductive reasoning, will show the determinant will only change by possibly a negative and a power of t.

Case i): Use row operations to add a row containing $1-t$ in this column to all rows containing -1 or t in this column. The resulting column will look like your old column, and the determinant of your matrix has not been changed.

This means that if before, the determinant of the matrix from expanding down this column was

$$-1[det(A_1)+...+det(A_n)]+t[det(B_1)+...+det(B_m)]+(1-t)[det(C_1)+...+det(C_p)],$$

where A_i, B_i, C_i are the $(n-1)\times(n-1)$ matrices obtained by removing the row and column of each respective -1, t, and $1-t$ term, then now your determinant is

$$-1[det(A'_1)+...+det(A'_n)]+t[det(B'_1)+...+det(B'_m)]+(1-t)[det(C'_1)+...+det(C'_p)],$$

where A'_i, B'_i, C'_i are obtained from A_i, B_i, C_i by replacing -1s with ts and vice-versa. Clearly, if we can demonstrate that something similar can be done with every single column, then we have shown that the matrix of the reverse-oriented knot has the same determinant as the matrix of the original orientation of the knot. To show this, we simply have to examine the other two cases.

Case ii): Simply expand down this column, no changes necessary. If before, expanding down this column gave determinant $A(t)$, expanding down it now will either give $t^{-1}A'(t)$ in case iia) or will give $tA'(t)$ in case iib). [Note that $A'(t)$ will differ from $A(t)$ by no more than a power of t and possibly a negative sign, by the same inductive reasoning used in i).] Since we are allowed to multiply an Alexander polynomial by a power of t, this doesn't actually change the polynomial.

Case iii) If there is only one -1 and t term, simply switch the two rows containing these terms. This will result in the original column, and while a row swap changes the sign of the determinant this isn't something we care about. If there is more than one -1 and t term, subtract a row with a -1 in this column from every other row with a -1 in this column, and do the same for rows which have ts in this column. This does not change the derivative, and will result in a column with only one -1 and one t term, which was just discussed.

This covers every case. We've now demonstrated that there is a way to manipulate the matrix obtained from the reverse oriented knot in such a way so that finding the determinant by expanding down any column will yield the same determinant we had originally, ignoring changes of sign and powers of t. This shows that the Alexander polynomial obtained from the reverse oriented knot is identical to the one obtained from the originally oriented knot.

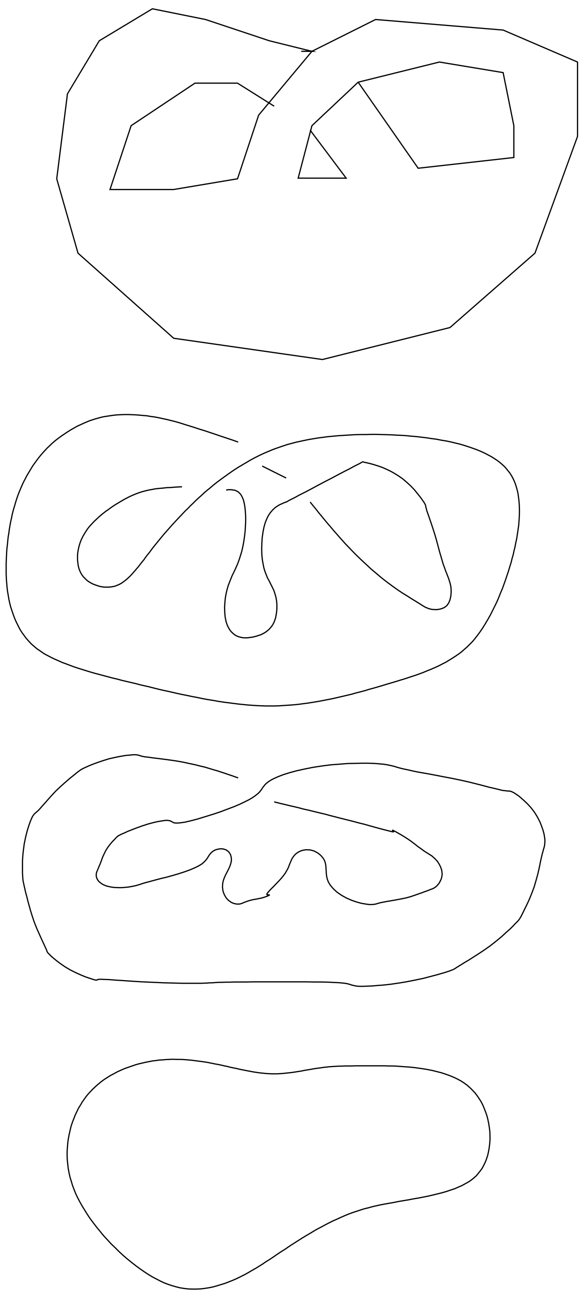
Chapter 4

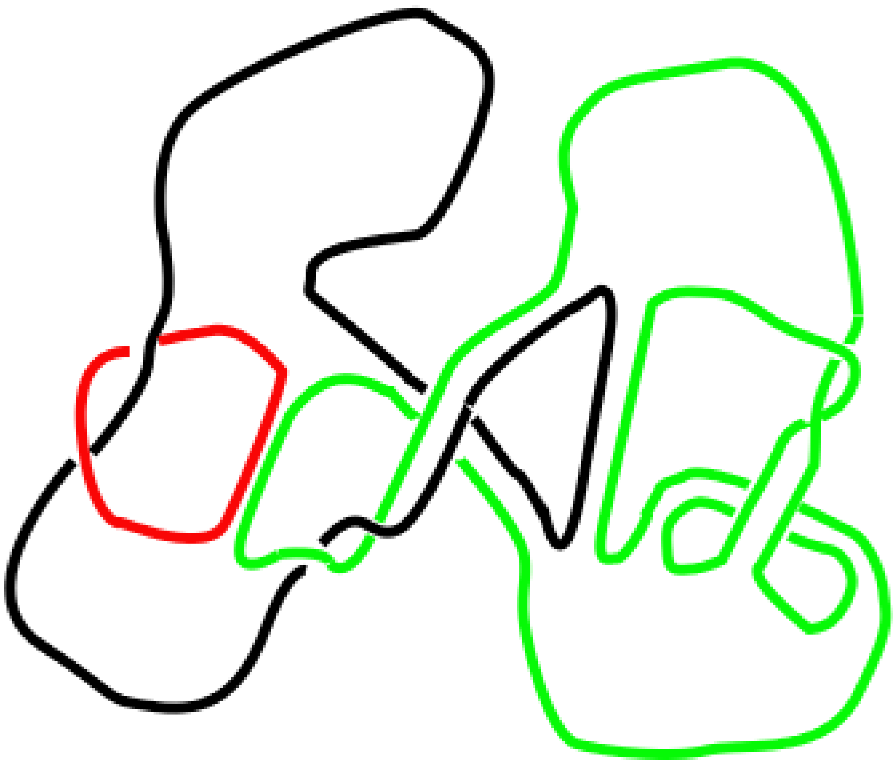
1.2 4.1.2 The surface in figure 4.1 is a disk with two twisted bands attached. This surface is homeomorphic to the same surface with the bands untwisted. This is because there exists a one-to-one and onto map that cuts, untwists and reattaches the bands in the first surface so that it looks, in 3-space, like the second surface. This mapping is continuous because points that are close to each other on the first surface map to points that are close to each other on the second surface.

