6 Exponents and Exponential Functions

Essential questions

a. Why are exponents useful? How are they used in real world applications?

b. What does a negative exponent mean?

c. Where do rules for exponents come from?

d. How does exponential function compare with polynomial functions, linear in particular.

e. How do we undo the exponential function?

6.1 Population growth

Definition 1 An exponent is a convenient way to write repeated multiplication. Given a natural number b the following notation represents a product of b many a's.

$$a^b = \overbrace{a \cdot a \cdot a \cdot .... \cdot a}^{b \text{ many } a\text{'s}}$$

Question 6.1 Use exponents to represent the following:

a. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

b. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

c. $x \cdot x \cdot x$

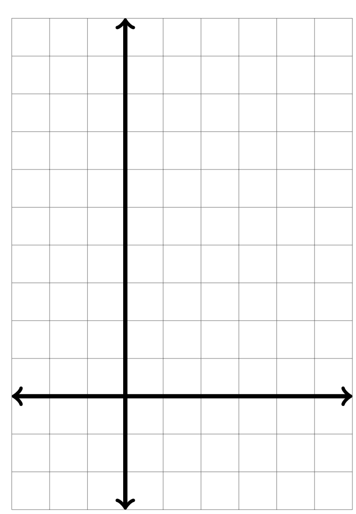
d. $a \cdot a \cdot a \cdot a \cdot a \cdot a$

Question 6.2 A colony of bacteria is being grown in a laboratory. It contains a single bacterium at $12:00$ noon (time 0), and the population is doubling every hour.

a. How long do you think it would take for the population to exceed 1 million? 2 million? Write down your guesses and compare with other students' guesses.

b. Make a table of values showing how this population of bacteria changes as a function of time. Find the population one hour from now, two hours from now, etc. Extend your table until you can answer the questions asked in Question 6.2 a. and graph your points. How close were your guesses?

t	Number of bacteria	
0	1	
1		
2		
3		
4		
5		
6		



c. In the third column in Question b. write the population each time as a power of 2 (for example, 4 is 2^2).

d. What would the population be after x hours? (Write this as a power of 2.)

e. Compare the population after 8 hours with the population after 5 hours.

   a. How much more is the population after 8 hours? (Compare by subtracting.)

   b. How many times as much is it? (Compare by dividing.)

   c. Which of your answers is a power of 2? What power of 2 is it?

f. How many bacteria would there be after three and a half hours?

g. Why does Question f. demand that we depart from thinking of this as a sequence?

h. What does $2^{3.5} = 2^{\frac{7}{2}}$ mean? Can you use the graph to estimate this number?

i. How long exactly do we have to wait to see at least 1000000 bacteria?

6.2 Rules for Exponents

Question 6.3 Rewrite the following expressions using just one exponent. To answer the question, think about how many twos would appear after you multiplied everything out.

a. $(2^2)^3$

b. $(2^4)^5$

c. $(2^5)2^7$

d. $(2^9)2^{10}$

Question 6.4 Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $(5^5)^3$

b. $(5^4)^6$

c. $(5^4)5^6$

d. $(5^2)5^{10}$

Question 6.5 Rewrite the following expressions using just one exponent. To answer the question, think about how many twos (or xs) would appear after you multiplied everything out. Think about a and b as positive integers.

a. $(2^a)^b$

b. $(2^a)2^b$

c. $(x^a)^b$

d. $x^a x^b$

Question 6.6 Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $\frac{5^5}{5^2}$

b. $\frac{6^4}{6^3}$

c. $\frac{x^a}{x^b}$

Question 6.7 Evaluate this expression in two different ways: using the rule you just developed and by multiplying everything out:

$$\frac{5^7}{5^8}$$

Question 6.8 For any $x \neq 0$, define x^{-1} to be the number such that $x^{-1}x = 1$. This makes x^{-1} the multiplicative inverse of x.

a. What number is 2^{-1}?

b. What number is 3^{-1}?

c. What number is 2^{-2}?

d. What number is 3^{-2}?

e. The rule you developed for Question 6.5 Part d. is a rule we want to be true in general. Use that rule and the definition of $2^{-1}2 = 1$ to decide the value of 2^0.

f. This is the table you filled out recently. Use the patterns apparent in the table to decide why this definition makes sense:

x	2^x
5	32
4	16
3	8
2	4
1	2
0	1
-1	
-2	
-3	
-4	

Let's go back to the

Question 6.9 What does $2^{3.5} = 2^{\frac{7}{2}}$ mean?

a. Calculate $(2^{\frac{7}{2}})^2$. Assume the rules for from 6.5 apply.

b. Explain what $\sqrt{2^7}$ means.

c. Combine Parts 1 and 2 to make sense of $2^{\frac{7}{2}}$

Question 6.10 A colony of bacteria is being grown in a laboratory. It contains a single bacterium at $12:00$ noon (time 0), and the population is doubling every hour. How many bacteria are there after 3.5 hours.

