


A Workbook for Math1010

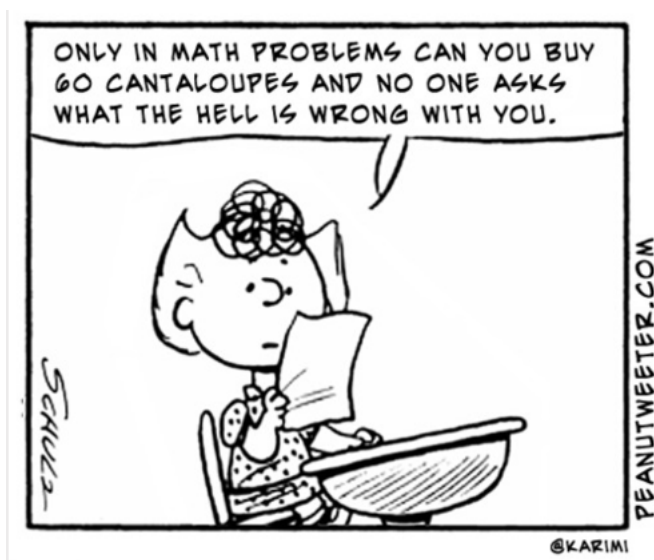
Intermediate Algebra

The material in this workbook has been written and compiled by several instructors of the course.

We are grateful to the *Urban School of San Francisco* and *The Park School of Baltimore* for their willingness to share their work. 

1 Introduction

All of you have different reasons for being in this class. Most of you are here because either you or someone else decided that you need to refresh or acquire some skills that are needed in a course that you may need either for your major or graduation. Ordinarily, that is how the class was treated: an opportunity to remind you of skills and procedures that you have learned and forgotten, or never quite learned. We have different goals for this class. While it's important to develop certain skills, it is more important to know what to do with those skills. In all of the classes that will follow, and not just mathematics ones, as well as in all the interactions with the world around you, it's important to have an ability and skill to think through a problem you encounter, make a plan for solving the problem, execute it and then look back, think through your answer and decide on its reasonableness. It is also important to realize the power mathematics holds in both dealing with the world around us, but also in its own right. We need to get away from the view that mathematics looks like this:



In this section you will find some big ideas that we think are important to keep in mind as you're working through this course. You will also find the learning outcomes that can serve you as a guide to what you should be learning. Each section will contain a list of essential questions you should be able to answer at the end of the unit.

1.1 Big Ideas

1. We can talk about many different instances of a situation at the same time: variability can be described and used productively.
2. The number systems developed from our need to solve various problems. We choose to extend the existing number systems so that the properties of operations are retained.
3. Problems come from various areas and not all of them can be solved, but much can be learned from attempts at solution, successful or not.

1.2 Learning Outcomes

1. Students are willing to engage with problems which are unfamiliar to them and to which the solutions or paths to solutions are not immediately obvious.
2. Students can extract relationships between quantities and describe them in different ways: tables, expressions, graphs, words, and can translate between these representations in order to answer questions most efficiently.
3. Students can answer questions about quantities given relationships between two or more by solving equations, whether it be algebraically, using tables, graphs or approximating.
4. Students understand how different growth patterns influence shape of the graph.
5. Students can recognize linear, exponential and polynomial from verbal descriptions, tables, and graphs.

1.3 Warming up for the semester

Question 1.1 (Brazil) Two mothers and two daughters sleep in the same room. There are only three beds and exactly one person sleeps on each of them, yet all people are accounted for. How is this possible?

Question 1.2 (Ireland) One day three brothers were going past a graveyard. One of them said, “I shall go in so that I may say a prayer for the soul of my brother’s son.” The second man said the same thing. The third brother said, “I shall not go in. My brother’s son is not there.” Who is buried in the graveyard?

Question 1.3 (Puerto Rico) Who is the sister of my aunt, who is not my aunt, but is the daughter of my grandparents?

Question 1.4 (Russia) An old man was walking with a boy. The boy was asked, “How is the old man related to you?” The boy replied, “His mother is my mother’s mother-in-law. What relation is that?”

Question 1.5 On your calculator:

- Put in first 3 digits of your phone number
- Multiply by 80
- Add 1
- Multiply by 250
- Add the last four digits of your phone number
- Add the last four digits of your phone number
- Subtract 250
- Divide by 2
- What did you get?

Was that surprising? Try to explain why that happened.

Question 1.6 A pot and a lid cost \$11 (this was once upon a time). The pot costs \$10 more than the lid. How much does each item cost individually?

Question 1.7 What do these questions have to do with mathematics? What do they have to do with algebra? Describe the process you used to solve these questions.

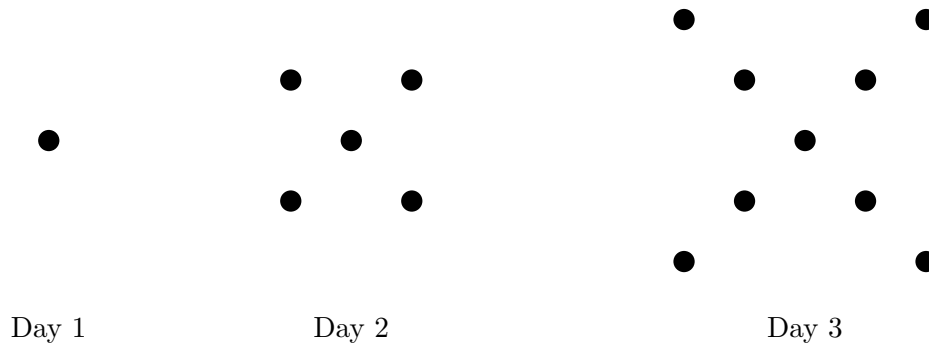
2 Sequences

Essential questions

1. How do we describe a pattern?
2. How can patterns be used to make predictions?
3. What are some ways to represent, describe, and analyze patterns?

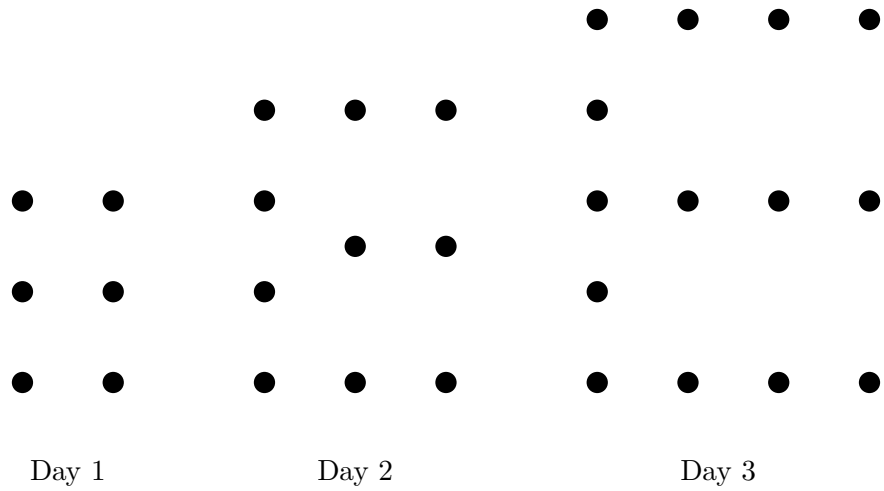
2.1 Visual Patterns

Question 2.1 Look at the pattern below and answer the questions:



- a. Describe the pattern that you see in the sequence of figures above.
- b. Assuming the sequence continues in the same way, how many dots are there on the fourth day? On the fifth day? On the tenth day?
- c. How many dots are there on the 100^{th} day?

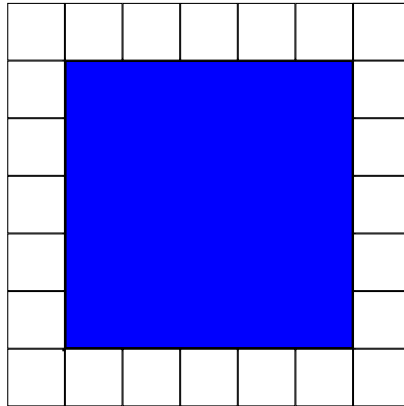
Question 2.2 Look at the pattern below and answer the questions:



- Describe the pattern that you see in the sequence of figures above.
- Assuming the sequence continues in the same way, how many dots are there on the fourth day? On the fifth day? On the tenth day?
- How many dots are there on the 100th day?

2.2 Tiling a Pool

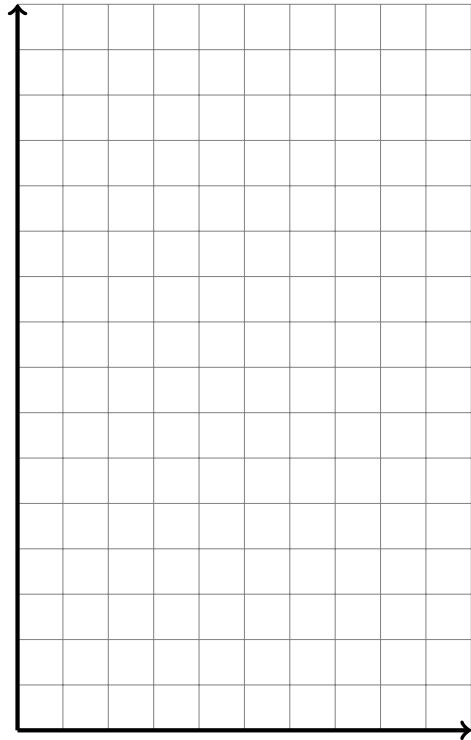
Question 2.3 The summer season is nearly over and the owner of the local pool club is thinking of what all needs to be done once the pool closes. One of the common things in need of repair are the tiles around the perimeter of the pool. In the picture below a 5 foot square pool has been tiled with 24 square tiles (1 foot by 1 foot).



- a. Make sketches to help you figure out how many tiles are needed for the borders of square pools with sides of length 1, 2, 3, 4, 6, 10 feet without counting. Record your results in a table.
- b. Write an equation for the number of tiles N needed to form a border for a square pool with sides of length s feet. How do you see this equation in the table? How do you see the equation in your pictures?
- c. Try to write at least one more equation for N . How would you convince someone that your expressions for the number of tiles are equivalent?

d. Use your work to decide how many tiles you would need for a square pool whose sides are 127 feet long. What about a square pool whose sides are 128 feet long?

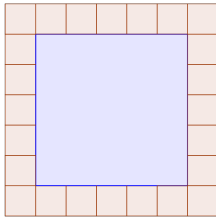
e. Graph the relationship you observed between s (the side length) and N (the number of tiles needed).



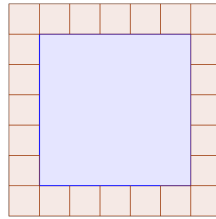
s side length	N number of tiles
1	24
2	
3	
4	
5	
6	
10	

f. Relate the growth pattern in each of the representations of the pattern (table, equation, graph).

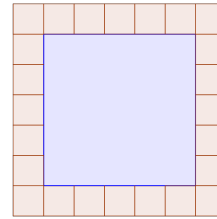
Question 2.4 Some students calculated the number of tiles differently. Show their solution in the diagram and explain their thinking.



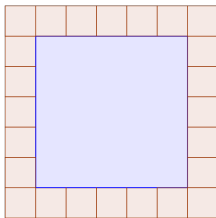
$$4(s+1)$$



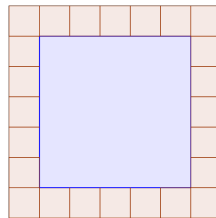
$$4s+4$$



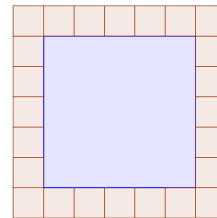
$$2s+2(s+2)$$



$$4(s+2)-4$$



$$4(s+2 \cdot \frac{1}{2})$$



$$(s+2)^2 - s^2$$

How can you convince someone that all of the expressions are equivalent? Use both properties of operations on whole numbers as well as the diagrams.

Definition 1 An infinite list of numbers is called a **sequence**. Sequences are written in the form

$$a_1, a_2, a_3, \dots$$

a_n is called the n^{th} **term of the sequence**.

The third term in the sequence listed above is a_3 . The "3" refers to the position of the member within the sequence and a_3 refers to the number that is in that position.

For example, if we look at the sequence $\{b_n\}$ whose members are listed below:

$$17, 13, 9, 5, 1, -3, -7, \dots$$

We can tell that $b_1 = 17$, $b_6 = -3$. What is b_4 ? What is the 10th term of this sequence?¹

Question 2.5 If p_1, p_2, p_3, \dots is a sequence such that

$$p_n = \# \text{tiles around a square pool of side length } n,$$

- What is the value of p_5 ?
- What is the value of p_{15} ?
- What is the value of p_n ?
- What is the relationship between p_n and p_{n+1} .

Definition 2 A sequence a_1, a_2, a_3, \dots is an **arithmetic sequence** if there is a number d such that you obtain any member of the sequence by adding d to the member that came before it. Symbolically, we'd write that:

$$a_n = a_{n-1} + d.$$

Question 2.6 Is the sequence p_1, p_2, p_3, \dots (from Question 2.5) an arithmetic sequence?

¹You would write that: $b_{10} = \quad$.

2.3 Geometric Sequences

Question 2.7 Social media has created a way to quickly share information (articles, videos, jokes, ...). Gangnam Style is a YouTube video that became popular in July 2012. On September 6th, the video had 100,000,000 views. On December 21st the video was the first video in history to have over 1,000,000,000 views. If Gangnam style was released on July 15, how many days did it take to for the video to hit 100,000,000 views? How many days did it take for the video to breach 1,000,000,000 views?