Although the two surfaces are homeomorphic, they cannot be deformed into each other in 3-space. We show this by demonstrating that their boundaries are different. In exercise 4.1.1, we saw that the boundary of the surface with twisted bands is a trefoil knot. In the following diagram we see that the boundary of the surface with untwisted bands is the unknot.

2.1 (Miles)

4.2.1 (Miles) Problem: Use Theorem 3 to find the genus of the surface illustrated below (note: the boundaries for the knot have distinguished by coloring them different colors).





Theorem 3: The genus of a connected orientable surface, which is formed by attaching bands to a collection of disks, is given by: (2−disks+bands−boundary components)/2

Our surface as four disks, seven bands, and three boundary components. Thus the genus $= (2-4+7-3)/2 = 2/2 = 1$

2.3 (Tim) Prove corollary 2.

Corollary 2. if two connected orientable surfaces intersect in a single arc contained in each of their boundaries the genus of the union of the two surfaces is the sum of the genus of each.

the genus of a surface is $g = \frac{2-\chi-B}{2}$ Since we are connecting the surfaces along the boundary of each the union of the surfaces has a number of boundary components $B_1 + B_2 - 1$ Since the boundary that we connect the two along becomes the same boundary in both surfaces and so would be counted twice. By theorem 1 in section 4.2 we have that the euler characteristic χ of the union of the two surfaces is the sum of both minus 1 since they meet along a single arc. Thus for the genus of the union of the two surfaces we have $g = \frac{2-\chi_1-\chi_2-B_1-B_2+2}{2}$ which is clearly the sum of the genuses of the two surfaces alone $g_1 \frac{=2-\chi_1-B_1}{2}$ and $g_1 = \frac{2-\chi_1-B_1}{2}$

2.4 **Problem 4.2.4: Use Theorem 5 to prove that the only genus 0 surface with a single boundary component is the disk.**

Theorem 5: Every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk.

Theorem 3: If a connected orientable surface is formed by attaching bands to a collection of disks, the genus of the resulting surface is given by

$$G = (2 - disks + bands - boundary)/2$$

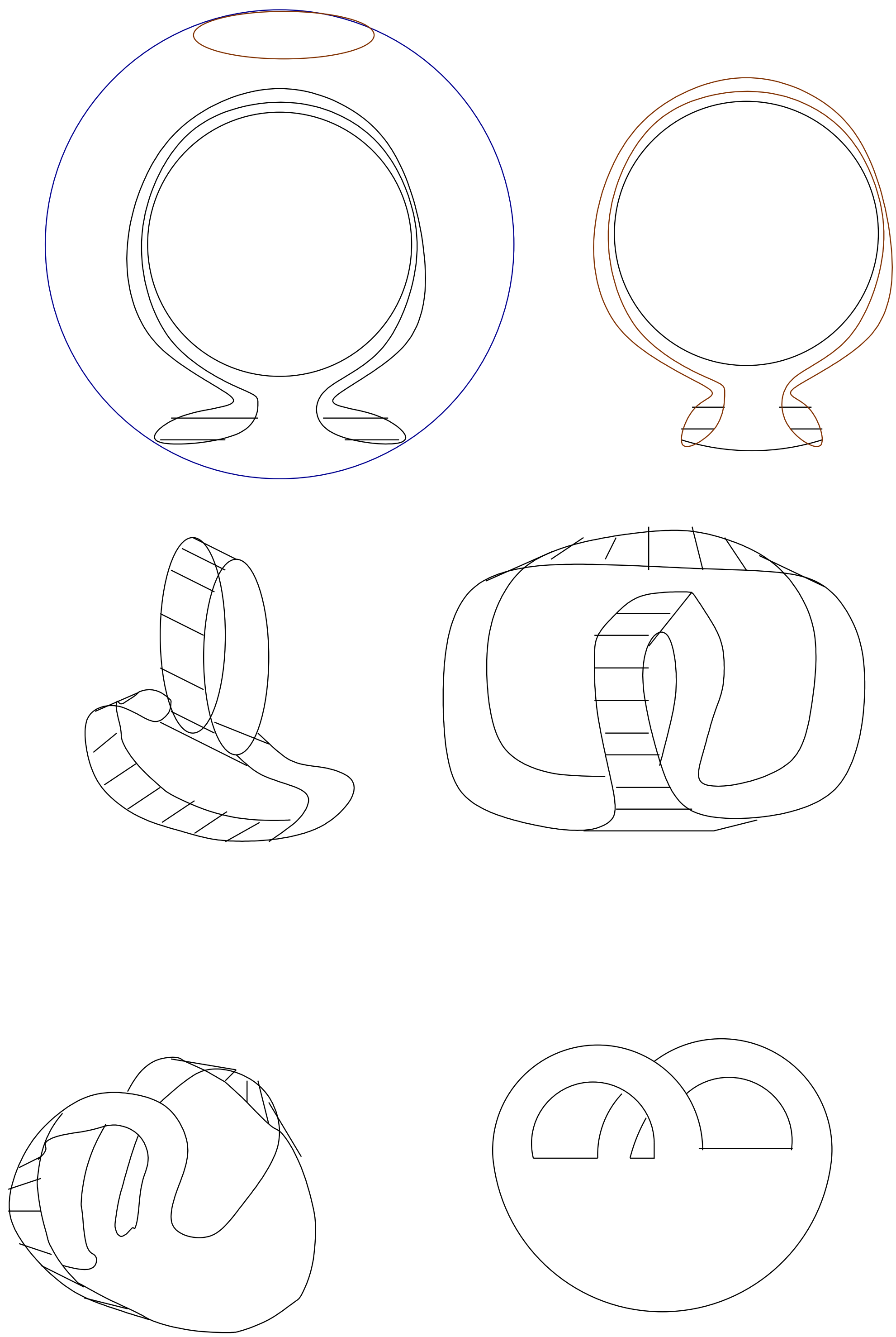
.