Question 6.11 Let us redo this for $a^{\frac{1}{2}}$:

a. Calculate $(a^{\frac{1}{2}})^2$

b. Explain what $\sqrt{a}$ means.

c. Combine Parts a. and b. to make sense of $a^{\frac{1}{2}}$:

Question 6.12 Think about how $f(x) = x^{\frac{1}{2}}$ is the inverse function of $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2$.

a. Why is the domain of g limited to $[0, \infty)$?

b. What would be the inverse function of $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3$?

c. What would be the inverse function of $l : [0, \infty) \to \mathbb{R}$ given by $h(x) = x^4$?

d. What would be the inverse function of $p : [0, \infty) \to \mathbb{R}$ given by $p(x) = x^n$?

Question 6.13 What does $2^{\frac{7}{5}}$ mean?

1. Calculate $(2^{\frac{7}{5}})^5$. Assume the rules for from 6.5 apply.

2. Explain what $\sqrt[5]{2^7}$ means.

3. Combine Parts 1 and 2 to make sense of $2^{\frac{7}{5}}$

Question 6.14 Use Question 6.13 to come up with a good definition of $5^{\frac{m}{n}}$.

Question 6.15 In the following exercises, we will write the expression in a simplified version, which means that every power will be written using only positive exponents.

a. $6w^5(2w^{-2})$

b. $(3a^{-2}b^{-4})^2$

c. $\dfrac{2^{-3}r^{-2}(r^{-1})^{-2}}{r(r^3)^{-3}}$

d. $\left(\dfrac{3q}{4p^2}\right)^2\left(\dfrac{2p}{5q}\right)^{-2}$

Question 6.16 A patient is administered 75 mg of DRUGX. It is known that 30% of the drug is expelled from the body each hour.

a. How many mg of DRUGX are present after 2 hours?

b. How many mg of DRUGX are present after 3 hours?

c. Develop an exponential function that models the amount of DRUGX in the body after t hours.

d. Use your model to calculate the amount of DRUGX in the body after 2.5 hours?

e. What does the fractional exponent you used in d. mean?

f. A patient needs to take another dose once the amount of DRUGX is less than 20 mg. How long should the patient wait before the first and second dose?

g. How long will it be when the model predicts that there will be exactly 20 mg of the drug in the body?

6.3 Graphs of Exponential Functions

Question 6.17 Let's make some predictions.

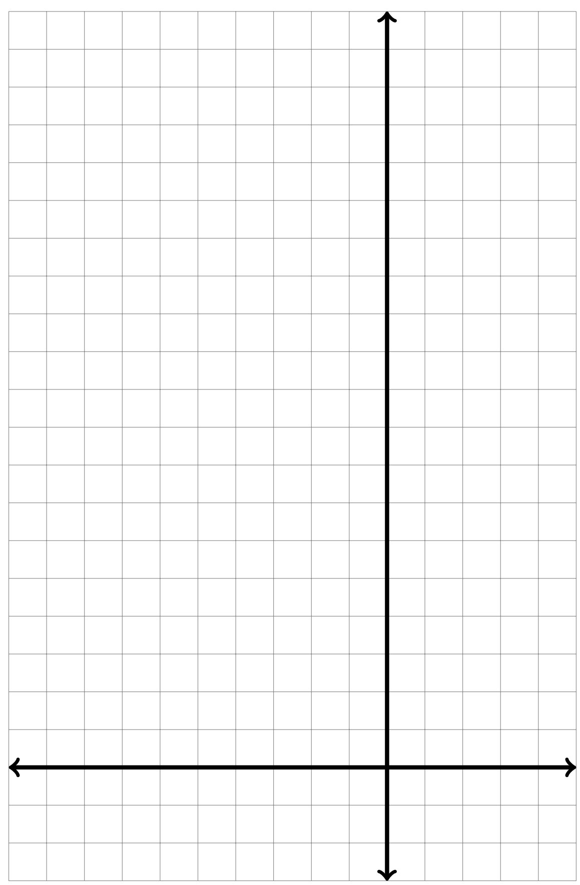
	Similarities	Differences
$f(x) = 2^x$ & $g(x) = 5^x$		
$f(x) = 2^x$ & $h(x) = (\frac{1}{2})^x$		
$h(x) = (\frac{1}{2})^x$ & $k(x) = (\frac{1}{5})^x$		