Question 2.8 To model the sensation of "viral videos", assume that on day one there was one view, that every new view corresponds to a new person seeing the video and on average a new viewer shows the video to 2 new people.

- a. How many times was the video viewed on day 2?
- b. How many times was the video viewed on day 3?
- c. How many times was the video viewed on day 5?
- d. How many times was the video viewed on day n ?
- e. Let v_1, v_2, v_3, \dots be a sequence such that

$$v_n = (\text{the number of times the video is viewed on the } n^{\text{th}} \text{ day}).$$

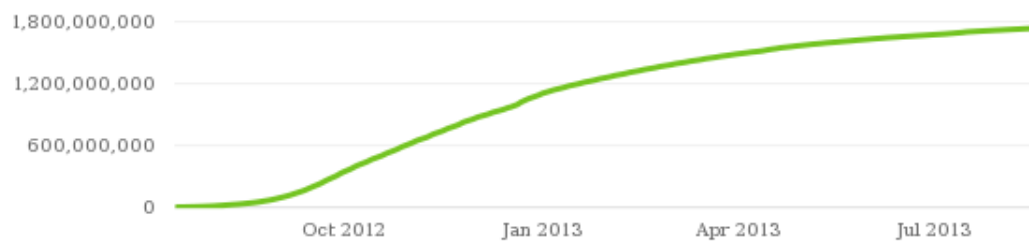
Write down an algebraic relationship between v_n and v_{n+1} .

- f. How many times is the video viewed in the first two days?
- g. How many times is the video viewed in the first three days?
- h. How many times is the video viewed in the first five days?
- i. How many times is the video viewed in the first n days?
- j. Let t_1, t_2, t_3, \dots be a sequence such that

$$t_n = (\text{total number of views from day 1 to day } n).$$

Write down an algebraic relationship between t_n and t_{n+1} .

Question 2.9 The graph below is data from YouTube about the actual number of views of Gangnam Style. Does our model accurately describe the behavior of the viral video phenomena? What do you think some limitations of our model are?



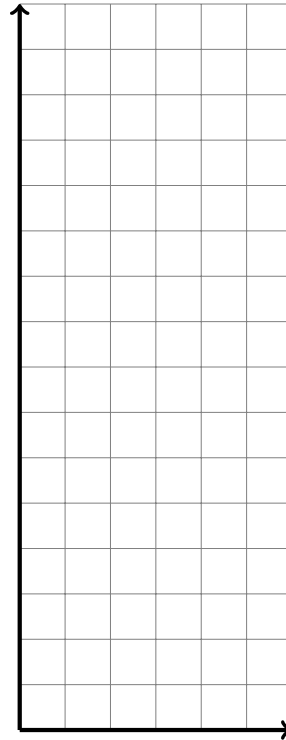
Question 2.10 A ball is dropped from a height of 10 feet. The ball bounces to 80% of its previous height with each bounce.

- a. How high does the ball bounce after the first bounce? e. Record (n, b_n) in a table and graph the relationship between n and b_n .

- b. How high does the ball bounce after the third bounce?

- c. How high does the ball bounce after the n^{th} bounce?

- d. Let b_1, b_2, b_3, \dots be a sequence where b_n is the height the ball bounces after the n^{th} bounce. What is the relationship between b_n and b_{n+1} ?



n	b_n
1	8
2	
3	
4	
5	
6	
10	

- f. The sequence b_n models the height of a ball bouncing. How many times does the model predict the ball will bounce? Is this realistic?

Question 2.11 Assume you invest \$1,000 in a savings account that pays 5% a year.

a. How much money will you have after one year?

b. How much money will you have after two years?

c. How much money will you have after 50 years?

d. How much money will you have after n years?

n	m_n
1	
2	
3	
4	
5	
6	
10	

e. Let m_1, m_2, m_3, \dots be the sequence such that m_n = dollars in account after n years. What is the relationship between m_n and m_{n+1} ?

Question 2.12 The height of a ball bouncing, the number of viral video daily views, and the amount of money in the bank account are all examples geometric sequences. Describe similarities and differences among these three examples.

2.4 Counting High-Fives

After a sporting event, the opposing teams often line up and exchange high-fives. Afterward, members of the same team exchange high-fives. In this problem, you will explore the total number of high-fives that take place at the end of a game.

- Every player exchanges exactly one high-five with every other player.
- When two players exchange a high-five, it counts as one exchange, not two.

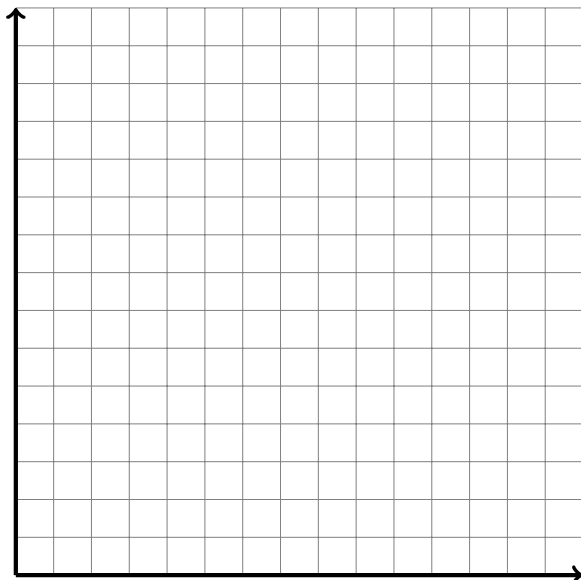
Question 2.13 If everyone in this room exchanged high-fives, guess how many high-fives there would be.

Question 2.14 Let n be the *combined* number of players on each of the two teams. Let H_n be the number of high-fives that are exchanged at the end of the game. Complete the following table.

n players	H_n high-fives
1	
2	
3	
4	
5	
6	
7	

Question 2.15 Is the sequence H_1, H_2, H_3, \dots an arithmetic sequence? Is the sequence geometric?

Question 2.16 Sketch the graph of the high-five sequence, then describe what you see, comparing and contrasting it to the patio sequence from Question 2.3.



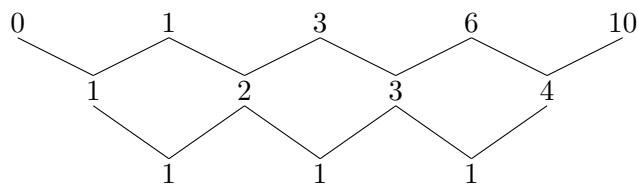
Question 2.17 What is the relationship between H_n and H_{n-1} .²

Question 2.18 Find an explicit formula for the sequence H_n .

Question 2.19 The defining quality of an arithmetic sequence is the constant difference between consecutive terms in the sequence.

- a. Is there a constant difference between terms in the high fives sequence?

The following picture describes how to find the second difference of a sequence:



- b. Add 3 terms to the top row and complete the picture.
- c. What is the pattern?

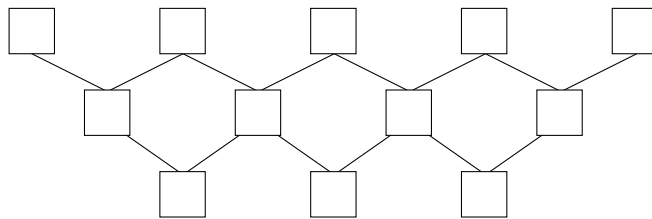
²Think about the context of the problem. If the 4th player enters the scene, how many players are there with whom she has to exchange high fives? What about when the 5th player enters?

Question 2.20 Let $\{t_1, t_2, \dots, t_n, \dots\}$ be a sequence where the n -th term is given by $t_n = 3n^2 + n + 4$.

- a. Fill in the following table:

n	t_n
1	
2	
3	
4	
5	
6	
7	

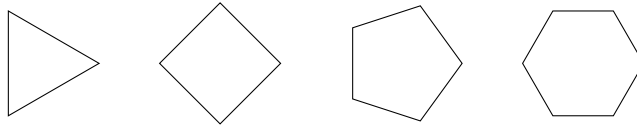
- b. Calculate a few second differences of this sequence. To help you organize your thoughts, use the following diagram:



Question 2.21 Make up your own sequence which has a constant second difference.

Question 2.22 Make up your own sequence which has a constant third difference.

Question 2.23 A polygon is a closed shape consisting of line segments which pairwise share a common point. Below are drawn 3-sided, 4-sided, 5-sided and 6-sided polygons which you may know under different names.



A diagonal of a polygon is a line segment which connects non-adjacent vertices of the polygon. Draw all diagonals for each polygon pictured above. Let's consider the sequence $\{d_n\}$ where d_n is the number of diagonals of a polygon with n sides.

a. Fill in the following table and sketch a graph.

n sides	d_n diagonals
1	
2	
3	
4	
5	
6	
7	



b. Can you come up with a recursive formula for the sequence?

c. Can you come up with an explicit formula for the sequence? ³

³Once again, it's useful to think about the context of the problem.

2.5 Summary

Definition 3 An infinite list of numbers is called a **sequence**. Sequences are written in the form

$$a_1, a_2, a_3, \dots$$

a_n is called the n^{th} term of the sequence.

Generally people think of sequences as having a nice pattern that one can describe either using words or mathematical expressions. For example, $a_n =$ "the number of rainy days in each month starting with January of 3017" describes a sequence (this is of course assuming that people or a natural disaster do not swipe the Earth away). $b_n = 3n - 4$, where n is a whole number, also describes a sequence. The first sequence, however, does not have a nice symbolic description. If you wrote down the sequence by listing its terms, you wouldn't notice any patterns that you can describe by a neat little formula. Many sequences are just lists of random numbers. Other sequences can neatly be described in mathematical, symbolic, language.

Definition 4 A sequence a_1, a_2, a_3, \dots is an **arithmetic sequence** if there is a number d such that

$$a_n = a_{n-1} + d.$$

We can say that this is the sequence where each term is obtained from the previous one by adding a constant number, d . We call this a *recursive* definition of a sequence. The n -th term of an arithmetic sequence can be described *explicitly* as well:

$$a_n = a_1 + (n - 1)d.$$

Here you see that, in order to describe the sequence completely it is necessary to give its first term a_1 as well as the common difference d .

Some people like to start their sequences with 0^{th} term of the sequence: a_0, a_1, a_2, \dots . In that case the explicit formula for the arithmetic sequence seems less complicated:

$$a_n = a_0 + nd.$$

Another example of a special type of a sequence is the one where each element is obtained from the previous one by multiplying it by a constant number, r .

Definition 5 A sequence a_1, a_2, a_3, \dots is a **geometric sequence** if there is a number r such that

$$a_n = r \cdot a_{n-1}$$

The explicit formula for the n -th term of a geometric sequence is given by

$$a_n = a_1 \cdot r^{n-1}.$$

Here, too, we might get somewhat simpler expression if we start with 0^{th} term instead of the 1^{st} :

$$a_n = a_0 \cdot r^n.$$

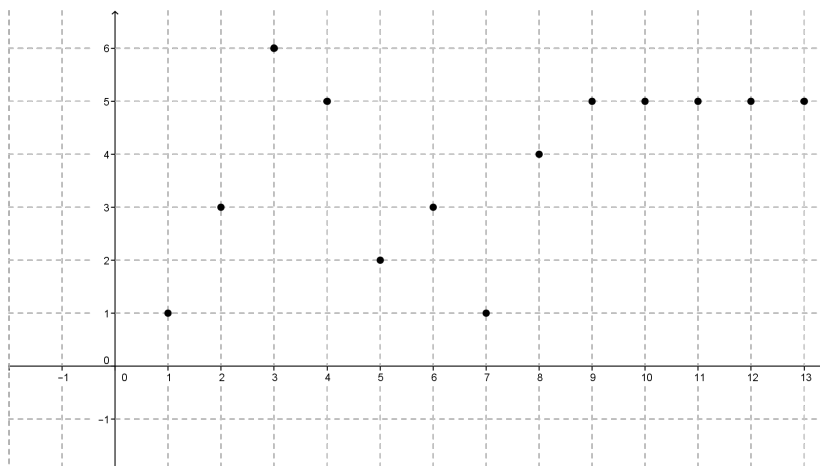
Apart from describing a sequence by giving either recursive or explicit formula that tells us what a general term looks like, we note that we can also simply use a list $\{a_1, a_2, a_3, \dots\}$. For example: $\{1, 3, 6, 5, 2, 3, 1, 4, 5, 5, \dots\}$.

Sequences can be organized in tables:

n	a_n
1	a_1
2	a_2
3	a_3
4	a_4

n	1	2	3	4	5	6	7	8	9	10	...
a_n	1	3	6	5	2	3	1	4	5	5	...

We can graphically represent a sequence by graphing ordered pairs (n, a_n) where the position in the sequence is represented on the x -axis, and the corresponding term of the sequence is represented on the y -axis. For example:



2.6 Student learning outcomes

1. Students will be able to identify arithmetic and geometric sequences.
2. Students will be able to use algebraic expressions, graphs, tables and verbal cues to identify and work with sequences.
3. Students will be able to compute the n -th term in a geometric/arithmetic sequence.
4. Students will be willing to engage and work with a pattern that they are unfamiliar with.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.