Suppose there were a surface S with one boundary componenet, S not homeomorphic to the disk, and $G(S) = 0$. By Thm 5, we know S is homemorphic to a disk with bands attached. Since S is not homemorphic to the disk, there is at least one band attached. But if we substitute anything but now let us compute the genus of S using Thm 3. We find that $G(S) = (2-1+r-1)/2 = r/2$, where r is the number of arcs. But then the genus of S is not 0 as we had assumed, so we have a contradictin.

2.6 4.2.6 A punctured torus can be deformed into a disk with two bands attached. Begin by stretching the hole around the surface until the torus looks like the subsurface in the picture.

4.2.8 (Miles) Problem: Prove the genus of a surface is always nonnegative.

We know (from theorem 3) that the genus of a knot can be found using the following equation: $g(K) = (2-\text{disks}+\text{bands}-\text{boundary components})/2$
We also know (from theorem 5) that every connected surface with boundary is homeomorphic to a surface constructed by attaching bands to a disk. So it is sufficient to conscider surfaces that consist of only a disk with bands attached. Let's first look at the simplest surface, just a disk. The genus of this surface is 0.
Now let's consider what happens when we add disks or bands to it. What we want to do is look at the worst case scenarios, and make the genus as small as possible (maybe even negative).Looking at the formula we see that this is done by adding as many disks as possible, adding as few of bands as possible, and making as many boundary components as possible.
Notice that anytime another disk is added at least one band is required (otherwise you would have more than one surface). Also notice that to add more boundary components requires adding bands, or bands with disks. If it is done with just bands, then the most boundary components can introduce is 1, but this is cancelled out by the band added. So what if we increase the number of boundary components by adding a collection of bands with disks? We run into the same problems we had earlier. Consider adding boundary components by adding another disk with two bands running to the first disk. This will increase the disk and boundary number by one, but also the band number by one. The same thing happens no matter how many disks and bands we may add to try and increase the number of boundary components.



Thus the biggest number that we can reduce the genus number by is 0. It follows that the smallest make the genus is 0.

3.3

4.3.3 (Miles) Problem: Why does Seifert's algorithm always produce an orientable surface?

In short, the reason this occurs is because Seifert's algorithm produces a surface which boundary is an orientable knot.
Suppose Seifert's algorithm produced a nonorientable surface. We already know that the boundary of the surface created is a knot. However, we reach a contradiction because all knots are orientable and this would imply that the knot wasn't orientable. Thus it must be that the Seifert surface is in fact orientable.

3.4 **Problem 4.3.4: In applying Seifert's Algorithm, a collection of Seifert circles is drawn. Express the genus of the resulting surface in terms of the number of these Seifert circles and the number of crossings in the knot diagram.**

This is simply an elementary applicatino of Thm 3, shown above. Denoting the number of seifert circles by s and the number of crossings by x, we have $G = (2 - s + x - 1)/2$. This follows by noting that each seifert circle is a disk, each crossing is a band joining a disk to another disk, and the resulting surface has one boundary component, namely the knot from which the surface was obtained.

5.1 (Charlotte)

4.5.1 (Charlotte) If K is nontrivial, there does not exist a knot J such that $K\#J$ is trivial. If $K\#J$ is trivial, it is the unknot, with genus 0. The additivity of knot genus states that the genus of the connected sum of two knots is the sum of the genus of the two knots. So for the connected sum of two knots to be 0, both knots would have to be trivial.

5.2 (Tim) Use the connected sum of 3 distinct knots to find an example of a knot which can be decomposed as a connected sum in two different ways.

Since the knot is made up of three knots it follows that you can break it up as a connected sum of one prime knot and the other two together or you can break it up with that prime knot connected to one of the other two and a third prime knot left out. Of course the above only holds if we are dealing with 3 different prime knots since otherwise the knots we are separating into would be indistinguishable.

5.5 **4.5.5: Use the genus to prove that there are an infinite number of distinct knots. As a harder problem, can you find an infinite number of prime knots?**

Two knots are distinct if they have different genus. But we can take a knot K with genus g and construct $K \times K$ with genus $2g$. Then $K \times K$ is distinct from K. We can do this an infinite amount of times, finding an infinite number of distinct knots $K \times K \times ... \times K \times K$.

Will think about the prime bit some more.

Chapter 5

1.2 (Tim) Show that S_6 is not commutative.

Two elements of S_6 are (1, 3, 4) and (2, 4, 6) just quickly checking we get (1, 2, 4)(2, 4, 6) = (1, 4)(2, 6) (2, 4, 6)(1, 2, 4) = (1, 2, 4)(4, 6)

so clearly (1, 3, 4) and (2, 4, 6) do not permute and so S_6 is not commutative.

5.1.3 (Miles)

(a) Verify that the inverse to (1,4,2,5)(3,6) is (1,5,2,4)(3,6)
To do this we simply look at the product of these permutations and see if we get the identity permutation. Consider the product (1,4,2,5)(3,6)(1,5,2,4)(3,6). In the first cycle 1 goes to 4, then in the third 4 goes to 1. Thus 1 goes to 1. Notice 2 goes to 5, then five goes to 2. If you do this you will get the permutation (1)(2)(3)(4)(5)(6), which is the identity permutation.

(b)Find the inverses to (1,3,6,4,5,2), (1,6,4)(2,5,3), and (1,2,3,4)(3,4,2)(3,5,6,1).
The inverse to (1,3,6,4,5,2) is (2,5,4,6,3,1), the inverse to (1,6,4)(2,5,3) is (4,6,1)(3,5,2), and the inverse to (1,2,3,4)(3,4,2)(3,5,6,1) is (6,5,1)(3,4,2). Each of these can be verified why taking the product of the permutation and the permutation that is inverse to it and noticing that that identity is the result.

(c) In general how does one write down the inverse of a permutation given in cyclic notation?
First write the permutation down as a product cycles such that no two of the cycles will have an element in common (we know we can do this by theorem 1). The inverse of the permutation is simply the set of cycles that were just found, but with all the elements in each cycle written in reverse order.