Question 6.18 Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by the rule $f(x) = 2^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

b. Sketch the graph for f:

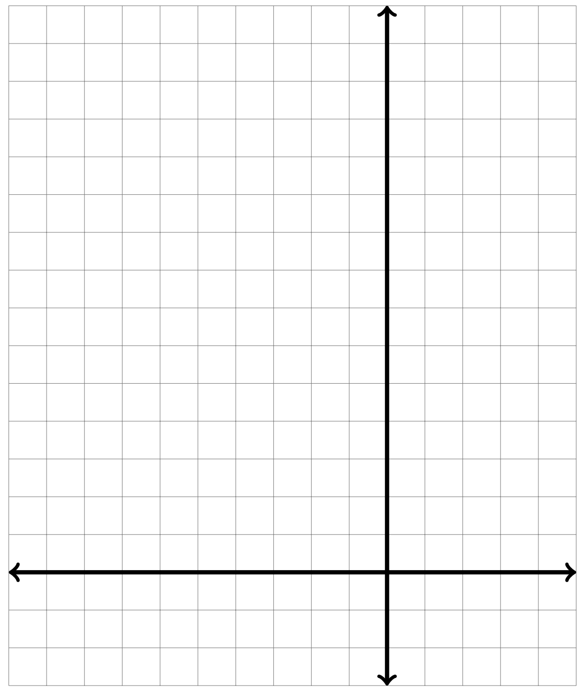


Question 6.19 Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by the rule $f(x)=(\frac{1}{2})^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

b. Sketch the graph for f.

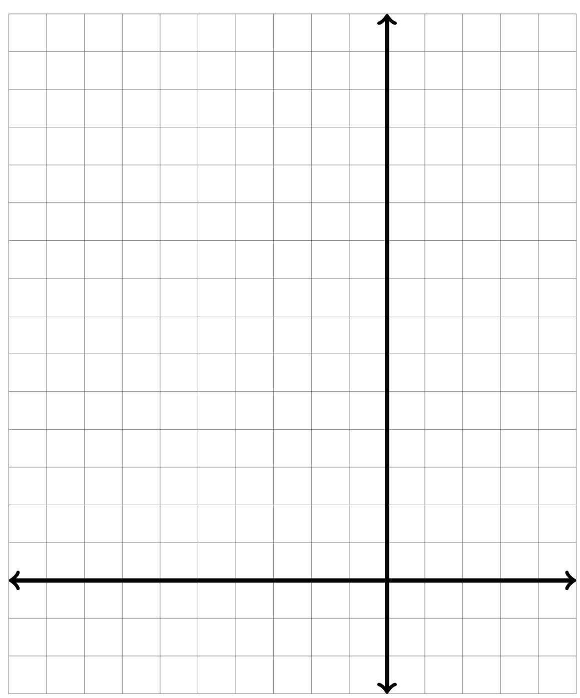


Question 6.20 Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by the rule $f(x)=5^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

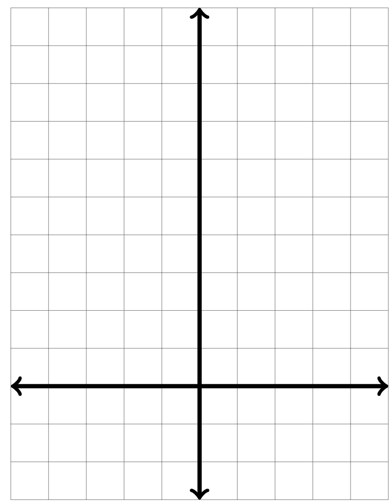
b. Sketch the graph for f.



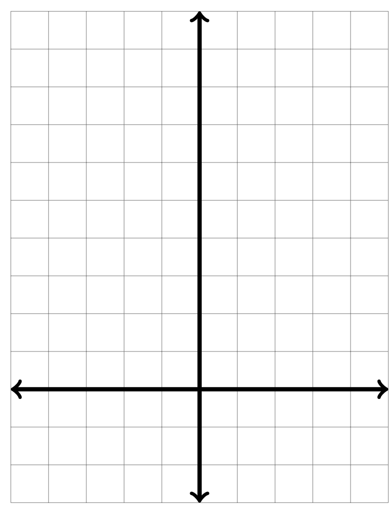
Question 6.21 Look at the graphs you drew in Questions 6.18, 6.19, and 6.20.

a. All three graphs share a common point. Which point is this?

b. Let $a > 1$. Use Questions 6.18 and 6.20 to help you sketch a graph of $f(x) = a^x$. Articulate why this is the general shape.



c. Let $0 < b < 1$. Use Question 6.19 to help you sketch a graph of $g(x) = b^x$.



d. From looking at the graphs are the functions $f(x) = a^x$ and $g(x) = b^x$ invertible? Explain.