3 Functions

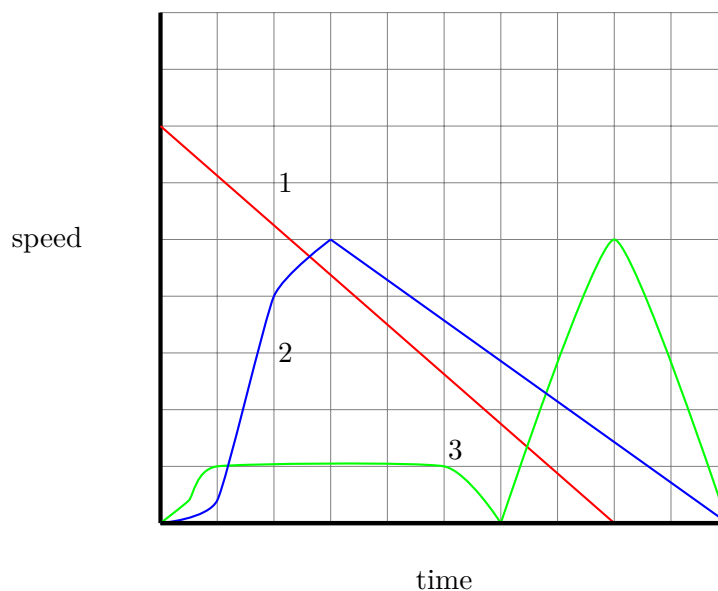
Essential Questions

1. How do we represent relationships?
2. How do you recognize if a relationship is a function?
3. How do we express a function?
4. How is a function related to its graph?
5. In what ways can functions be combined?
6. Given a discrete set of data how could we use this to describe a function?

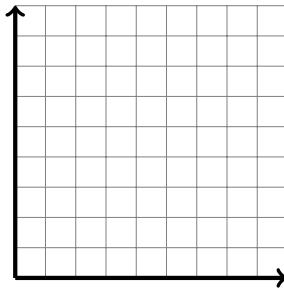
3.1 Describing relationships

You've all taken photographs, and you know that those accurately represent some occurrence: a child sledding, a car driving by, a marble rolling on the floor. Those photographs depict a situation, but don't tell us a whole story: how fast was the child going, was the car stopped or moving, was the marble gaining speed or rolling to a stop. In order to describe certain characteristics of an object or a relationship between two quantities, the graphical representation is more telling. In the next few problems, we will try to decide what the graphs tell us about relationships.

Question 3.1 You remember those good ol' days when you sledded with your best buddies and pushed them off the sled? That was not a nice thing to do! Well, remember one of those times, and imagine yourself climbing that awesome hill, then sledding down. Which of the graphs depicted below best represents your trip up and down the hill?

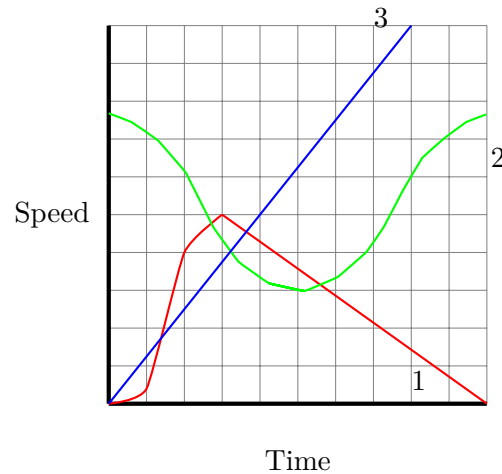


- a. Write a couple of sentences explaining why the graph you chose describes the situation above.
- b. Write a couple of sentences explaining what the sledding experience would have been like if the other two graphs were the correct graphs.
- c. Put a scale on your graph. On the horizontal axis place 0-10 minutes, and on the vertical axis place 0-5 miles per hour.
- How fast were you moving after 2 minutes? 5 minutes? After 7?
 - What was the time when you moving 3 miles per hour? 5 miles per hour? 1 mile per hour?
 - On the new set of coordinate axes and graph speed on the x -axis and time on the y -axis. Draw the same relationship on this new coordinate graph.



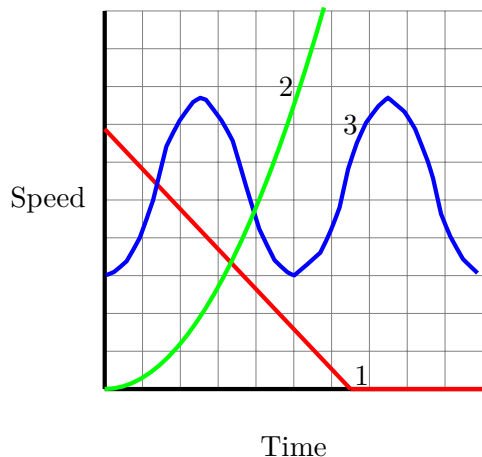
- How are the two graphs related? What kind of information does the new graph convey?

Question 3.2 Let's say you are sitting outside and a car passes by. It slows down as it passes you and then speeds up. Which of the graphs depicted here best represents the speed of the car?



- Write couple of sentences explaining why the graph you chose describes the situation above.
- Write couple of sentences explaining what situation you would observe if the car's speed had been described by the other graphs.
- Put a scale on your graph. On the horizontal axis place 0-60 seconds, and on the vertical place 0-35 miles per hour.
 - How fast was the car moving after 30 seconds? After 45 seconds ? After 60 seconds?
 - At what time was the car going 20 miles per hour? At what time was the car going 0 miles per hour?
 - Place any point at all on the graph and describe the moment that corresponds to that point.

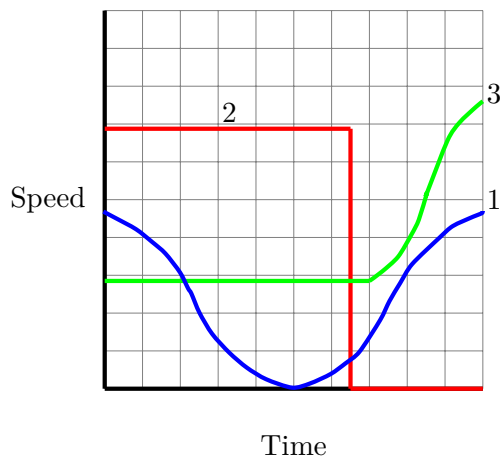
Question 3.3 Imagine you roll a marble on the floor and you watch it roll to a stop (yes, you have an exciting life). Which of the graphs depicted below best represents the speed of the marble?



Put a scale on your graph. On the horizontal axis place 0-2 minutes, and on the vertical axis place 0-2 meters per second.

- How fast was the marble moving after 30 seconds? After 1 minute 30 seconds? After 1 minute 45 seconds?
- What was the time when the marble was moving 1 meter per second? At what time was the marble stopped?

Question 3.4 You're watching Formula 1 and the lead car slams into the wall at full speed. Which graph best depicts this sad story?



How fast was the car moving after 30 seconds? After 60 seconds? After 90 seconds? Put a scale on your graph. On the horizontal axis place 0-100 seconds, and on the vertical axis place 0-120 meters per second.

- How fast was the car moving after 30 seconds? After 60 seconds? After 90 seconds?
- What was the time when the car was moving 90 meter per second? 60 m/s? 45 m/s? 30 m/s? 0 m/s?

3.2 Relations and Functions

The previous sections have described many relationships in different ways. We used words, graphs, tables, and expressions. What was common in all of those examples was that we knew which objects were related to each other and how. In other words, we knew how the objects were paired up. Let's look at some examples, old and new.

Example 3.1 In our pool problem we had the filled out the following table:

n	T_n
1	8
2	12
3	16
\vdots	\vdots
n	$4 + 4 \cdot n$
\vdots	\vdots

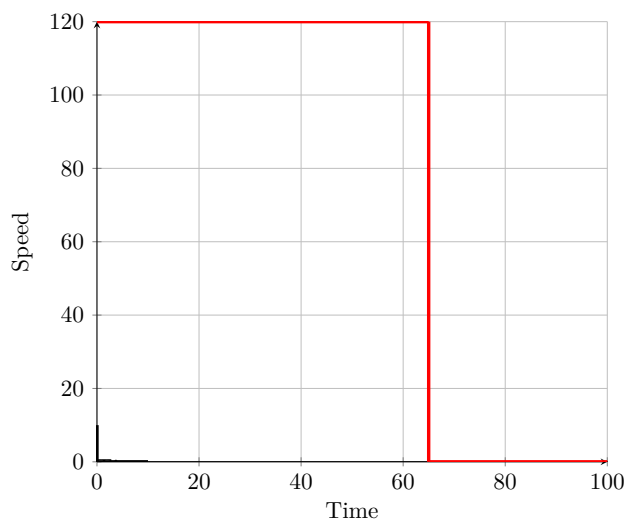
The objects were counting numbers: $\{1, 2, 3, 4, \dots\}$, and they were paired up: to each counting number (which represented the side length of a pool) we associated another counting number: the number of tiles needed to tile the pool with that side length.

For example, for a pool of side length 3 feet we needed 16 tiles. So, we put the numbers 3 and 16 together in a “pair”, $(3, 16)$. The pair is “ordered” because we first thought about the length of the pool side length, and then proceeded to figure out the number of tiles needed; as a result, we chose to write the “3” as the first coordinate in our ordered pair, and the “16” as the second coordinate: $(3, 16)$. In this way, our Pool Problem table is really the set of ordered pairs

$$\{(1, 8), (2, 12), (3, 16), (4, 20), (5, 24), (6, 28), \dots, (n, 4 + 4 \cdot n), \dots\}.$$

Secretly, thinking about this set of ordered pairs is probably how most of us created our graphs of this sequence: We typically think of the first coordinate as the horizontal axis coordinate, and the second coordinate as the vertical axis coordinate.

Example 3.2 In the Formula 1 example the red graph (graph 2) showed how speed depended on time:



Here we associated numbers between 0 and 100, they represented the seconds during which the car was moving, with numbers between 0 and 120, which represented the speed in meters per second. The pairs we formed according to the graph above.

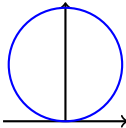
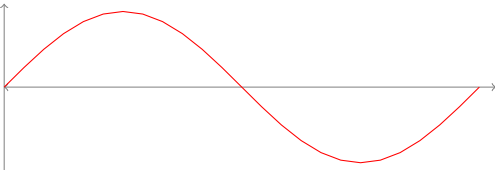
1. List at least three pairs from this relation:
2. Can you list all the pairs for this relation? Explain.

Definition 1 A **relation** is a collection of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target

We often call the first coordinates **inputs** and the second coordinates **outputs**.

Question 3.5 The following table gives several examples of relations, but they have been grouped into two categories. The ones in Group 1 share features that ones in Group 2 do not have. Study the two groups, and decide what the reason for such grouping may have been.¹

Group 1	Group 2								
a. $D = \{1, 2, 3, \dots\}$, $T = \{1, 2, 3, \dots\}$, $\{(1, 2), (2, 3), (3, 4), (4, 5), \dots\}$	A. $D =$ all the parents, $T =$ all the people, pairs (p, c) are formed whenever p is a parent of c .								
b. $D =$ all the people, $T =$ all the women, pairs (p, w) are formed whenever w is the biological mother of p .	B. $D =$ all the numbers between -1 and 1 , $T =$ all the numbers between 0 and 2 , pairs (x, y) whenever they lie on the blue circle: 								
c. $D = \{1, 2, 3\}$, $T = \{1, 2, 3\}$, $\{(1, 2), (2, 3), (3, 3)\}$	C. $D = \{a, b, c, d\}$, $T = \{1, 2, 3\}$ <table border="1" data-bbox="764 1241 898 1388"> <tr> <th>in</th><th>out</th></tr> <tr> <td>a</td><td>1</td></tr> <tr> <td>b</td><td>3</td></tr> <tr> <td>d</td><td>2</td></tr> </table>	in	out	a	1	b	3	d	2
in	out								
a	1								
b	3								
d	2								
d. $D =$ all the numbers between 0 and 2π , $T =$ all the numbers between -1 and 1 , pairs (x, y) whenever they lie on the red curve: 	D. $D = \{1, 2, 3\}$, $T = \{1, 2, 3\}$, $\{(1, 2), (1, 3), (2, 3), (3, 2)\}$								

¹It may help to graph some of these relations, or to give examples of pairs that belong to them.

Question 3.6 Based on your reasons, into which group would you place these relations:

1. The Formula 1 example from above.
2. $D = \{1, 2, 3, 4, \dots, 26645\}$, $T = \{0, 1, 2, 3, \dots, 500\}$, an input is some number d from D which represents the d^{th} day in the life of Joe, and the corresponding output is the number of text messages Joe sent on day d .

Explain why you categorized them this way:

The relations in the first group are called functions. Informally, we can say: A function is a set of ordered pairs, such that each input is paired with exactly one output.

Question 3.7 Come up with two examples of relations one of which is a function and one which is not.

1. Function:

2. Non-function:

More formally, we have this definition:

Definition 2 A function is a set of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target
- every element in the domain is paired with exactly one element of the target.

We say that we have a function $f: D \rightarrow T$, and if an ordered pair (a, b) is in our function, then we say that $f(a) = b$.

You may be more familiar with the words "input" and "output". If the ordered pair (a, b) belongs to our function f , then, in addition to saying that $b = f(a)$ (reads: b is f of a), we say that a is the input, and b is the output for a . a comes from the set D , domain, and b comes from the set T , target. We also often write $a \mapsto b$.

3.3 Ways to represent a function

Question 3.8 For each description below make a table of at least five (x, y) pairs that fit the description. Then write down the algebraic equation that describes the relationship.

- a. The y -coordinate is always equal to the x -coordinate.

x					
y					

- b. The y -coordinate is always four less than the x -coordinate.

x					
y					

- c. The y -coordinate is always opposite of the x -coordinate.