(d) Let g be the permutation (1,6,3)(2,4,5). Compute $g^{-1}(2,3,4)g$, and $g^{-1}(1,3)(4,5,2,6)g$. What is a short cut for computing $g^{-1}fg$ in general?
Doing some simple computation we see that $g^{-1}(2,3,4)g$ is the permutation (1,5,4)(2)(3)(6), and also $g^{-1}(1,3)(4,5,2,6)g$ is the permutation (1)(2,4,6,3,5). I don't know if this is what they were looking for, but the following is a shortcut (kind of). Conscider $g^{-1}(2,3,4)g$, or (3,6,1)(5,4,2)(2,3,4)(1,6,3)(2,4,5). Instead of working with this huge mess, we can just look at $g^{-1}f$, and when we get to the cycles in f just work backwards over g^{-61}. Looking at only (3,6,1)(5,4,2)(2,3,4) we see that 1 goes to 3, then three goes to 4, then 4 goes to 5. Thus 1 goes to 5. Now looking at 5: 5 goes to 4, then 4 goes to 2, and then 2 goes to 4. Therefore 5 goes to 4. Now look at 4: 4 goes to 2, then 2 goes to 3, then 3 goes to 1. So 4 goes to 1.Then the first cycle in the permutation is (1,5,4). This can be continued to get the rest.

1.4 (Charlotte)

5.1.4 (Charlotte) It is not possible in S_8 for the product of two 4-cycles to be an 8-cycle.

Either the two 4-cycles are disjoint or they have at least one common element. If they are disjoint then their product is the same two disjoint 4-cycles. If they have one element in common then their product can have at most 7 distinct elements. If they have more than one element in common their product will have fewer than 7 distinct elements, so it is not possible for an 8-cycle to be the product of two 4-cycles.

1.5 **Problem 5.1.5: In S_n, a) what is the order of $(1,3,4,6,2)$? b) Verify that the order of $(1,3,5)(2,4)$ is 6. c) What is the largest order of an element in S_7? S_{10}? S_{20}?**

a). $(1,3,4,6,2)$ is a 5-cycle and so has order 5.

b). $(1,3,5)$ and $(2,4)$ are disjoint. $(1,3,5)$ has order 3, and $(2,4)$ has order 2. So any n with $2|n$, $3|n$ will result in $[(1,3,5)(2,4)]^n = e$. The smallest such n is 6.

c). The largest order of S_7 is 12, which results from a 3-cycle and a 4-cycle permutation. In S_{10} the largest order is 30, which results from a 5-cycle, 3-cycle, and 2-cycle. In S_{20} the largest order is 420, obtained from a 7-cycle, 5-cycle, 3-cycle, and 4-cycle.

5.1.7 (Charlotte) a) Check that the 5-cycle (1,2,3,4,5) is equal to the product of transpositions, (1,2)(1,3)(1,4)(1,5).

It is straightforward to check that (1,2)(1,3)(1,4)(1,5)=(1,2,3,4,5). In each transposition (1,n), n goes to 1 and in the next transposition 1 goes to n+1, so the result is (1,2,3,4,5).

b) Write (1,5,3,4) as a product of three transpositions.

$$(1,5,3,4)=(1,4)(4,5)(4,3)$$

Write (1,2,4)(3,5,6) as a product of transpositions.

$$(1,2,4)(3,5,6) = (1,2)(1,4)(3,5)(3,6)$$

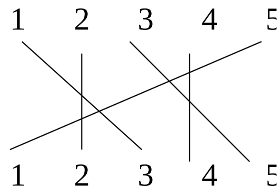
c) Argue that every permutation is the product of transpositions, and more specifically is a product of transpositions of the form (1,n).

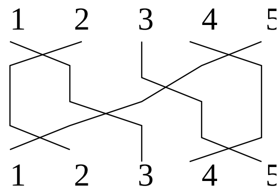
Given a set of elements $(a_1, a_2, a_3, ...)$, some element a_i can be moved to any position in the permutation of the set by a series of transpositions. Similarly, each element of the set can be moved to a different position by moving it one position at a time, trading places with a neighboring element each time. By such a series of transpositions we can make any permutation of the set. Thus, all permutations are the product of transpositions.

Show that every permutation is the product of transpositions of the form $(1,n)$.

We know that the set of transpositions generates S_K, so now we need to show that every transposition can be written as a product of transpositions of the form (1,n).

The transposition (a,b) may be written as the product of transpositions (1,a)(1,b)(1,a). So from transpositions of the form (1,n) we can build transpositions from which we can build any permutation.





d) Show that every permutation can be written as the product of transpositions taken from the set $\{(1,2),(2,3),(3,4),(4,5),...,(n-1,n)\}$. To show this we will consider an example. Write (1,3,5) as the product of transpositions. To find a solution we draw a diagram showing each element from 1 to 5 and where it is sent by the permutation. The first row of numbers represents the original elements. The second row is the image under the permutation. The lines match each element to its image under the permutation.

The second diagram is drawn to show more clearly that each move is simply made up of transpositions. Each crossing represents a transposition of two elements. The six transpositions (1,2),(4,5),(3,4),(2,3),(1,2) and (4,5) occur. The result of these transpositions is the desired permutation. We can draw such a diagram for any permutation, thus showing that any permutation can be written as a product of transpositions.

5.1.9 (Miles)

(a) Show that the set of 4-cycles generates S_4.
We know that the symmetric group can be generated by the set of transpositions, thus if we can find a way to write any transposition as the product of 4-cycles then we know that the set of 4-cycles will generate S_4.
Let A be a set in S_4, with A_i an element in the set. Suppose you want to transpose the $i = m$ element with the $i = n$ element. Then let B be the set with $B_i = A_i$ if $i \neq m, n$. If $i = m$ let $B_m = A_n$, likewise if $i = n$ let $B_n = A_m$. Then AB is the transposition of the $i = m$ element of A with the $i = n$ element of A.

(b) Show that the 4 cycles (1,2,3,4) and (1,2,4,3) generate S_4.
Notice that:
$(1, 2) = (1, 2, 3, 4)(1, 2, 4, 3)^2(1, 2, 3, 4)^2(1, 2, 4, 3)$
$(1, 3) = (1, 2, 3, 4)(1, 2, 4, 3)^2$
$(1, 4) = (1, 2, 3, 4)(1, 2, 4, 3)^2(1, 2, 3, 4)(1, 2, 4, 3)$
$(2, 3) = (1, 2, 3, 4)^2(1, 2, 4, 3)$
$(2, 4) = (1, 2, 3, 4)(1, 2, 4, 3)^2(1, 2, 3, 4)^2$
$(3, 4) = (1, 2, 3, 4)(1, 2, 4, 3)$
Thus (1,2,3,4) and (1,2,4,3) generate S_4.