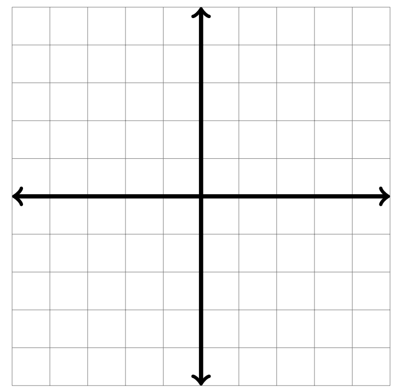
e. Why does it not make sense to talk about functions of the form $h(x) = c^x$ when $c < 0$?

6.4 Inverse function

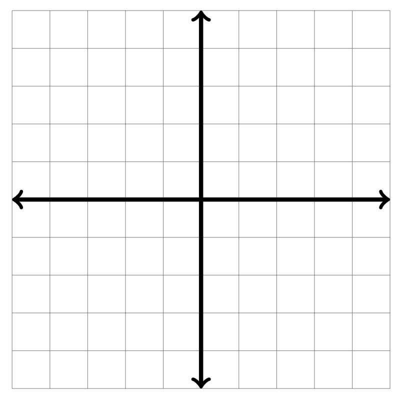
You have already discovered that exponential functions are invertible. Before we think about their inverse functions, let's solve a few problems as a warm up.

Question 6.22 Graph each of the functions. Decide if the following functions have inverses. If a function has an inverse calculate the inverse and graph the inverse function.

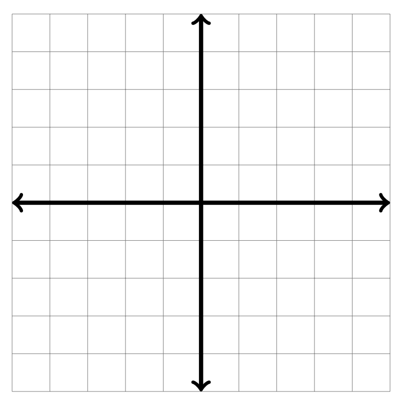
a. $f(x) = 4x + 5$



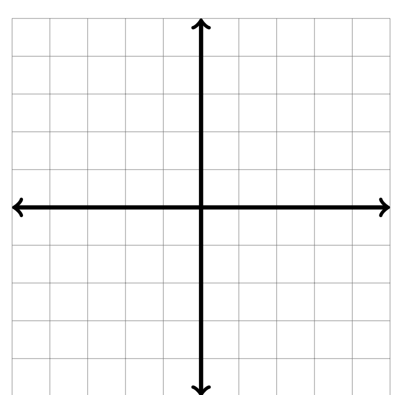
b. $g(x) = (x+5)(x-4)$



c. $h(x) = x^3 + 4$



d. $f_2(x) = 2^x$



Question 6.23 Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined by $f_2(x) = 2^x$.

a. For what value of x does $f_2(x) = 4$?

b. For what value of x does $f_2(x) = 16$?

c. For what value of x does $f_2(x) = 128$?

d. For what value of x does $f_2(x) = \frac{1}{2}$?

e. For what value of x does $f_2(x) = \frac{1}{4}$?

f. For what value of x does $f_2(x) = -5$?

With this knowledge fill out the following table:

x	$f_2^{-1}(x)$
4	
16	
128	
$\frac{1}{2}$	
$\frac{1}{4}$	
-5	
1	
2	
8	

Question 6.24 Evaluate the following:

a. $f_4^{-1}(16)$

b. $f_3^{-1}(81)$

c. $f_5^{-1}(125)$

d. $f_{\frac{1}{2}}^{-1}(4)$

e. $f_{\frac{2}{3}}^{-1}(\frac{9}{4})$

6.5 Solving Exponential and Logarithmic Equations

Exponential equation is an equation of the form: $y = ab^x$. If you know a, b, and x, it is easy to calculate y, but sometimes you need to find the one of the other three variables is the unknown. Let's consider the three examples below.

Question 6.25 a:

a. You want to know how much someone needs to deposit in an account so that after seven years the amount in the account is \$287.17. The interest rate 2%, compounded annually. Write and solve the equation.

b. Solve $y = ab^x$ for a.