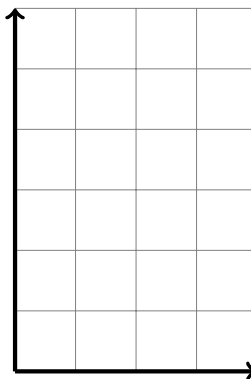
x					
y					

- d. The y -coordinate is always the square of the x -coordinate.

x					
y					

Question 3.9 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function. Use the table below to guess which function f is.

x	$f(x)$
1	3
2	4
3	5
4	6

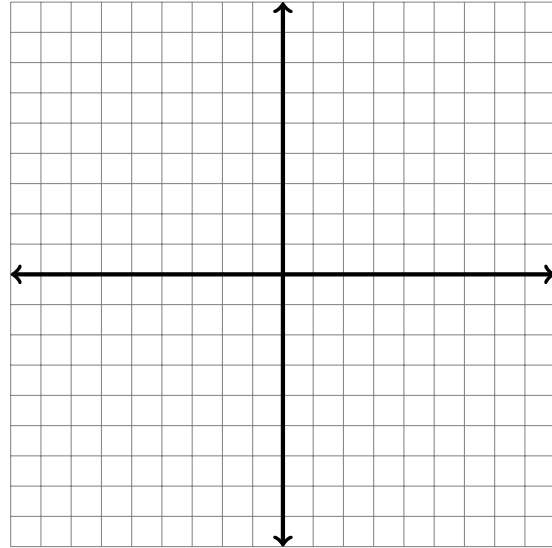


- Graph the points listed in the table.
- Draw in a possible graph for a function f such that the points in the table are included in the graph.
- How many different choices for f are there in Question b.?
- If this were data collected in a laboratory, which function would you choose for f ?

Question 3.10 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = 2x + 1$.

- Construct a table for $f : \mathbb{R} \rightarrow \mathbb{R}$.
- Plot the entries from your table on a graph.

x	$f(x)$

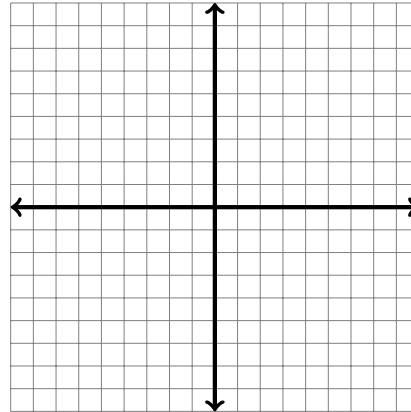


- Evaluate $f(-5)$, and $f(a)$, where a represents any number.
- Is there an input for which the output is 12? -13 ?
- Use your graph to estimate $f(-3)$. Explain how this is done.
- Use your graph to estimate the input whose output is $\frac{9}{2}$. Explain how this is done.

Question 3.11 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = x^2 + 1$.

- a. Construct a table and graph $f : \mathbb{R} \rightarrow \mathbb{R}$.

x	$f(x)$

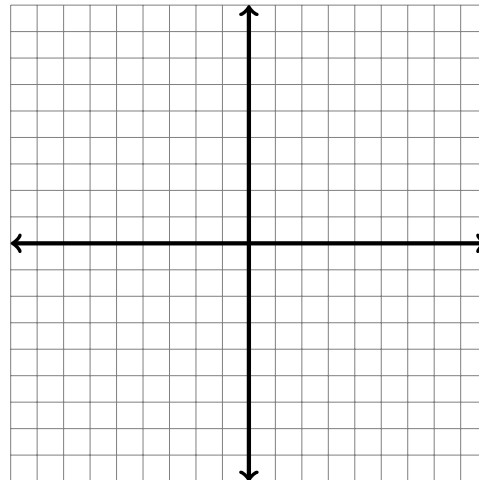


- b. Evaluate $f(-2)$, and $f(a-1)$, where a represents any number.
- c. Is there an input for which the output is 37? -10 ?

Question 3.12 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = -3x + 1$.

- a. Construct a table and graph $f : \mathbb{R} \rightarrow \mathbb{R}$.

x	$f(x)$



- b. Evaluate $f(-1)$, and $f(b+1)$, where b represents any number.
- c. Is there an input for which the output is 25? -7 ?

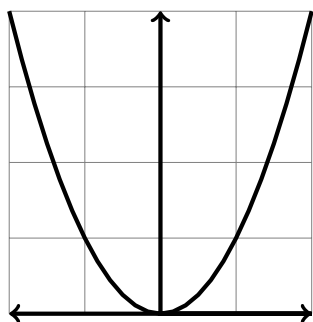
Question 3.13 Below is the graph of a function $g : \mathbb{R} \rightarrow \mathbb{R}$.

a What is $g(0)$?

b What is $g(1)$?

c For what values of x does $g(x) = 4$?

d Fill in the table.



x	$g(x)$

Question 3.14 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$. Is the point $(3, 11)$ on the graph of f ? How do you know?

Question 3.15 Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a function. How many times could a vertical line intersect the graph of h ? Explain.

3.4 Combining Functions

Question 3.16 You are in the market to buy a new TV. After carefully looking at consumer reports you choose the TV you want to buy:

Sony - 50" Class (49-1/2" Diag.) - LED - 1080p - 60Hz - HDTV

Luckily for you, you also just got a \$100 coupon for this model from BestBuy. When you walk in the store you receive a special promotion for 10% off. We will consider four different scenarios:

1. You are allowed to take one discount only.
2. You are allowed to take both discounts off the original price.
3. You will use the \$100 off coupon and then the 10% off coupon.
4. You will use the 10% off coupon and then the \$100 off coupon.

Part 1: The current price tag on the TV is \$599. For each of the scenarios above calculate the sale price of the TV as well as the amount you saved.

Scenario 1

Price

Discount

Scenario 2

Price

Discount

Scenario 3

Price

Discount

Scenario 4

Price

Discount

Part 2: How will the situation change as we vary the price of the TV? Let the cost of the TV be represented by the variable x . In each case articulate a reasonable domain and target for the function that represents the cost of the TV after the discount(s) have been applied as well as their real world meaning.

Scenario 1:

$f(x)$ is the cost of TV with \$100 off.

$g(x)$ is the cost of TV with 10% off.

Scenario 2:

$p(x)$ is the cost of TV with both discounts off the original price

Scenario 3:

$s(x)$ is the cost of TV with \$100 then 10% off

Scenario 4:

$r(x)$ is the cost of TV with 10% then \$100 off

We saw that we can combine functions using same operations we used for numbers. There is, however, a new way of combining functions that we have not had available to us until now: composition. Remember how Scenario 3 and 4 differed:

$$x \xrightarrow{\text{"-100"}} x - 100 \xrightarrow{\text{"\cdot 0.9"}} 0.9(x - 100)$$

$$x \xrightarrow{\text{"\cdot 0.9"}} 0.9x \xrightarrow{\text{"-100"}} 0.9x - 100$$

Definition 3 Given two functions $f : D \rightarrow T$ and $g : T \rightarrow S$, we define a composition of functions f and g to be a new function which consists of ordered pairs (a, c) whenever a pair (a, b) belongs to f and pair (b, c) belongs to g , or

$$(g \circ f)(a) = g(f(a))$$

$$a \xrightarrow{f} f(a) \xrightarrow{g} g(f(a))$$

Question 3.17 We are given partial tables for two functions, $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$. Use these tables to fill out the partial table for $f \circ g : \mathbb{R} \rightarrow \mathbb{R}$ given below. In the event that information you need is not available explain what additional information you would need.

x	$f(x)$
-14.2	2
$-\sqrt{2}$	-4
-1	$-\frac{5}{8}$
0	-7
2	12
4	2
$\sqrt{5}$	-1

x	$g(x)$
-14.2	3
1	-14.2
-1	2
0	$-\sqrt{2}$
2	4
4	4
$\sqrt{5}$	0

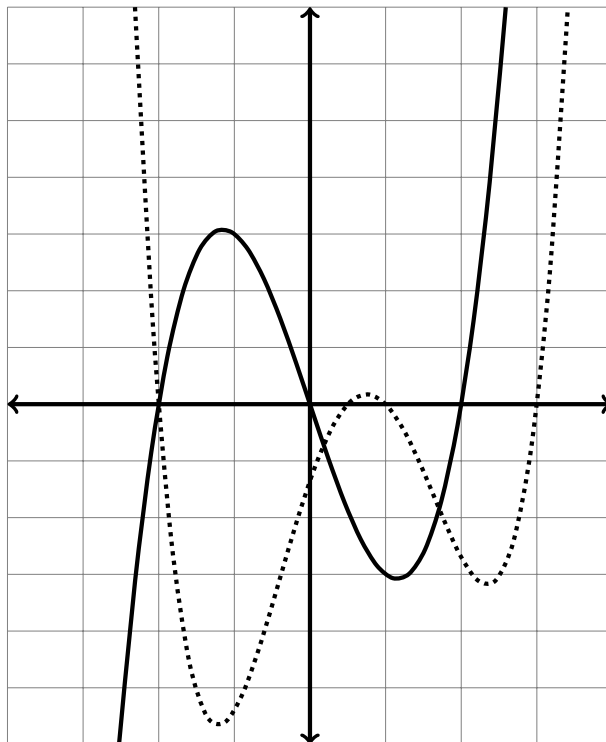
x	$(f \circ g)(x)$
-14.2	
1	
-1	
0	
2	
4	
$\sqrt{5}$	

Question 3.18 For the two functions $h : \mathbb{R} \rightarrow \mathbb{R}$ and $l : \mathbb{R} \rightarrow \mathbb{R}$ given by their algebraic rules $h(x) = 2x + 1$ and $l(x) = x^2 - 1$ find the algebraic expressions for:

a. $h \circ l : \mathbb{R} \rightarrow \mathbb{R}$

b. $l \circ h : \mathbb{R} \rightarrow \mathbb{R}$

Question 3.19 Two functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$ are given by their graphs. The graph of f is dotted.



Fill out the following tables as accurately as possible using the graphs of these two functions. In the event that information you need is not available, explain what additional information you would need.

x	$(f+g)(x)$
-2	
-1	
0	
1	
2	
3	

x	$(f-g)(x)$
-2	
-1	
0	
1	
2	
3	

x	$(f \circ g)(x)$
-2	
-1	
0	
1	
2	
3	

x	$(g \circ f)(x)$
-2	
-1	
0	
1	
2	
3	

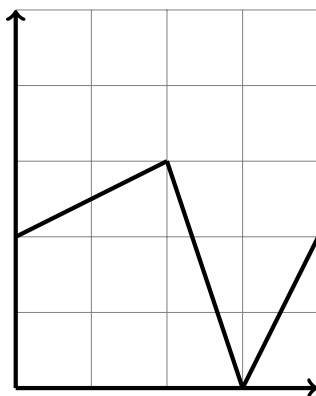
Question 3.20 We are given two functions $f: \mathbb{R} \rightarrow \mathbb{R}$ and $g: \mathbb{R} \rightarrow \mathbb{R}$.

a. Does $(f \circ g)(x) = (g \circ f)(x)$ sometimes, always, never?

b. Find an example and/or counterexample.

Question 3.21 Below is the portion of the graph for the function $f : \mathbb{R} \rightarrow \mathbb{R}$.

- a. What is $f(2)$?
- b. What is $(f \circ f)(0)$?
- c. What is $(f \circ f \circ f)(3)$?



Question 3.22 We have two functions $f, g : \mathbb{R} \rightarrow \mathbb{R}$ given by their algebraic expressions: $f(x) = x^2$ and $g(x) = x^3$.

- a. Write a formula for $f \circ g$.
- b. Write a formula for $f \cdot g$.
- c. How are they different?

3.5 Inverse Functions

Question 3.23 In Question 3.16 we found that we could calculate the sale price of any TV whose original price we knew using the following function rule: $r(x) = 0.9x - 100$.

a. Sam paid \$549 for her TV. What was the original price?

b. Sharmita paid \$269 for her TV. What was the original price?

c. Z paid \$ y for his TV. What was the original price?

Question 3.24 The following table gives information about function $f : \mathbb{R} \rightarrow \mathbb{R}$.

x	3	4	-3
$f(x)$	5	6	-1

a. Write a possible algebraic rule for f (The rule I have in mind is of the form $f(x) = ax + b$).

b. Suppose you want to find another function that will undo the effects of this one. That is, it will take 5 and turn it back into 3. Write a rule for this new function and call it g .

c. Evaluate $f(7)$.

d. Evaluate g at the answer you got for Question c.. Explain what happened.

Question 3.25 Let $h : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by the rule $h(x) = 2x + 1$. Complete the table for the function h and the function that will **undo** h .

h		undo h	
input	output	input	output
2	5	5	2
1		3	
0		1	
-1		-1	

- a. Call the “undo h ” g . What can you say about the pairs in h and g ?

We call g the *inverse relation* of h .

- b. Find algebraic expression for g .
- c. Calculate $(h \circ g)(x)$.
- d. Calculate $(g \circ h)(x)$.

Question 3.26 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = x^2$. Fill out the tables below:

f		inverse relation of f	
input	output	input	output
-2			
-1			
0			
1			

- a. Is the inverse relation of f also a function?
- b. Can you find an algebraic expression for the inverse relation of f ?

At this moment, we have two different ways of thinking about inverse functions.