2.1 (Tim) Check that the consistency condition is satisfied at all the crossings in the labeled knot diagrams illustrated by figures 5.2 and 5.3

For the labelings in 5.2 we have the conditions

$(1, 3)(1, 2)(1, 3) = (2, 3)$ $(2, 3)(1, 3)(2, 3) = (1, 2)$ $(1, 2)(2, 3)(1, 2) = (1, 3)$

$(4, 3, 2, 1)(1, 3, 2, 4)(1, 2, 3, 4) = (1, 2, 4, 3)$ $(4, 2, 3, 1)(1, 2, 4, 3)(1, 3, 2, 4) = (1, 2, 3, 4)$ $(3, 4, 2, 1)(1, 2, 3, 4)(1, 2, 4, 3) = (1, 3, 2, 4)$

all of which hold

and the conditions obtained from 5.3 are

$(3, 2, 1)(5, 4)(3, 5)(1, 2, 4)(4, 5)(1, 2, 3) = (2, 4)(1, 5, 3)$ $(3, 5, 1)(4, 2)(4, 5)(1, 2, 3)(2, 4)(1, 5, 3) = (1, 2)(3, 4, 5)$ $(5, 4, 3)(2, 1)(2, 4)(1, 5, 3)(1, 2)(3, 4, 5) = (1, 3)(2, 4, 5)$ $(5, 4, 2)(3, 1)(1, 2)(3, 4, 5)(1, 3)(2, 4, 5) = (3, 5)(1, 2, 4)$ $(4, 2, 1)(5, 3)(1, 3)(2, 4, 5)(3, 5)(1, 2, 4) = (4, 5)(1, 2, 3)$

which also all hold so the consistency condition is satisfied at the crossings.

2.2 **Problem 5.2.2: In figure 5.5, two of the labelings satisfy the consistency condition while one does not. Find the inconsisten labeling.**

Figure b is inconsistent. The bottom intersection, read off the diagram, implies $(12)(345)(35)(124)(543)(12) = (13)(245)$. This equation actually equals $(152)(34)$, however.

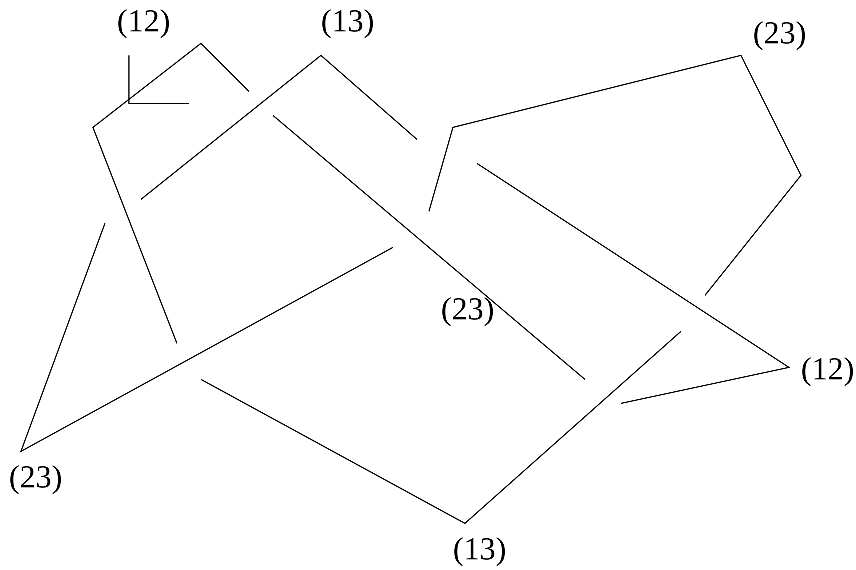


Figure A

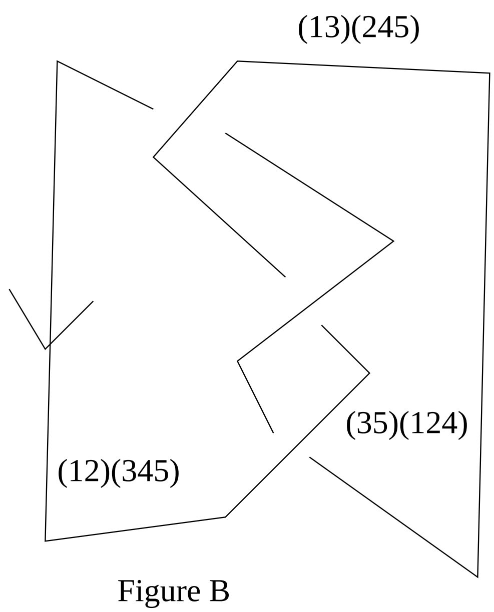
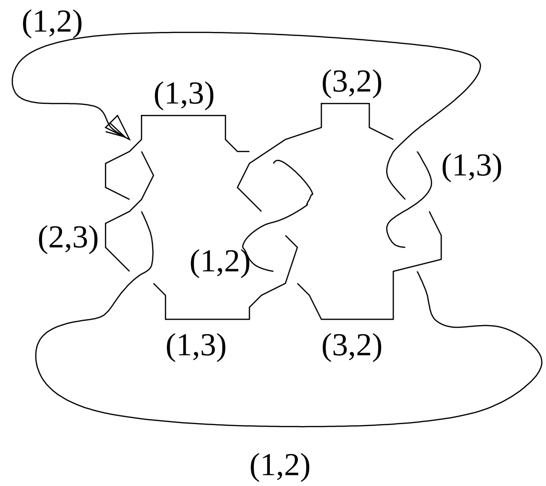


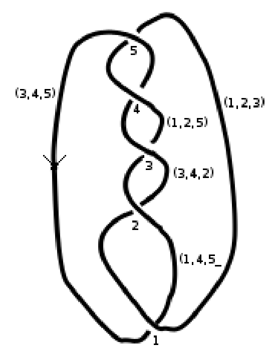
Figure B

5.2.4 (Charlotte) Find a labeling of the (3,3,3)-pretzel knot illustrated in Figure 5.7 using transpositions from S_4. Your labeling should have every transposition appear, so it is clear that the labels generate the group.