Question 6.26 b:

a. You want to know the yearly decay rate of a chemical that is decaying exponentially. At time 0, there was 300 grams of the substance. 10 years later there was 221 grams left. Write and solve the equation.

b. Solve $y = ab^x$ for b.

Question 6.27 x:

a. You want to know how long it will take for a bacteria population to triple, if the hourly growth rate is 160%.

b. Solve $y = ab^x$ for x.

Question 6.28 Solve the following equations:

a. $2^x = 16$

b. $5^x = 125$

c. $3 \cdot 2^x = 24$

d. $2 \cdot 5^{x-2} + 1 = 51$

Question 6.29 Solve the following equations:

a. $\log_3 27 = x$

b. $\log_4 x = -2$

c. $2 \log_3 x = 4$

d. $3 \log_4 x + 1 = 7$

6.6 Applications

Question 6.30 In 1975, the population of the world was about 4.01 billion and was growing at a rate of about 2% per year. People used these facts to project what the population would be in the future.

a. Complete the following table, giving projections of the world's population from 1976 to 1980, assuming that the growth rate remained at 2% per year.

Year	Calculation	Projection (billions)
1976	$4.01 + (.02)4.01$	4.09

b. Find the ratio of the projected population from year to year. Does the ratio increase, decrease, or stay the same?

c. There is a number that can be used to multiply one year's projection to calculate the next. What is that number?

d. Use repeated multiplication to project the world's population in 1990 from the 1975 number, assuming the same growth rate.

e. Compare your result to the previous problem with the actual estimate of the population made in 1990, which was about 5.33 billion.

  (a) Did your projection over-estimate or under-estimate the 1990 population?

  (b) Was the population growth rate between 1975 and 1990 more or less than 2%? Explain.

f. Write an algebraic expression for $f(x)$ which predicts the population of the world x years after 1976.

g. At a growth rate of 2% a year, how long does it take for the world's population to double? We call this *doubling time*.

h. Complete the following table:

x Years passed since 1975	n number of doubling times	f Projection (billions)
1975	0	4.01

i. Give an algebraic expression for the function f as a function of the number of doubling times n.

j. Give an algebraic expression for the function n as a function of years x passed since 1975.

k. Find the compostion $f \circ n$ and explain what it represents in terms of the population projection.

Question 6.31 This is what Wikipedia tells us: Radiocarbon dating (or simply carbon dating) is a radiometric dating technique that uses the decay of carbon-14 to estimate the age of organic materials, such as wood and leather, up to about 58,000 to 62,000 years. Carbon dating was presented to the world by Willard Libby in 1949, for which he was awarded the Nobel Prize in Chemistry. Basically, the way it works is that we know how long carbon-14 takes to decompose to half the initial amount (this is called *half-life*), and by observing how much carbon is in a given sample, we can decide how old the sample is. It is known that carbon-14 has a half-life of 5730 years.

a. What kind of function do you expect will model the decay of carbon 14? Explain what evidence you have for your claim.

b. Write an algebraic expression (rule) for the function that models the decay of carbon-14.

6.7 Homework

6.7.1 Population Growth

See Applications.

6.7.2 Rules for Exponents

Exercise 1 Rewrite the following using one exponent. Justify your answers.

1. a^5a^6
2. $(a^4)^5$
3. $(5^5)^2 5^3$
4. $-5y^2y^3$
5. $3z \cdot z^3$
6. $3t^2 \cdot (-2t^6)$
7. $(vw^3)^5 \cdot v^2$
8. $3y^3 \cdot 2y^2$
9. $(-3t^4)^3$
10. $q^3p^2 \cdot (p^2q^3)^4$
11. $(-5pr)(r^2p^3)^3$
12. $a \cdot a^2 \cdot (-a)^4$
13. $(t^3r^2)^3$

Exercise 2 Rewrite the following using positive exponents. Justify your answers.

1. 5^{-1}
2. 7^{-3}
3. $\frac{1}{4^{-2}}$
4. $\frac{3^4}{3^5}$
5. 8^{-2}
6. $\frac{27m^5n^6}{9mn^3}$
7. $\frac{28x^2y^3}{2xy^2}$
8. $(\frac{3x}{4y})^2$
9. $\frac{5x^2y}{10x^4y^2}$
10. $\frac{7t^2}{tr^3} \cdot \frac{2tr^5}{3r}$
11. $\frac{(-4xy)^3}{8xy^2}$
12. $(\frac{7}{8})^0$
13. $\frac{5}{xy^{-1}}$
14. $\frac{5}{5^{-2}}$

Exercise 3 Rewrite the following expressions using one exponent:

1. $(a^5a^{-3})^3$
2. $(a^{-2}a^3)^2$
3. $(\frac{x}{x^2})^3$

Exercise 4 Looking back at the entire chapter on exponential functions, answer the following essential questions:

a. Why are exponents useful? How are they used in real world applications?

b. What does a negative exponent mean? What does a fractional exponent mean?

c. Where do rules for exponents come from?