Definition 4 ("Undoing") We say that a function $f : D \rightarrow T$ is invertible (has an inverse function) if there is a function $g : T \rightarrow D$ for which:

$$(g \circ f)(a) = a, \quad \text{for all } a \text{ in } D$$

and

$$(f \circ g)(b) = b, \quad \text{for all } b \text{ in } T$$

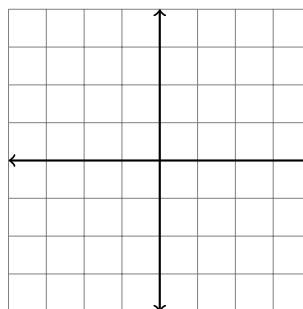
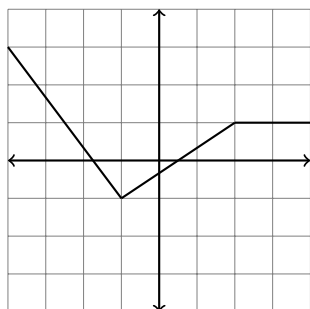
Definition 5 ("Switching") For a given function $f : D \rightarrow T$ we form an inverse relation $g : T \rightarrow D$ by exchanging the coordinate pairs belonging to f : if (a, b) is in f , then (b, a) is in g . If g is also a function, then we say that f is invertible, and that g is its inverse.

Note: If f is invertible, then its inverse function is denoted with f^{-1} .

Question 3.27 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be an invertible function.

- If you know that the point $(10, 17)$ is on the graph of f , what will be a point on the graph of $f^{-1}(x)$.
- What is $f(f^{-1}(4))$?
- If $(x, f(x))$ is a point on the graph of f , what is the associated point on the graph for f^{-1} ?

Question 3.28 Below is a graph for $f : [-4, 4] \rightarrow \mathbb{R}$. Graph the inverse relation of f on the empty coordinate system. Is f invertible? Explain.



Question 3.29 Let $f : \mathbb{R} - \{\frac{3}{2}\} \rightarrow \mathbb{R}$ be a function given by $f(x) = \frac{-5-x}{3-2x}$. Find $f^{-1}(7)$, without finding $f^{-1}(x)$.

Question 3.30 Function $f : [2, \infty) \rightarrow \mathbb{R}$ is given by $f(x) = 2\sqrt{x-2}$.

- a. Why does not f have all real numbers in its domain?
- b. How would you determine whether f has an inverse function?
- c. What would be the expression for f^{-1} ?

Question 3.31 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by the algebraic rule $f(x) = x^2$.

- a. Is f invertible? How do you know?
- b. Can you change the domain of f so that it would have an inverse? Explain.

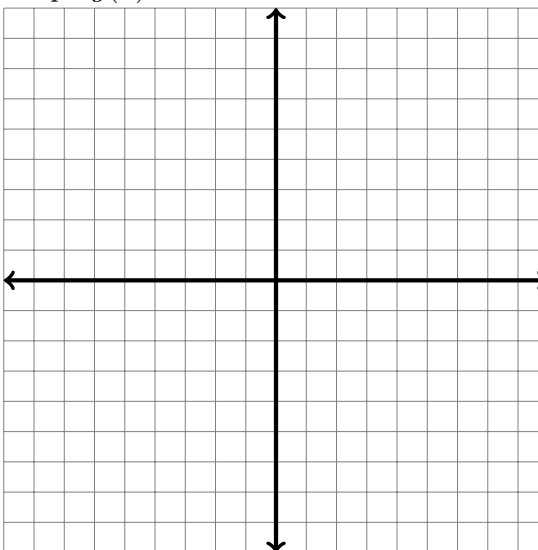
Question 3.32 Let $g : [0, \infty) \rightarrow \mathbb{R}$ be a function such that $g(x)$ is the side length of a square with area x .

a. Explain why the domain of g is $[0, \infty)$.

b. Fill in a table for $g(x)$.

x	$g(x)$
1	
4	
16	
$\frac{1}{4}$	
0	

c. Graph $g(x)$.



d. Is g invertible? Explain.

e. Find an algebraic formula for $g(x)$.

3.6 Summary

One of the most ubiquitous uses of mathematics in everyday life is describing relationships between quantities. You're used to seeing graphs in the newspapers that tell you how the stock prices change during a month, how temperature changes over time, the snowfall over the years, systolic blood pressure of people with various levels of glucose in their blood, etc. We are interested in seeing how quantities change, what inferences we can make about the relationships, and whether we can make predictions about future behavior. Most ordinarily we consider relationships between two quantities, which means that we are interested in pairs of values: the value of Facebook stock on 12/31/13 was \$54.54, the minimum temperature in Salt Lake City on 12/13/13 was 13°F, the total snowfall at Alta during 2010-2011 season was 723.5", the recorded systolic pressure was 130mmHg and 145mmHg for people with 110mg/dL glucose in their blood. This inspires us to make the following definition:

Definition 6 A **relation** is a collection of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target

We often call the first coordinates **inputs** and the second coordinates **outputs**.

Our examples are then: $(12/31/13, \$54.54)$, $(12/13/13, 13^\circ\text{F})$, $(2010 - 11, 723.5'')$, $(110\text{mg/dL}, 130\text{mmHg})$, and $(110\text{mg/dL}, 145\text{mmHg})$.

Notice that it is important for us to know what types of quantities we're interested in (domain and target) as well as in which order they appear (although we could consider either order, in which case we would have a different relation), but that there aren't any rules about how those pairs are made. Notice also that in all the examples but the last knowing the first value, determined the second one exactly. In the last problem, we couldn't claim that every person with the same blood sugars also has the same blood pressure. When we know exactly the output for a given input, we have a special type of relation which we call *function*:

Definition 7 A **function** is a set of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target
- every element in the domain is paired with exactly one element of the target.

We say that we have a function $f : D \rightarrow T$, and if an ordered pair (a, b) is in our function, then we say that $f(a) = b$.

If the ordered pair (a, b) belongs to our function f , then, in addition to saying that $b = f(a)$ (reads: b is f of a), we say that a is the input, and b is the output for a . a comes from the set D , domain, and b comes from the set T , target. We also often write $a \mapsto b$. Let us reiterate: in order to define a function we must know its domain, target and the pairings that belong to it.

We can represent functions in many different ways:

- We can use verbal descriptions, such as the ones at the beginning of this section.

- We can represent them by simply stating the ordered pairs that belong to the function, for example:

$$f = \{(a, 1), (b, 2), (c, 3), (d, 4), \dots, (z, 26)\}.$$

In this instance, even though it's not listed specifically, we can infer that the domain of f is the English alphabet and the target is the set of whole numbers $\{1, 2, 3, \dots, 26\}$.

- We can use tables:

x	$f(x)$
a	1
b	2
c	3
d	4

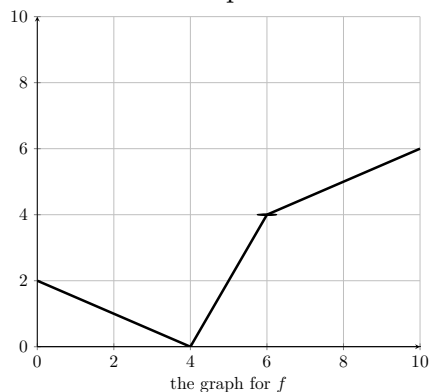
The domain here is less clear, and would need to be specified. It is possible that we only showed a partial table, or that the whole one is shown. This representation also clearly shows the ordered pairs.

- It is sometimes possible to represent a function with an algebraic expression. For instance, $f : \mathbb{R} \rightarrow \mathbb{R}_+$ is given by

$$f(x) = \frac{x^2 + 3}{2}.$$

The ordered pairs are less obvious here, but can be found. Remember that the pairs are made in such a way that the first coordinate is the input, an element from the domain, for example $x = -3$, and the second coordinate is its output, the corresponding element from the target: $f(-3) = \frac{(-3)^2 + 3}{2} = 6$, so one pair is $(-3, 6)$.

- Another useful representation of a function is the graph:



Here, as well, the pairs aren't immediately obvious, but can be found by looking at the points on the graph and reading their x and y coordinates. For instance, we can tell that $f(6) = 4$ because a point $(6, 4)$ lies on the graph of f .

Functions are not unlike numbers which you're used to, in the sense that we can combine them in various ways. Suppose for a minute that f and g are functions with equal domain and target $f, g : D \rightarrow T$. We can define new functions:

$$f + g : D \rightarrow T \quad \text{is defined by} \quad (f + g)(x) = f(x) + g(x)$$

$$f - g : D \rightarrow T \quad \text{is defined by} \quad (f - g)(x) = f(x) - g(x)$$

$$f \cdot g : D \rightarrow T \quad \text{is defined by} \quad (f \cdot g)(x) = f(x) \cdot g(x)$$

$$f \div g : D' \rightarrow T \quad \text{is defined by} \quad (f \div g)(x) = f(x) \div g(x).$$

In the last function the domain, D' , is either D or D with all the elements where the value of g is 0 excluded.

A combination of functions that does not have an analogous combination of numbers is *composition*:

Definition 8 Given two functions $f : D \rightarrow T$ and $g : T \rightarrow S$, we define a composition of functions f and g to be a new function which consists of ordered pairs (a, c) whenever a pair (a, b) belongs to f and pair (b, c) belongs to g , or

$$(g \circ f)(a) = g(f(a))$$

$$a \xrightarrow{f} f(a) \xrightarrow{g} g(f(a))$$

Regardless of which combination you need, you can find the values of the new function given different representations of the original functions. For example, if two functions are given by their graphs, you can find the graph or table of their sum or composition. Likewise, if you have algebraic expression, you could find the algebraic expression or a table of their product or the difference.

A closely related concept to that of composition is an inverse relation, and then inverse function. We can define an inverse relation of any relation: it simply consists of ordered pairs which when the order is reversed belong to the original relation. In other words, if (a, b) belongs to a given relation, then (b, a) belongs to the inverse relation. When our original relation is a function, it is possible that the inverse relation is also a function, but it is not always the case. When it is, we talk about invertible function.

Definition 9 ("Undoing") We say that a function $f : D \rightarrow T$ is invertible (has an inverse function) if there is a function $g : T \rightarrow D$ for which:

$$(g \circ f)(a) = a, \quad \text{for all} \quad a \quad \text{in} \quad D$$

and

$$(f \circ g)(b) = b, \quad \text{for all} \quad b \quad \text{in} \quad T$$

This really tells us that the composition of the function and its inverse is the identity function, a function that returns the output which is the same as the input.

Definition 10 ("Switching") For a given function $f : D \rightarrow T$ we form an inverse relation $g : T \rightarrow D$ by exchanging the coordinate pairs belonging to f : if (a, b) is in f , then (b, a) is in g . If g is also a function, then we say that f is invertible, and that g is its inverse.

Note: If f is invertible, then its inverse function is denoted with f^{-1} . Once again, different representations of the original function can be used to find the inverse function.

3.7 Student learning outcomes

1. Students will be able to determine if a relationship is a function.
2. Students will be able to seamlessly move between different representations of functions.
3. Given two functions $f(x)$ and $g(x)$ students will be able to calculate the composition of $f \circ g(x)$ using a table, graph and algebraic expressions.
4. Student understands a meaning of inverse function, can determine when it exists based on different representations of the given function and find some representation of the inverse.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.

4 Linear Functions

Essential questions

1. If a function $f(x)$ has a constant rate of change, what does the graph of $f(x)$ look like?
2. What does the slope of a line describe?
3. What can be said about the intersection of two lines?

4.1 Interpolating a Discrete Set of Data

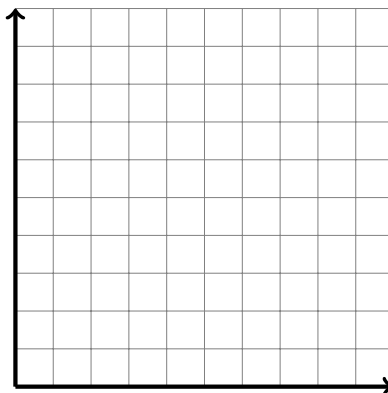
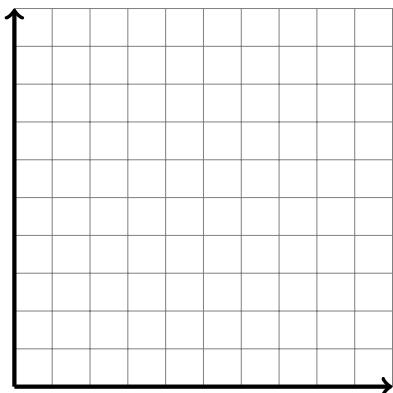
Zion Bank on the corner of 4th South and 7th East has a sign that reports the time and temperature. The temperature is given in two ways, using both the Celsius and Fahrenheit temperature scales. Here is a log of the temperature at different times of the day for August 29, 2013:

Time	Temp (C)	Temp (F)
11:03	31	87
12:00	32	90
2:00	35	95
3:04	35	95
4:08	34	93
8:03	27	81

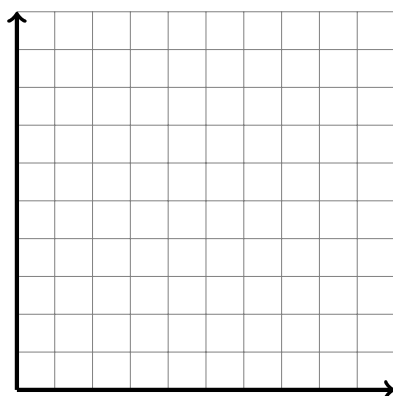
The weather report said that the low for the night had been 74° F at 4 : 30 am and the high for the day was 97° F at 3 : 30pm. Using the information in the table, estimate what you think the Celsius readings on the bank sign would have been at those two times. Explain how you got your answers.