5.2.6 (Miles) Problem: Show that in some of the previous examples the labeling becomes inconsistent if the orientation of the knot is reversed. Show, however, that is the orientation of the knot is reversed and each label is replaced with its inverse, then the labeling will again become consistent.

Conscider the diagram in Figure 5.5. Here it is with the orientation reversed:





Looking at the following vertices we see that the labelings are not consistent.
1: $(1,2,3)(3,4,5)(3,2,1) = (1)(2,4,5)(3) \neq (1,4,5)$
2: $(1,4,5)(1,2,3)(5,4,1) = (1)(2,3,5)(4) \neq (3,4,2)$
3: $(3,4,2)(1,4,5)(2,4,3) = (1,3,5)(2)(4) \neq (1,2,5)$
4: $(1,2,5)(3,4,2)(5,2,1) = (1,3,4)(2)(5) \neq (3,4,5)$
5: $(3,4,5)(1,2,5)(5,4,3) = (1,2,4)(3)(5) \neq (1,2,3)$

Now if we also replace each label with its inverse we get the following equations at each crossing:
1: $(3,2,1)(5,4,3)(1,2,3) = (1,5,4)(2)(3) = (5,4,1)$
2: $(5,4,1)(3,2,1)(1,4,5) = (1)(2,4,3)(5) = (2,4,3)$
3: $(2,4,3)(5,4,1)(3,4,2) = (1,5,2)(3)(4) = (5,2,1)$
4: $(5,2,1)(2,4,3)(1,2,5) = (1)(2)(3,5,4) = (5,4,3)$
5: $(5,4,3)(5,2,1)(3,4,5) = (1,3,2)(4)(5) = (3,2,1)$
Thus when the orientation was reversed and the labelings were replaced with their inverses the labelings became consistent.

3.2 Problem 5.3.2: Prove it is impossible to label the trefoil with trans-

positions from S_4.

The trefoil has only three arcs. Once the labelings on two of these arcs are determined, the third is forced. But to be consistent, this third labeling will have to have the form (xy), where the first labeling takes the form (xa) and the second takes the form (yb). But then the third labeling is generated by the first two, and it takes more than two transpositions to generate S_4.

5.3.4 (Charlotte) Check the claims about labelings for diagrams 6_1 and 9_{46}.

Diagram 6_1 cannot be labeled with transpositions from S_4 If we label arcs a and b with the same labeling, for example both (1,2), all arcs in the diagram are forced to be labeled (1,2). If the labels for a and b overlap, such as (1,2) and (2,3) this forces a labeling in transpositions from S_3. If the labels for a and b are disjoint we reach a contradiction. For example, if a is (1,2) and b is (3,4), we will find that arc d must be both (1,2) and (3,4). Therefore, 6_1 cannot be labeled with transpositions from S_4.

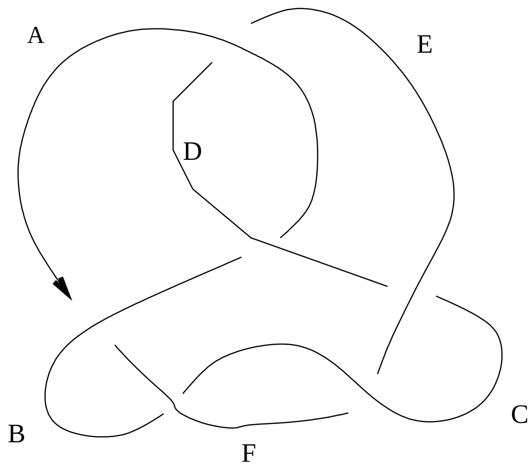
Diagram 9_{46} can be labeled with transpositions from S_4 as follows.

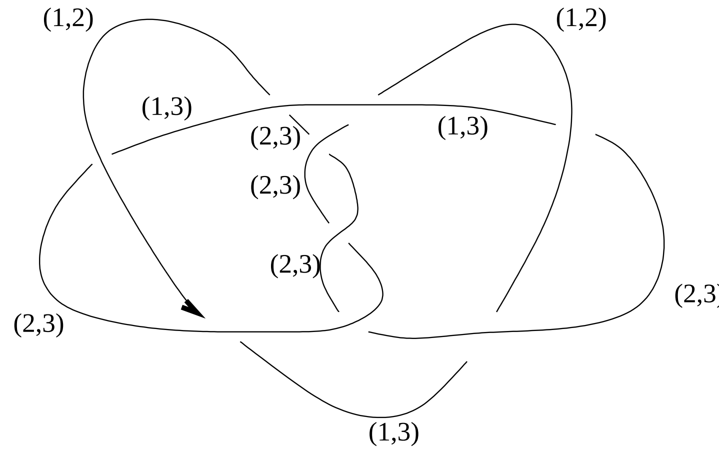
Diagrams 6_1 and 9_{46} can be labeled with 4-cycles from S_4 as follows.

5.4.1 (Miles) Problem: Why is $xyx^{-1} = yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1}$ equivalent to $yx^{-1}zxy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx^{-1} = 1$?

We know that $((yx^{-1}zxy^{-1})x(yx^{-1}z^{-1}xy^{-1}))$ has an inverse, in particular: $(yx^{-1}zxy^{-1})x^{-1}(yx^{-1}z^{-1}xy^{-1})$.
Multiplying both sides of the first equation on the left by this we get: $yx^{-1}zxy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx$
$= yx^{-1}zxy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}yx^{-1}zxy^{-1}xyx^{-1}z^{-1}xy^{-1} = 1$
$\Rightarrow yx^{-1}zxy^{-1}x^{-1}yx^{-1}z^{-1}xy^{-1}xyx^{-1} = 1$

Questions:
Can we find the cases that causes a certain invariant to not be able to distinguish two knots? Put another way, can we combine a couple of invariants we know to obtain a true invariant?
Do any of the invariants we've covered so far apply to linking number?
Is there a "simplest" diagram for a knot? How about a link?
Can prove what the least number of crossings in a diagram (equivalent to some knot) is?