Exercise 5 Explain how the exponent $^{-1}$ means different things when we think about a^{-1} and f^{-1}, where a is a real number, and f is a function.

Exercise 6 Give an example to show how the Quotient of Powers rule can explain why $b^0 = 1$.

Exercise 7 Given an example to show how the Quotient of Powers rule can explain why $b^{-2} = \frac{1}{b^2}$.

Exercise 8 The Earth is about $93 \cdot 10^6$ miles from the sun. Light travels at about $1.86 \cdot 10^5$ miles/second. About how long does it take light from the sun to reach the Earth? Show how one or more of the laws of exponents is useful in solving this problem.

Exercise 9 In 1980, the population of the U.S. was about $227 \cdot 10^6$. If the land of the U.S. was about $35 \cdot 10^6$ miles^2, what was the average number of people per square mile of land? Show how one or more of the laws of exponents is useful in solving this problem.

Exercise 10 For which figure is the ratio of the volume to surface area greater: a sphere or a cube? (Volume of a sphere:$\frac{4}{3}\pi r^3$, where r = radius; Surface area of a sphere: πr^2.)

6.7.3 Graphs of Exponential Functions

Exercise 11 What can you say about the shape of the graph of $f(x) = ab^x$ if $a > 0$ and $0 < b < 1$? As b increases in value from 0 toward 1 how does the graph change?

Exercise 12 What can you say about the shape of the graph of $f(x) = ab^x$ if $a > 0$ and $b > 1$? As b increases in value how does the graph change?

Exercise 13 Suppose $b = 1$ in the equation $f(x) = ab^x$, where $a > 0$. What does the graph of this function look like? In what way(s) is this graph fundamentally different from the graphs you sketched in the previous two problems? Answer the same question for when $b = 0$.

Exercise 14 Now suppose $b = -2$ and $a = 1$, so the equation is $f(x) = (-2)^x$.

a. Without using the graphing or table features of your calculator complete the following table. [1]

x	-2	-1	0	1	2	3	4
$f(x)$							

b. Plot the points you found in Part a and connect them with a smooth curve so as to indicate what the graph of $f(x) = (-2)^x$ might look like.

c. Describe how the graph in Part b differs from all the other graphs you've sketched.

d. Explain why we do not use negative numbers for b when we speak of exponential functions $f(x) = ab^x$.

Exercise 15 Determine the value of a and b in $f(x) = ab^x$ if the following facts are known about f:

a. $(0,3)$ and $(2,5)$ are on the graph of the function.

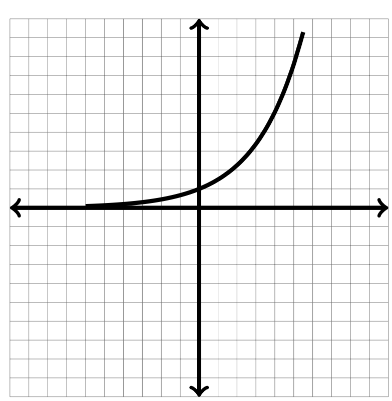
b. $(1,2)$ and $(3,10)$ are on the graph of the function.

c. $(2,3)$ and $(8,1)$ are on the graph of the function.

d. $(-3,8)$ and $(1,1)$ are on the graph of the function.

e. $f(0) = 5$ and $f(3) = 20$.

Exercise 16 The graph of $y = f(x)$ is shown below. Use this graph to quickly sketch a reasonable graph for each of the following functions:



a. $y = f(x) + 2$

b. $y = f(-x)$

c. $y = -f(x)$

d. $y = -f(x) + 2$

[1] Make sure to enclose -2 in parentheses when doing your calculations.

6.7.4 Inverse Functions

Exercise 17 Evaluate each of the following logarithms:

a. $\log_{10} 1000$

b. $\log_{10} \frac{1}{100}$

c. $\ln \frac{1}{e}$

d. $\ln \sqrt{e}$

Exercise 18 Graph the inverse function of $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 5^x$. Make sure that you clearly state the domain and the target of f^{-1}.

Exercise 19 Explain why there is no function given by the rule $f(x) = log_{-3}(x)$.