Question 4.1 Use the coordinate systems below to plot the data. There are few issues that you should be paying attention to:

- a. Choose an appropriate scale and plot the points that show how the Celsius temperature changes with time. Your first point will be (11 : 03,31).
- b. Plot the points that show how the Fahrenheit temperature changes with time. Your first point will be (11 : 03,87).
- c. Write a short description of what your graphs show. Compare the two graphs.



Question 4.2 So far we have observed how the temperature reported in different scales depended on time. Now we will see how the Fahrenheit temperature changes with respect to the Celsius temperature. As before, choose an appropriate scale and plot the points from the table. Your first point will be $(31, 87)$.



- The points of your graph should fall approximately in a straight line. Draw a straight line that seems to go through most of the points.
- What is the Fahrenheit temperature when the Celsius temperature is 25° ?
- What is the Celsius temperature when the Fahrenheit temperature is 50° ?
- Is there a temperature where a Fahrenheit and Celsius thermometer show the same number? If so, what is it?

Question 4.3 If you increase the Celsius temperature by one degree, by how much does the temperature increase on the Fahrenheit scale?

- a. Explain how you know whether your answer to the previous question is accurate.

- b. How would your answer be different if you knew that 25°C is 77°F , and that 50°C is 122°F ?

Question 4.4 We want to come up with a general rule such that if we know the temperature in Celsius we can calculate the temperature in Fahrenheit.

- a. From Question 4.3 we know the effect of increasing the Celsius temperature by one degree on the temperature in Fahrenheit.

- b. Write down a rule that converts the temperature in Celsius to the temperature in Fahrenheit.

Question 4.6 There is another temperature scale called Kelvin. The scale is used because 0° Kelvin is the minimum temperature a system can have. If the temperature increases by one degree Kelvin, then the temperature also increases by one degree Celsius. Use the fact that $-273.15^\circ C = 0^\circ K$.

- Find a function g that represents the conversion between Celsius and Kelvin.
- Find a function h that represents the conversion between Fahrenheit to Kelvin.
- If you haven't already, how can you use composition of functions to answer the previous question?

4.2 Slope

Question 4.7 Kingda Ka is a steel accelerator roller coaster located at Six Flags Great Adventure in Jackson, New Jersey, United States. It is the world's tallest roller coaster, the world's second fastest roller coaster, and was the second strata coaster ever built. The steepest portion of Kingda Ka is a 418 foot drop. During the 418 foot drop the train moves 25 feet horizontally.

Your friends Nancy and John are debating if Kingda Ka is steeper than Wicked, a roller coaster at Lagoon Amusement Park in Farmington, Utah. Lagoon does not advertise the specs of Wicked as well as Six Flags does. However, Nancy and John have a photograph of them on the ride. They measure the drop in the photograph 15 cm and after the drop the train has only been displaced 1 cm.

a. Is there enough information to determine which roller coaster is steeper?

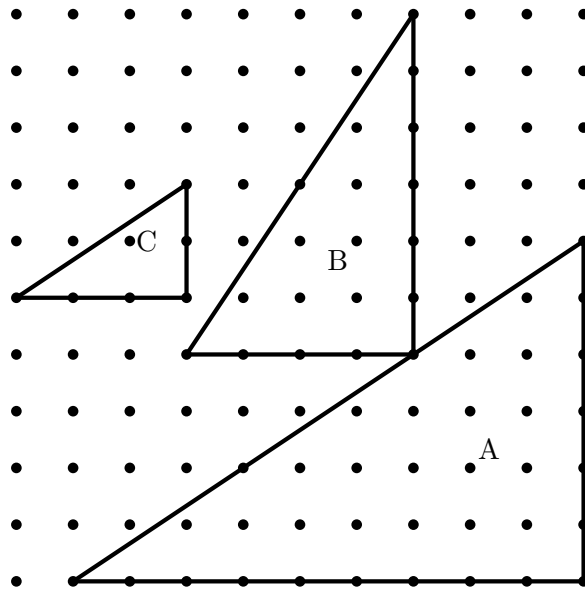
b. If so calculate which coaster is steeper.

c. Is steepness all you look for in a roller coaster?

Question 4.8 Steep roads sometimes have a sign indicating how steep they are. For example, the sign may say 5% Grade. This means that you gain 5 units of altitude (the rise) for every 100 units you move in the horizontal direction (the run).

- a. On a 5% grade, how many units of altitude do you gain for every 200 units you move in the horizontal direction.
- b. On a 5% grade, how many units in the horizontal direction would you have to move to increase your altitude by 100 units?
- c. How would a mathematician report a 5% grade? What is the corresponding slope?
- d. If the road up Little Cottonwood Canyon travels 8.26 miles horizontally and the elevation change is about 4000 feet, what is the average grade of canyon road? What is the average slope? (Use the fact that there are 5280 feet in a mile)
- e. What is the grade when you are driving on the Salt Flats?

Question 4.9 Consider the following :



- a. Find the slope of each hypotenuse in the above figure.

A :

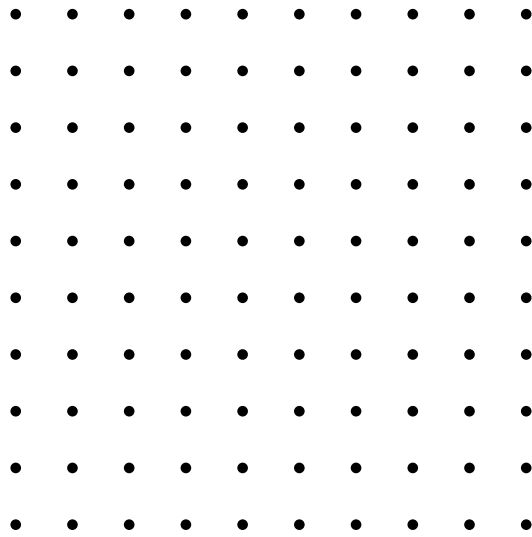
B :

C :

- b. Which triangle has the steepest hypotenuse?

- c. Two of the triangles' hypotenuse have the same slope. Why might someone make the mistake and report all three of the triangles have the same slope?

Question 4.10 Here is a geoboard:



- Draw a triangle on the geoboard that would have a hypotenuse with the largest possible slope. Calculate the slope of the figure you drew. Explain how you know it is the requested triangle.
- Draw a triangle on the geoboard that would have a hypotenuse with the smallest possible slope. Calculate the slope of the figure you drew. Explain how you know it is the requested triangle.
- List all the possible slopes of the triangles you can draw on the geoboard. Report them as fractions.

Question 4.11 What can you say about the slope of a line if, when you follow the line from left to right

- a. It goes up?
- b. It goes down?
- c. It doesn't go up or down?

Question 4.12 What can you say about the slope of a line that does not contain any points in the

- a. First quadrant.
- b. Second quadrant.
- c. Third quadrant.
- d. Fourth quadrant.

Question 4.13 The slope between two points is the quotient of the difference between their y -coordinates and the difference between their x -coordinates ($\frac{\Delta y}{\Delta x}$).

- a. What does this mean for the slope of a vertical line?
- b. What does this mean for the slope of a horizontal line?

4.3 Lines

Question 4.14 For each equation below find two pairs of numbers, (x, y) , that satisfy the equation. Label the two points and calculate the slope of the line segment that connects the two points.¹

a. $y = 1.5x + 3$

b. $y = -1.5x + 3$

c. $y = 2x + 3$

d. $y = -3x + 3$

e. How did your answer compare to people who chose different points?

¹Do not simply copy down the slope from the equation!

Question 4.15 Find two (x, y) pairs that satisfy the equation $y = mx + b$ (your pairs should be in terms of m and b). Use the pair of points to calculate the slope of the line segment connecting the two points.

Question 4.16 Given an equation for a line $y = .5x + 3$, how do you calculate the y -intercept? Explore both a geometric technique and an algebraic technique.

Question 4.17 Given an equation for a line $y = mx + b$, calculate the y -intercept. Did you use an algebraic or geometric approach?

Question 4.18 For each of the following linear equations, fill out the following tables.

a. $y = x + 2$

x	y
0	
1	
2	
3	

x	y
0	
2	
4	
6	

x	y
1	
3	
6	
8	

When $x = 0$, what is y ?

When x increases by 1, how much does y increase?²

b. $y = -4 - 3x$

x	y
0	
1	
2	
3	

x	y
0	
2	
4	
6	

x	y
1	
3	
6	
8	

When $x = 0$, what is y ?

When x increases by 1, how much does y increase?

c. $y = 9$

x	y
0	
1	
2	
3	

x	y
0	
2	
4	
6	

x	y
1	
3	
6	
8	

When $x = 0$, what is y ?

When x increases by 1, how much does y increase?

Where do you find these numbers in each of the tables for each equation?

²If y decreases, think of it as a negative increase.

Question 4.19 In the 2013-2014 academic year the tuition to attend the University of Utah is \$6400 a year (for 12 credits a semester). In the 2012-2013 academic year the cost of tuition was \$6000 a year (for 12 credits a semester).

- a. Suppose that a linear function can model the tuition at the U. What will the tuition cost for the academic year 2014 – 2015 (for 12 credits a semester)?

- b. Write down a function f such that $f(t)$ represents the tuition in the academic year t (for 12 credits a semester). Discuss what a reasonable domain might be for your function by thinking about what $f(0)$ and $f(10,000,000)$ would represent.

- c. For what values of t will $f(t)$ be most accurate?

- d. In what year will tuition cost \$10,000 per semester? (according to our model)

4.4 Summary

One of the simplest, and very useful, functions are linear functions. You're used to seeing them as linear equations such as this one $y = 2x + 1$, although those aren't the only linear equations. Other examples include $2x - 3y = 4$ or $2(y - 3) = 3(x + 1)$. Each of these equations have one thing in common: if we choose any two pairs of solutions, and find the slope between them:

$$\frac{\text{change in } y \text{ coordinates}}{\text{change in } x \text{ coordinates}} = \frac{\Delta y}{\Delta x}$$

we will inevitably get the same number! Further, each of the equations can be placed into the following format: $y = ax + b$, for some real numbers a and b , which motivates us to give the following definition:

Definition 1 A **linear function** $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function given by an algebraic rule of the form:

$$f(x) = ax + b$$

Here the number a represents the slope, the rate of change, of the function f . It tells us how much the output changes when the input changes by 1.

Line through two points We know that to completely determine a line it is enough to know two points that lie on it. Suppose then, that a line passes through two points: (x_1, y_1) and (x_2, y_2) . We know that the slope between those two points is:

$$a = \frac{y_2 - y_1}{x_2 - x_1}.$$

Our function, f , then has the following rule:

$$f(x) = \frac{y_2 - y_1}{x_2 - x_1}x + b,$$

or, if you prefer, your equation of the line is:

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + b.$$

We still need to know b . Since both points, (x_1, y_1) and (x_2, y_2) , lie on this line and belong to the function, we can use either of them to find b by substituting the values of its coordinates for x and y in the equation. For example, let's use the first point:

$$y_1 = \frac{y_2 - y_1}{x_2 - x_1}x_1 + b.$$

Now we have an equation in which only b is an unknown and we know how to solve those.

Line with a given slope through a given point We can similarly find an equation of a line if we know its slope and one point that lies on it. In essence, half of the work we did above has already been done for us. Say we know that the slope is a and a point that belongs to the line is (x_1, y_1) . We have $y = ax + b$, we know a , so we just need to find b . Since we know our point satisfies the equation of the line we can easily find b from the following equation: $y_1 = ax_1 + b$.

4.5 Student learning outcomes.

1. Students will be able to use a discrete set of data and draw an interpolating graph of a function.
2. Students will be able to recognize linear functions from graphs, equations, tables and verbal descriptions.
3. Given two points in the plane a students will be able to write an equation for the line that passes through the two points.
4. Given a linear function $f(x)$, students will be able to draw the graph of $f(x)$.
5. Given two linear functions $f(x)$ and $g(x)$, students will be able to determine if the graphs of the functions intersect.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.

5 Quadratic Functions

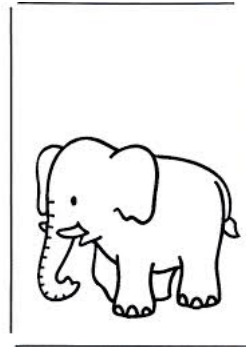
Essential Questions

1. What is the shape of the quadratic function and how can we use its features productively?
2. How can we find the zeros of a quadratic function?
3. How do we calculate the max or min of a quadratic function?

5.1 Rectangular fences

Question 5.1 You want to make a rectangular pen for Ellie, your pet elephant. What?! You don't have a pet elephant? That's rather unfortunate; they're quite cute. Well, imagine you have one. You want to make sure Ellie has as much space as possible. Unfortunately, you only have 28 feet of fencing available. If you use all of your fencing to make the pen, what is the biggest possible area you can achieve?

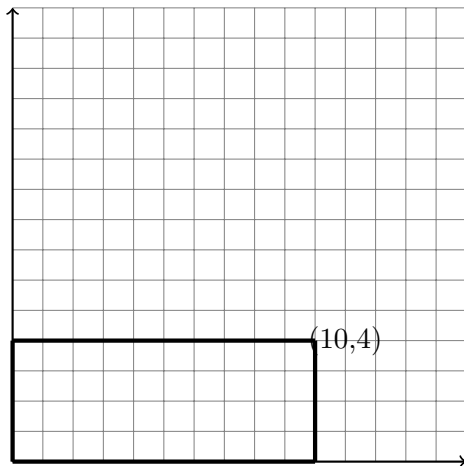
Outline here possible approaches to answering this question. What might you, or someone else, try to do to solve this problem?