A
E
D
B
F
C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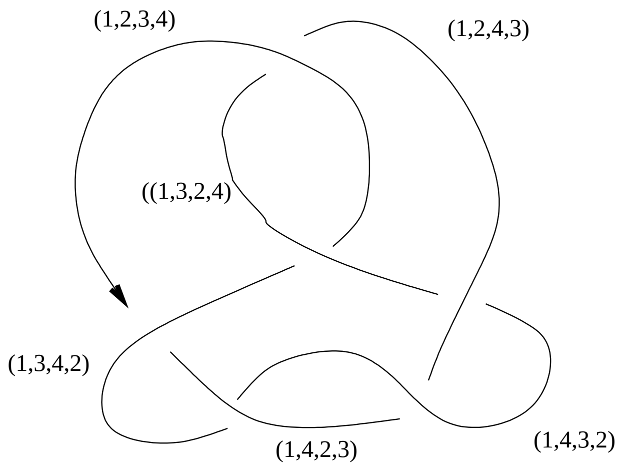
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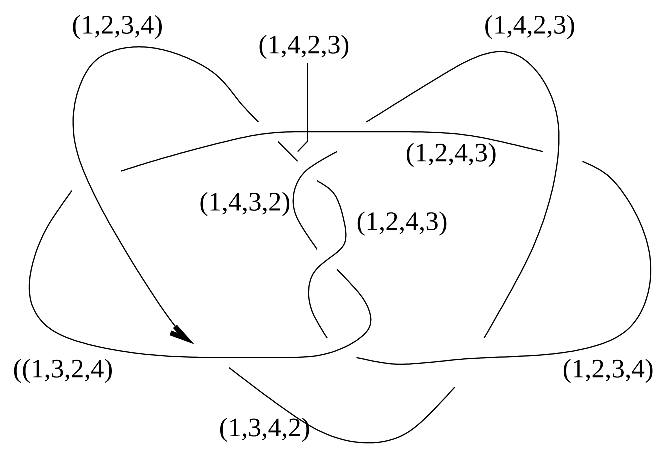
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Chapter 6

1.2 **Problem 6.1.2: Find the Seifert matrix of the knot in figure 6.1.**

6.1.4 (Charlotte) Find the Seifert matrix corresponding to (p,q,r)-pretzel knots with p, q and r odd.

As in figure 4.8 in the text (page 67), a pretzel knot can be deformed to look like a disk with bands attached by pushing the boundary down through each twist of the center band, and leaving a space at its base between the two newly formed sides of the center band. Call the bands p, q and r, where the name indicates the number of twists on the band. In the new surface, we have only two bands which we will call P and R. It will be useful to refer to the twists that were originally on p as P_p and the twists that were originally part of q but are now part of P as P_q. We will do the same for R. What was formerly q is now made up of the q twists of P_q and q twists of R_q.

Smoothing the bands in the center will either increase or decrease the number of twists on p and r, depending on the direction of the twists. (We will describe the case of q and p only, realizing that the same is true for q and r.) If p and q are twisted in the same direction, then for every twist we push off of P_q we automatically loose a twist on P_p. That is, each twist on P_q undoes a twist on P_p.

If p and q are twisted in opposite directions, then for every twist we push off of P_q we will add a twist to P_p. In this case, each twist on P_q simply becomes instead a twist on P_p.

We said that the number of twists on P_q and R_q is q. So now we can say that the number of twists on P is the number of twists on P_p plus the number of twists on P_q, or simply $p+q$. Similarly, the number of twists on R is $r+q$. Remember to designate one direction of twists as negative and the other direction as positive.

Because p, q and r are odd, P and R will necessarily be even. This ensures that our Seifert surface will be orientable. We draw our surface in the standard way with curves P, P*, R and R*.

Since P is counted in half twists, the linking number of P and P* is $\frac{P}{2}$. Similarly, the linking number of R and R* is $\frac{R}{2}$. These are integers since P and R are even.

For q odd, the linking number of P and R* will be either $q-1$ or $q-2$. From looking at figure 4.8 you can see that this depends on whether P is above or below R at the top of band q. For the purpose of the matrix we are trying to describe, we will say we have arranged the knot so that P is above R at the top, which will mean that the linking number of P and R* is $q-1$ and the linking number of R and P* is $q-2$.

This gives us the Seifert matrix:

$$\begin{pmatrix} \frac{P}{2} & q-1 \\ q-2 & \frac{R}{2} \end{pmatrix}$$

6.1.6 (Miles) Problem: What would be the effect of changing the orientation of the Seifert surface on the Seifert matrix?

We knot that the (i, j) entry in the Seifert matrix is given by $lk(x_i, x_j{}^*)$. Clearly the magnitude of the linking number is going to stay the same. What is going to change is the sign. If a crossing is left handed and the orientation is reversed it will become right hand and vice versa. This corresponds to a changing sign. Thus thus if a Seifert matrix is computed for a surface and then computed again with the same surface with the orientation reversed, the matrices will be the same, except the signs on each entry will be opposite.

6.2.3 (Charlotte) Check the calculation of the determinant that gives the Alexander polynomial of the knot in Figure 6.1.

The matrix associated with the knot in Figure 6.1 is $V - tV^t$ where V is the Seifert matrix of the knot:

$$\begin{pmatrix} 2-2t & 1 & 0 & 0 \\ -t & -5+5t & 1-t & 0 \\ 0 & 1-t & 2-2t & -1+2t \\ 0 & 0 & -2+t & -2+2t \end{pmatrix}$$

Calculating the determinant by expanding on the first column we get

$$\begin{gathered} (2-2t)[(-5+5t)(2-2t)(-2+2t) \\ -[(-5+5t)(-1+2t)(-2+t)+(1-t)(1-t)(-2+2t)]] \\ +t[(2-2t)(-2+2t)-(-1+2t)(-2+t)] \\ = 64t^4 - 272t^3 + 417t^2 - 272t + 64 \end{gathered}$$

which is the Alexander Polynomial of the knot.

6.2.5 (Miles) Problem: Use the result of exercise 6.1.7 to show that the Alexander polynomial of the conneted sum of knots is the product of their individual Alexander polynomials.

Result of exercise 6.1.7: Let M_J and M_K me Seifert matrices for knots J and K (respectively). Then the Seifert matrix for $J\#K$ is the matrix

$$\begin{bmatrix} M_J & 0 \\ 0 & M_K \end{bmatrix}$$

The Alexander polynomial of $J\#K$ is then equal to

$$\begin{vmatrix} M_J - tM_J & 0 \\ 0 & M_K - tM_K \end{vmatrix}$$

Which is equal to $det(M_J - t{M_J}^T)det(M_K - t{M_K}^T)$ which equals the Alexander polynomial of J multiplied by the Alexander polynomial of K.

Questions

Brian

For this I think an interesting one would be the one the book mentions: What is the least number of Reidemeister moves that is required to transition the knot for Prob. 3.1.2 into the unknot and can we prove that this is the least number required. I see a method of 5 moves so far.