Exercise 20 A function $g : \mathbb{R} \to \mathbb{R}$ is given by the rule $g(x) = 15 \cdot \left(\frac{1}{2}\right)^x - 3$. Decide whether g has an inverse function. If it does, what is the algebraic expression for the inverse?

Exercise 21 Find the expressions for the inverse functions of the following functions:

a. $a(x) = 3\log_{10} x - 1$

b. $b(x) = \log_{10} x^2 - 2$

c. $c(x) = \frac{1}{4} \cdot 3^{2x-1}$

d. $d(x) = \frac{1500}{2+3^{x-1}}$

6.7.5 Solving Exponential and Logarithmic Equations

Exercise 22 Solve the following exponential equations:

a. $4^{x-1} = 16$

b. $7^{2x+1} = 7^{3x-1}$

c. $3^{2x-1} = 27^x$

d. $5^{3x-8} = 25^{2x}$

Exercise 23 Solve the following exponential equations:

a. $5^x = 9$

b. $3^{2x+1} = 15$

c. $\left(\frac{1}{2}\right)^x = 3$

d. $e^x = 30$

e. $12 \cdot 5^{0.1x} = 30$

f. $3^{x-1} - 4 = 7$

Exercise 24 Solve the following logarithmic equations:

a. $\log_4 x = 2$

b. $\log_4(x+2)x = 2$

c. $3\log_7(x-1) = 6$

d. $\frac{2}{3}\log_4(x-1) + 3 = 6$

6.7.6 Applications

Exercise 25 Disease can spread quickly. Suppose the spread of a direct contact disease in a small town is modeled by the function:

$$P(t) = \frac{10000}{1+2^{3-t}}$$

where $P(t)$ is the total number of people infected after t days.

a. Estimate the initial number of people infected with the disease. Explain how you found your answer.

b. How many people will be infected after 1,2,3,4, and 5 days? (Fill out a table)

c. What is the maximum number of people who can become infected? How do you know?

d. The town officials must inform the citizens when 5000 people become infected. Which day will the town officials make an announcement?

Exercise 26 As soon as you drive a new car off the dealer's lot, the car is worth less than what you paid for it. This is called depreciation. Chances are that you will sell it for less than the price that you paid for it. Some cars depreciate more than others, but most cars do depreciate. On the other hand, some older cars actually increase in value. This is called appreciation. Suppose you have a choice between buying a 1999 Mazda Miata for $19,800 which depreciates at 22% a year, or a 1996 Honda Civic EX for $16,500 which only depreciates at 18% a year. In how many years will their values be the same? Should you instead buy a 1967 Ford Mustang for $4,000 that is appreciating at 10% per year? Which car will have the greatest value in 4 years? In 5 years?

Exercise 27 Suppose that the number of bacteria per square millimeter in a culture in your biology lab is increasing exponentially with time. On Tuesday there are 2000 bacteria per square millimeter. On Thursday, the number has increased to 4500.

a. Derive the particular equation.

b. Predict the number of bacteria per square millimeter that will be in the culture on Tuesday next week.

c. Predict the time when the number of bacteria per square millimeter reaches 10.000.

d. Draw the graph of the function.

Exercise 28 If you invest $1000 at 4% interest and it is compounded continuously, your balance is given by the function $b(t) = 1000(2.718)^{.04t}$ (where t is the number of years you have invested your money).

a. How much money do you have after 1 year?

b. How much money do you have after 4.5 years? What does the fractional exponent mean?

c. How long would you have to wait to have $10000?

6.8 Summary

Much like linear functions were generalizations of arithmetic sequences, the exponential functions are generalizations of geometric sequences. If we are observing equal sized intervals of inputs we will notice that there is a constant ratio of corresponding outputs. We can think of exponential functions as having a constant multiplicative "rate of change".

Definition 2 An exponential function is every function $f : \mathbb{R} \to \mathbb{R}$ of the form

$$f(x) = ab^x$$

where $a \neq 0$, and $b \neq 1$.

You might wonder how it is that the domain of the exponential function defined above is the set of real numbers. Indeed, if we think about exponents only as repeated multiplication, then this function can only be defined on the set of natural numbers. We need to make sense of what it might mean to raise a number to a negative power. Or rational power. Or irrational power. In order to do this, we investigate exponentiation and develop some properties that are obvious if we write out the given exponentiation in terms of multiplication. For this purpose, let us suppose that c is any real number, and that a and b are natural numbers.