Hi! I'm Ellie, your pet elephant for a few days!

Question 5.2 We will investigate our main problem in several steps.

- Draw 6 rectangular pens having a perimeter of 28 on the coordinate axis. Like below.
- Label the coordinate in the upper right hand corner of each pen.
- Make a table showing all the coordinates on your graph. Look for a pattern and make three more entries in the table.



length	height
10	4

- Write an equation for the function described by your graph and table. This is a function that will relate the height of the rectangle as a function of the length.

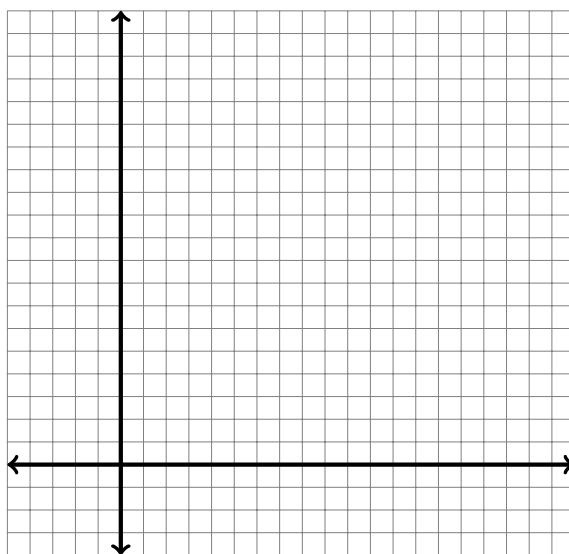
Question 5.3 The point (4,10) is the upper right corner of a plausible pen.

- What does the sum of these numbers represent in this problem?
- What does the product of these two numbers represent in this problem?
- Of all the rectangular pens recorded on your chart, which rectangular pen enclosed the largest area?
- How many rectangles are there whose perimeter is 28?

Question 5.4 For each rectangle from question c. compute the area.

length	height	area
10	4	

Question 5.5 Make a graph of the area as a function of length. Connect the points on your graph with a smooth curve¹. What kind of curve is it?



Question 5.6 On the same coordinate system, make a graph of the area as a function of height. Connect the points on your graph with a smooth curve. What kind of curve is it? What else do you notice?

Question 5.7 Why did it make sense to connect the dots of both graphs?

¹There should be no corners or sharp transitions;

Question 5.8 In this question we will interpret the graph.

- a. Label the highest point on your graph from 5.5 with its coordinates. Interpret these two numbers in terms of this problem. ²

- b. Where does the graph cross the x -axis? What do these numbers mean?

- c. If you increase the length by one foot, does the area increase or decrease? Does it change the same amount each time? Explain.

Question 5.9 We will now articulate our findings algebraically.

- a. Describe in words how you would find the area of the rectangular pen having perimeter 28, if you knew its length.

- b. If the perimeter of the rectangular pen is 28 and its length is L , write an algebraic expression for its area in terms of L .

- c. If you had 28 feet of fencing and wanted to make the largest possible rectangular pen, what would its length, width, and area be? Explain.

²This means: write a complete sentence explaining what your interpretation is.

a. Describe in words how you would find the area of the rectangular pen having perimeter P if you knew its length.

- b. If the perimeter of the rectangular pen is P and its length is L , write an algebraic expression for its area in terms of L and P .
- c. If you had P feet of fencing and wanted to make the largest possible rectangular pen, what would its length, width, and area be? Explain.

5.2 Graphs of quadratic functions

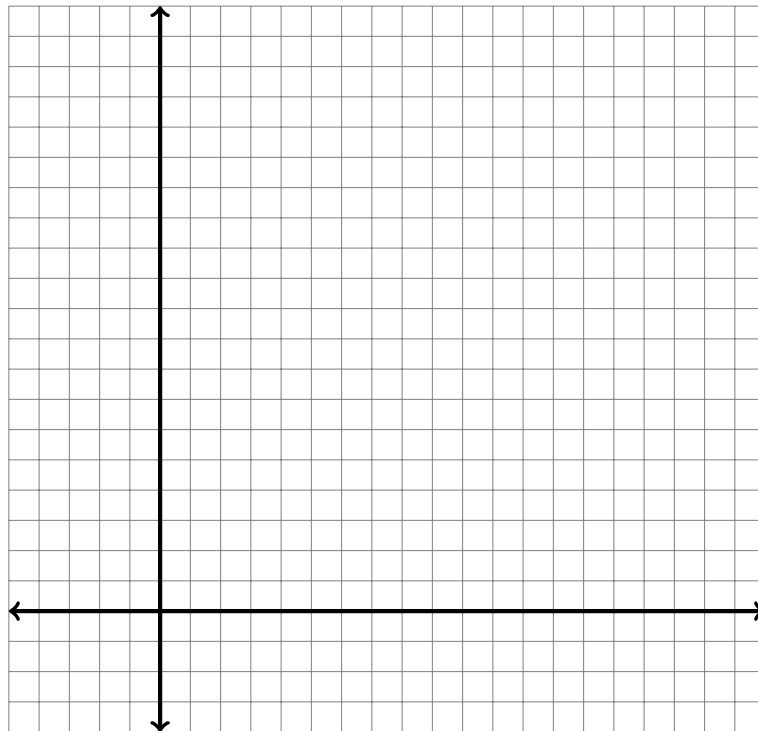
Question 5.11 Graph each of the following functions. Use a scale that will show values from -5 to 20 for the domain and from -20 to 100 for the target. To graph the functions, make a table and plot points.

a. $f(x) = x(8 - x)$

b. $g(x) = x(15 - x)$

c. $h(x) = x(12 - x)$

d. $k(x) = x(5 - x)$



Question 5.12 For each of the parabolas in question 5.11,

- label the graph with its equation;
- label the x -intercepts;
- label the y -intercepts;
- label the vertex;
- draw the line of symmetry;
- note if the graph opens up or opens down.
- by looking at the graph note if any of the functions have an inverse function.

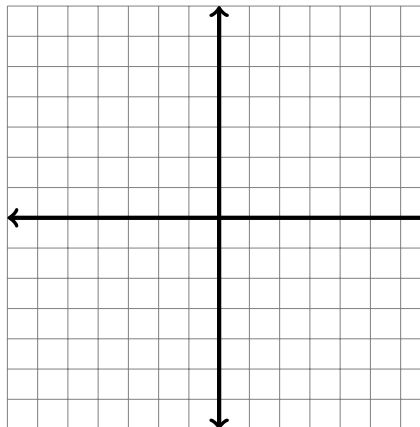
We will use the previous question to try to infer some general statements about the graphs of quadratic functions.

Question 5.13 You are given an arbitrary number b . Describe the graph of the quadratic equation $f(x) = x(b - x)$. Write an expression for the coordinates of its intercepts and maximum value in terms of b .

a. the x -intercepts:

b. the y -intercepts:

c. What is the line of symmetry for the graph?



Sketch the graph!

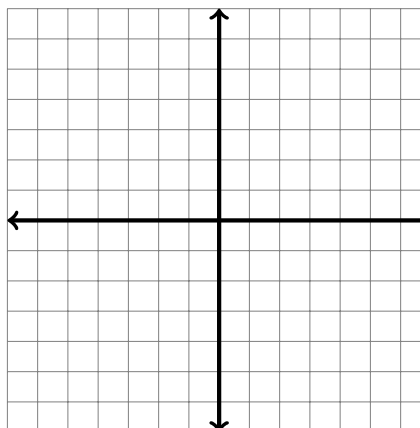
d. What is the maximum value of f ?

Question 5.14 For an arbitrary number q , describe the graph of the quadratic equation $f(x) = x(x - q)$. Write an expression for:

a. the x -intercepts:

b. the y -intercept:

c. What is the line of symmetry for the graph?

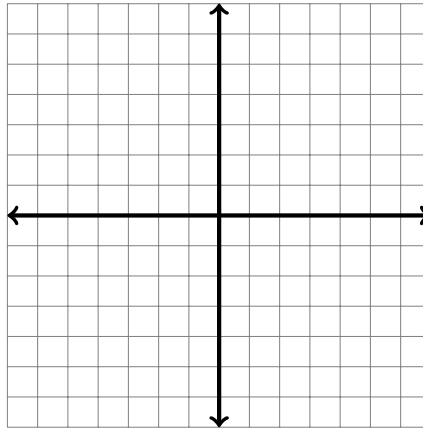


Sketch the graph!

d. What is the minimum value of f ?

Question 5.15 For arbitrary numbers a, b , describe the graph of the quadratic equation $f(x) = (x - a)(x - b)$. Write an expression for:

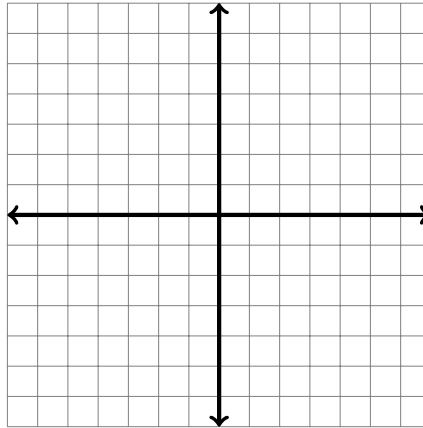
- a. the x -intercepts:
- b. the y -intercept:
- c. What is the line of symmetry for the graph?
- d. What is the minimum value of f ?



Sketch the graph!

Question 5.16 Graph the function $f(x) = x^2 + x - 6$. Write an expression for

- a. the x -intercepts;
- b. the y -intercept.
- c. What is the line of symmetry for the graph?
- d. What is the minimum value of f ?



Sketch the graph!

5.3 The Zero Product Property

Question 5.17 If $ab = 0$, which of the following is impossible? Explain.

- a. $a \neq 0$ and $b \neq 0$
- b. $a \neq 0$ and $b = 0$
- c. $a = 0$ and $b \neq 0$
- d. $a = 0$ and $b = 0$

Property 1 When the product of two quantities is zero, one of the quantities must be zero.

Question 5.18 If $(x - 6)(-2x - 1) = 0$, what are the possible values for x ? ³

Question 5.19 What would Property 1 say if the product of three quantities equaled 0?

$$a \cdot b \cdot c = 0$$

Question 5.20 Use Property 1 to solve the following equations:

- a. $(3x + 1)x = 0$
- b. $(2x + 3)(10 - x) = 0$
- c. $(3x - 3)(4x + 16) = 0$
- d. $6x^2 = 12x$

³Hint: use Property 1

Definition 1 An integer q is a **factor** of the integer p if there is a third integer g such that

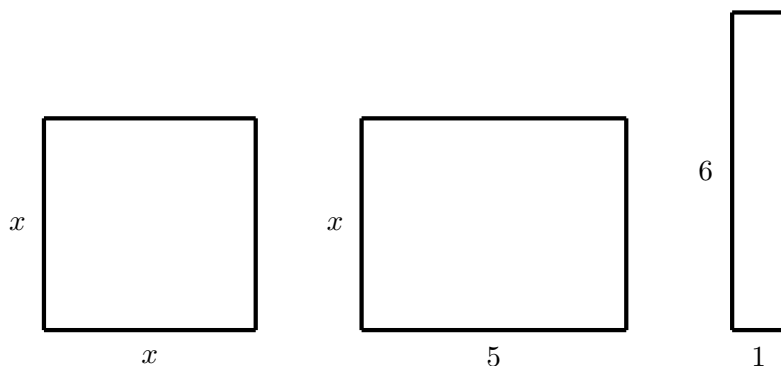
$$p = gq.$$

Definition 2 A polynomial $q(x)$ is a **factor** of the polynomial $p(x)$ if there is a third polynomial $g(x)$ such that

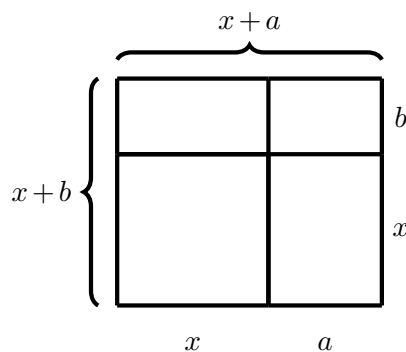
$$p(x) = q(x)g(x).$$

Question 5.21 Solve $x^2 + 5x + 6 = 0$. Our goal is to accomplish this by writing the left hand side as a product of two linear expressions, and then using the zero product property to find the solutions.

- a. Each term on the left hand side of the equation has a geometric meaning. Explain how the figures from left to right represent x^2 , $5x$, and 6, respectively.



- b. When we factor $x^2 + 5x + 6$ we are representing the above area as the area of a single rectangle.



- c. Find a and b such that $x^2 + 5x + 6 = (x + a)(x + b)$. Use the picture.
- d. Now that we have factored $x^2 + 5x + 6$, solve the equation $x^2 + 5x + 6 = 0$

Question 5.22 Solve the following equations by factoring. To help you factor draw the picture from Question 5.21.

a. $x^2 + 6x + 9 = 0$

b. $x^2 + 12x + 35 = 0$

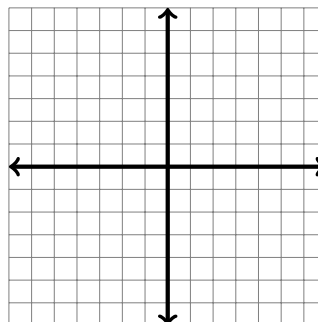
c. $x^2 + 9x + 20 = 0$

Question 5.23 Is $x = 4$ a solution to the equation $x^2 + 4x - 4 = 0$? Explain what it means to solve an equation.