Question:How many knots in the Appendix can taken from 1 link to 2 links or even n links? Of these new links, which ones are oriented equivalence?
Question:By cutting a knot at a point A, can you then make 2 more crossings and reconnect that knot and get all knots with +2 crossings?
Question Is there a 'pedigree tree' such that by creating one knot you can get to only some of the higher crossing knots?
Question If there is a tree does this help distinguish more knots apart?
Question Can you get 2^N distinct orienations in other ways? Does making a chain of n orientable equivalent knots such that it is not symmetric about the middle link give us an upper bound?
Question: Are there knots that have a mod p solution and a mod q solution.?

Miles

Questions:
Why are coloring numbers restricted to just the odd primes?
If two knots can both labeled mod p will they share any other characteristics?
If two knots are added together how will its coloring number relate to the two knots coloring numbers?
To used the equation $2x - y - z = 0$ (mod p) to find an invariant for knots. Could we perhaps use some other equation that would give us a stronger invariant? Questions:
1. Is there a easier way to check that a certain set generates a group?
2. How do you know if a set that generates a group is the smallest set (least number of elements) that generates that set?
3. In the previous chapter we learned that we could write down all possible surfaces (up to a homeomorphism). Can we do the same sort of thing with groups?

Is it necessarily the case that all surfaces which have that same knot K as their boundary are homeomorphic to each other?

Is the least sticks question for a knot essentially the same as the least triangles for a surface (with the knot as its boundary) question?
Is there a way to define a prime knot such that prime factorizations is unique, and a two prime knots determine a single knot?
Can we find how many diagrams there are with n crossings using stereotypical projection?
Can we show there are infinitely many distinct knots?
Can any diagram of the unknot be all the way unknotted using only the type 2 Reidemeister move?
Can a knot ever be labeled (the way we defined it in class) with an abelian group? If so, what are some examples?
Can we find the cases that causes a certain invariant to not be able to distinguish two knots? Put another way, can we combine a couple of invariants we know to obtain a true invariant?
Do any of the invariants we've covered so far apply to linking number?
Is there a "simplest" diagram for a knot? How about a link?
Can prove what the least number of crossings in a diagram (equivalent to some knot) is?

Tim

1. Are there arbitrarily many different knot invariants?

2. Would the existence of an algorithm to distinguish any two knots in a finite number of steps imply the existence of a finite number of invariants which would distinguish any two knots?

3. What is a simple way to enumerate the set of all knots allowing for duplicate knots?

4. What is the simplest algorithm to generate all possible knots without duplication. (does such an algorithm exist?)

5. The torus knots cover some portion of the set of all knots as do the (p, q, r) pretzel knots. Can you construct a finite set of simple classes of knots like these that would cover the set of all knots? (these classes of knots would be allowed to overlap)

6. Does the set of all pretzel knots of any number of strands cover the set of all knots?

7. Connecting the strands of a braid will give you some subclass of knots what knots are not covered by this class of knots?

8. Does the class of all knots which are generated by connecting the strands of a braid cover the set of all n stranded pretzel knots?

9. Is there a closed form for the number of knots which can be drawn with a minimum number of crossings n. (is it related to the number of partitions of n?)

10. A knot diagram creates a series of distinct regions cut off from one another by the arcs of the diagram. Clearly every knot diagram must have at least 2 regions and every non trivial diagram must have at least 5 regions. If we take into account the number of crossings on the boundary of each region can we use this information

to determine what the minimum possible number of crossings a knot with a diagram with these characteristics could have?

11. Traversing the regions of a knot diagram moving around them in a counter clockwise order and noting the order of the crossings gives a permutation of the crossings. Is there some natural way to translate cycles generated like this into a Sn labeling of the knot?

12. Does the Alexander polynomial suffice to distinguish all alternating knots (knots which go from over crossing to undercrossing to over crossing to...)

13. The alternating knots can all be drawn such that every crossing is on the outermost region of the knot. Is there any other class of knots which can be drawn with all points exterior?

14. Consider a knot to be constructed of a material that has a certain flexibility but which wants locally to be a straight line segment. This puts a certain tension into the knot. Such a knot will have a minimum energy configuration where the amount of curvature is least in some sense. Given a knot made of such an elastic material in any starting configuration will the knot always "uncoil" into this minimum energy configuration? (frictionless environment of course and the knot would be considered to have a fixed total length it cannot stretch) Can such a knot have more than one possible minimum energy configuration?

14. What is the set of groups which are the fundamental groups of some knot.

15. Treating knots as closed polygonal curves with integer vertices how many distinct knots can fit into a cube of side n? (for that matter how many different representations are there)

16. Knots can be represented as a language (a language is a set of strings and a string is just a series of symbols) Given some way of representing a knot with symbols the language of a knot would be the set of all possible representations using that system of representation which represent equivalent knots. The language of a knot with a particular representation system is then a knot invariant. Is it possible to have both a means of representing a knot which unambiguously determines the knot and generates a language of all possible representations which is finite?

17. What is the computational equivalent of the operation of finding knot equivalence? In other words what computational problems can we map onto the question of equivalence of knots. One such computational framework might be to take a proposition and translate it into two knots (or rather two knot representations) in such a way that the proposition is true if and only if the two knots are equivalent. Specifically can we do this with the operation of addition? so can we associate knots in such a way to the proposition $a + b = c$ so that the question can be answered by equivalence of knots.

18. Where on the Chomsky hierarchy is the language of knots?

19. The language of at least some subclass of knot representations is decidable, is it possible to find a class of knot representations such that the language of all knots in that representation is a context sensitive language or a context free language or even a regular language?

20. How many essentially different diagrams of a knot are there which share the same number of crossings? Is the set of minimum crossing diagrams of a knot different

from the set of minimum arc diagrams of a knot? If so in what cases?

Onye

1. When are two diagrams of the same knot described as isotopic?
2. Why are knots not colorable mod 2?
3. When do two diagrams represent the same link?
4. What is an alternating pretzel-knot?
5. What is knot addition and how does it work?
6. What is the mirror image of the figure 8 knot?
7. Does a non-trivial knot in R3 necessarily have four collinear points?
8. Are there any such things as ideal knots?
9. When is a knot or link described as prime?
10. What is the unknotting number for the figure 8 knot?
11. What does the term "family of knots" mean?
12. Are there links that are colorable mod 2?
13. Is there only a unique way by which a knot or link may be colorable?