$$c^a c^b = (\overbrace{c \cdot c \cdot c \cdot \ldots . \cdot c}^{a \text{ many } c\text{'s}})(\overbrace{c \cdot c \cdot \ldots . \cdot c}^{b \text{ many } c\text{'s}}) = \overbrace{c \cdot c \cdot c \cdot \ldots . \cdot c \cdot c \cdot c \cdot \ldots . \cdot c}^{a+b \text{ many } c\text{'s}} = c^{a+b}$$

In the similar way we obtain the familiar properties of exponents:

$$c^a c^b = c^{a+b}$$
$$(c^a)^b = c^{ab}$$
$$\frac{c^a}{c^b} = c^{a-b}$$

If we consider the last property when $b = a+1$ we'll notice that we have

$$\frac{c^a}{c^{a+1}} = c^{a-(a+1)} = c^{-1}$$

At the same time, we note that the denominator on the left hand side has one more factor of c, so

$$\frac{c^a}{c^{a+1}} = \frac{1}{c}$$

Since if two things are equal to the same thing, then they themselves must be equal, it seems reasonable to define

$$c^{-1} = \frac{1}{c},$$

for every real number c other than 0 (because division by 0 is undefined). We have now extended the domain of our exponential function to the whole set of integers because, for example:

$$f(-4) = ab^{-4} = a\left(b^4\right)^{-1} = a \cdot \frac{1}{b^4}$$

We would like to further allow the exponents to be fractions, so we think about what might happen if we did have a fractional power, say $\frac{1}{3}$. If we use the second property we listed above, then we can conclude that

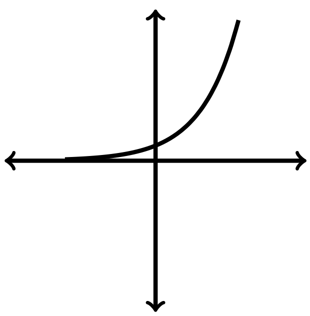
$$\left(b^{\frac{1}{3}}\right)^3 = b^{\frac{1}{3}\cdot 3} = b^1 = b.$$

We see that $b^{\frac{1}{3}}$ is the number which cubed gives us b. We know one such number already: $\sqrt[3]{b}$, and so conclude that these two numbers must be the same! In general then, it makes sense to define:

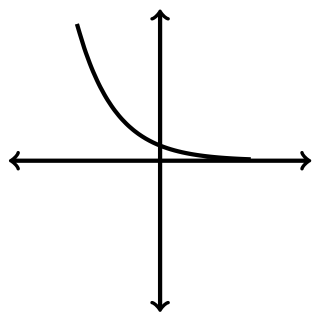
$$b^{\frac{m}{n}} = \sqrt[n]{b^m}.$$

To complete the definition of exponential function on irrational numbers, we'll postpone this story to little later when we have few other tools and understandings at our disposal. We will only say that a exponents thus defined will extend the properties we've had so far.

If we restrict ourselves to positive coefficients a, depending on whether b is smaller or larger than 1, the exponential function is either increasing or decreasing, respectively. (If $a < 0$, then the situation is reversed. You should be able to explain why that is the case.) For a brief period, let's think about the case when $a = 1$. Now, the graph of every exponential function $f(x) = b^x$ will pass through the point $(0, 1)$ and will have x-axis as the horizontal asymptote (a line which the graph will never intersect, but will get closer and closer to). It makes perfect sense that the graphs of exponential functions have no x-intercepts, since no power of b will be 0 or smaller than 0.



$f(x) = b^x, \quad b > 1$



$f(x) = b^x, \quad 0 < b < 1$

By looking at the graphs you should be able to tell that every exponential function has an inverse.

Definition 3 $\log_b : \mathbb{R}_+ \to \mathbb{R}$ is the inverse function of the exponential function with base b.

We can interpret that in following way:

$$\log_b(b^x) = x \qquad \text{and} \qquad b^{\log_b x} = x.$$

Or, we can think about the fact that the inverse function contains pairs whose inputs and outputs are reversed those of the original function, so the question

What is $\log_b a$?

is the same as wondering how to fill out the following table:

x	$\log_b(x)$
a	

But, we know how to do this!

It is the same as filling out the following table:

x	b^x
	a

Which the answer to the question:

To which power must we raise b to get a?

Or, in symbols:

$$\text{Saying} \quad \log_b a = c \quad \text{is the same as saying} \quad b^c = a.$$

6.9 Student learning outcomes

1. Students will be able to recognize exponential functions from graphs, equations, tables and verbal descriptions.
2. Students will understand that the exponential function increases or decreases faster than any linear (or polynomial) function.
3. Students will be able to explain how the rules of exponents arise and apply them to simplify various expressions.
4. Students will be able to use a table and graph to calculate the inverse of exponential functions.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.