Question 5.24 Not every quadratic polynomial can be factored. Which one of following polynomial functions can not be factored? You will want to graph each of them. Filling out a table should help.

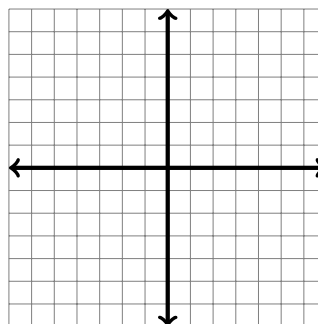
a. $f(x) = x^2 + 10x + 25$

x	$f(x)$



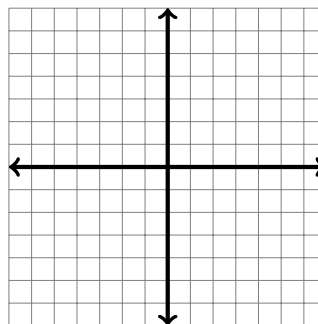
b. $g(x) = x^2 + 7x + 5$

x	$g(x)$



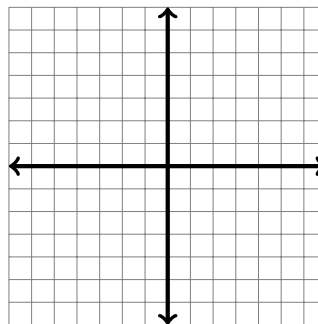
c. $h(x) = x^2 + 10x + 21$

x	$h(x)$



d. $k(x) = x^2 - 6x + 10$

x	$k(x)$



5.4 Completing the Square

Question 5.25 Solve the following equations⁴:

a. $x^2 = 4$

b. $x^2 = 25$

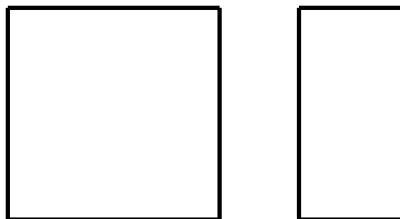
c. $x^2 = 7$

d. $(x + 3)^2 = 16$

e. $(x + 4)^2 = 5$

Question 5.26 In Question 5.25 we were able to solve the equation $x^2 + 8x + 11 = 0$ ⁵. Try and factor $x^2 + 8x + 11$. In this question we are going to investigate how to turn $x^2 + 8x + 11 = 0$ into the more convenient form of $(x + 4)^2 = 5$.

- a. Label the sides of the square and rectangle below so that the total area is $x^2 + 8x$.



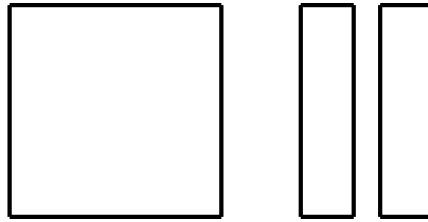
- b. Our goal is to cut and rearrange the pieces we have so that the new shape resembles a square as much as possible. What would you do?

- c. Why do we want a square?

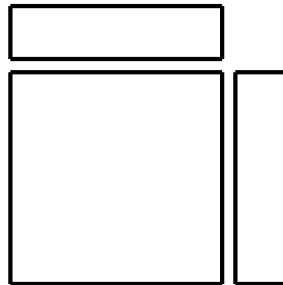
⁴Hint: Each equation has two solutions

⁵Really?! I didn't see it there. Did you?

- d. Here is how one student did this: she chopped the $8x$ rectangle in half. Label each side length. Has the area changed?



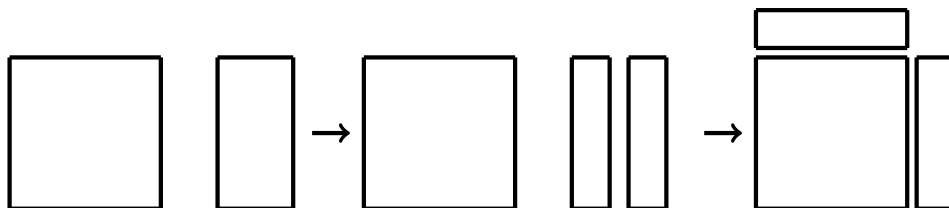
- e. In the picture below one of the rectangles has been moved to the top. Label the side lengths. Has the area changed?



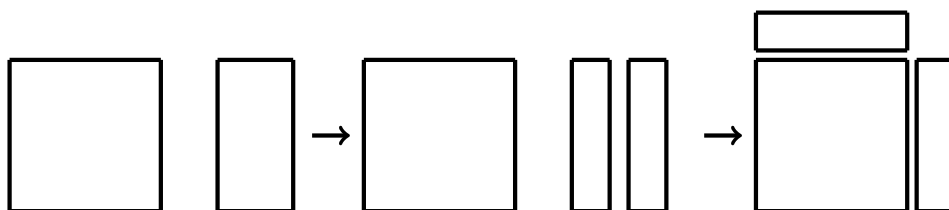
- f. Notice that this arrangement almost makes a square. What would be the area of the entire square?
- g. What is the area of the missing piece?
- h. Write an algebraic equation that relates: the area of the entire square, the area of missing piece, and $x^2 + 8x$.
- i. Use Part h. to substitute for $x^2 + 8x$ in the equation $x^2 + 8x + 11 = 0$.

Question 5.27 The process that we carried out in Question 5.26 is called completing the square. It is **wonderfully useful**. Complete the square for the following expression. Use the pictures provided to organize your thoughts.

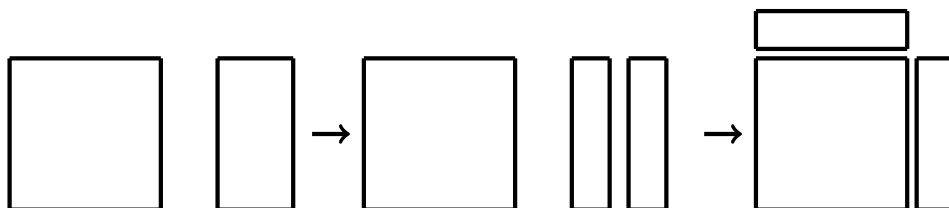
a. $x^2 + 10x$



b. $x^2 + 12x$



c. $x^2 + 5x$



Question 5.28 In Question 5.27 we completed the square for several expressions. Use that information to solve the following equations:

a. $x^2 + 10x = 10$

b. $x^2 + 12x = 14$

c. $x^2 + 5x = 7$

Question 5.29 Let's take this up a notch. Solve the following equations:

a. $2x^2 - 4x - 16 = 0$

b. $2x^2 + x - 6 = 0$

5.5 Calculating Maximum and Minimum Values of Quadratic Functions

Question 5.30 David Ortiz of the Boston Red Sox has an average off the bat speed of 102.2 miles per hour in the 2013 play off season. The average vertical speed off the bat is 67.5 miles per hour. This means that the height of the ball is given by $h(t) = -16t^2 + 99t$.

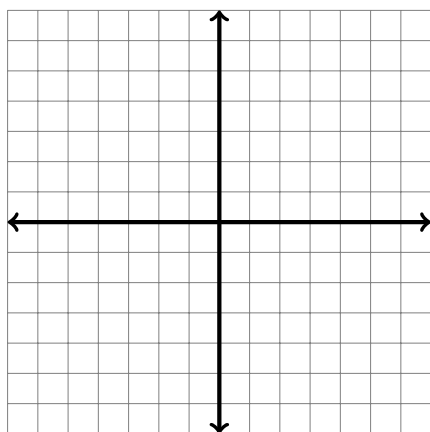
a. How long is the ball in the air?

b. What is the maximum height of the ball?

Question 5.31 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be given by $f(x) = x^2 + 3x - 4$. Does f have a maximum or a minimum? How were you able to tell?

Question 5.32 The following questions lead us to discover what the minimum value of $f(x) = x^2 + 3x - 4$ is.

- a. Given that $f(x) = x^2 + 3x - 4$ what are the values of x such that $f(x) = 0$?
- b. Use the symmetry of the graph of f to calculate the value of x where f achieves its minimum or maximum.



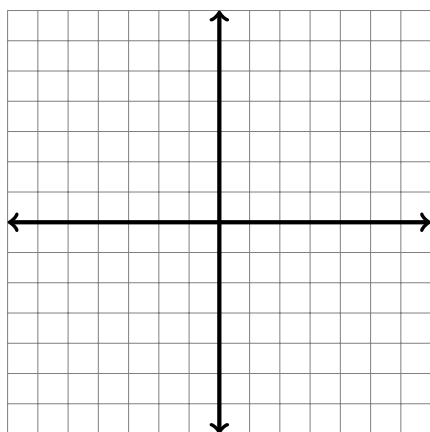
- c. Use Part b. to find the minimum or maximum value of f .
- d. Write $f(x)$ in a completed square form.

Question 5.33 Let $g(x) = -x^2 + 4x - 2$.

a. Does g have a maximum or a minimum? How are you able to tell?

b. What are the values of x such that $g(x) = 0$?

c. Use the symmetry of the graph of g to calculate the value of x where g achieves its maximum.



d. What is the maximum value of g ?

e. Write $g(x)$ in a completed square form.

Question 5.34 So why might you bother with different forms of the function expression?

a. In what circumstances would the completed square form be more useful?

b. In what circumstances would the factored form be more useful?

c. In what circumstances would the standard form be more useful?

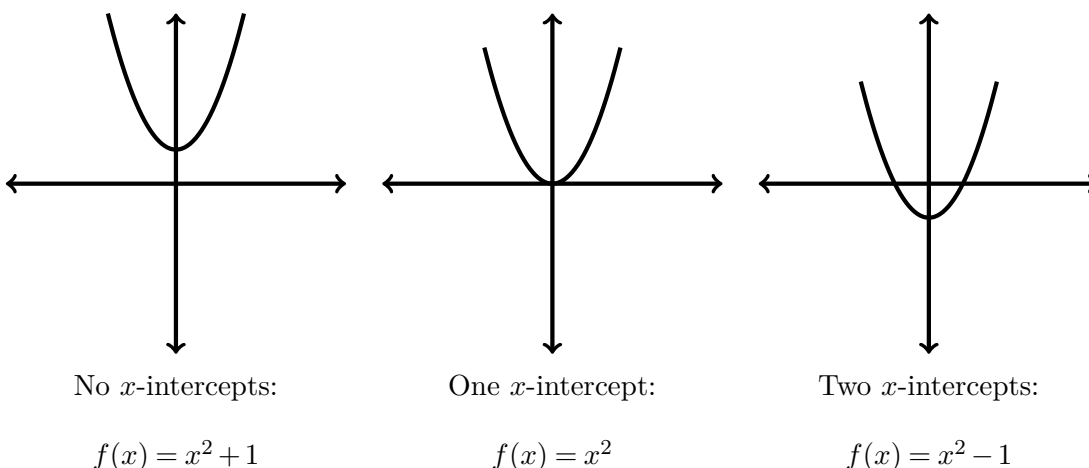
Question 5.35 Find a number between 0 and 1 such that the difference of the number and its square is maximum.

5.6 Summary

Definition 3 A **quadratic function** $f : \mathbb{R} \rightarrow \mathbb{R}$ is a function given by an algebraic rule of the form:

$$f(x) = ax^2 + bx + c \text{ with } a \neq 0.$$

The graph of a quadratic function f is a parabola. The y -intercept of the graph of a quadratic function f is the point $(0, f(0))$. This is a common feature with any function, since the y -intercept occurs when the x -coordinate is 0. A parabola can have 0, 1, or 2 x -intercepts. This is demonstrated by the following three pictures:



To find the x -intercepts of the graph of a quadratic function we solve the equation $f(x) = 0$. We have developed two techniques for solving a quadratic equation: factoring and completing the square, which we will outline below.

The parabola associated to a quadratic functions opens up if the leading coefficient is positive, and the parabola opens down if the leading coefficient is negative. A parabola has a vertical line of symmetry, and the location of the axis of symmetry can be calculated by looking at the average of the x -intercepts, if they exist. If the x -intercepts do not exist, it is still possible to locate the axis of symmetry by locating two inputs that yield the equal outputs. Again, averaging these two inputs will yield the x -coordinate of the points on the axis of symmetry.

Example 1 In this example we will work through how to graph a function $f(x) = x^2 + 2x - 3$.

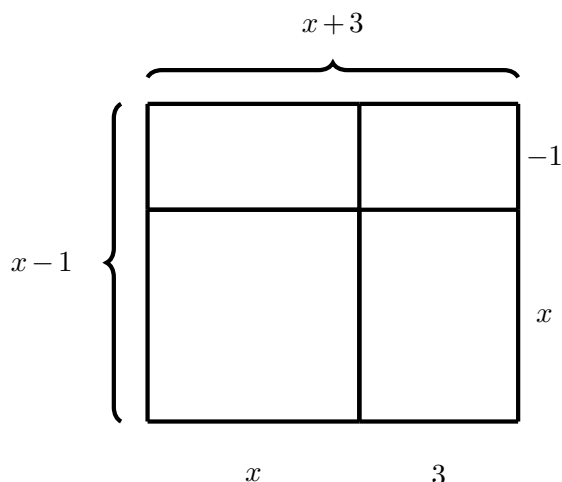
y -intercept: This is always the point $(0, f(0))$, so in this example we have the y -intercept:

$$(0, f(0)) = (0, 0^2 + 2 \cdot 0 - 3) = (0, -3)$$

x -intercept: To find the x -intercepts we need to solve the equation

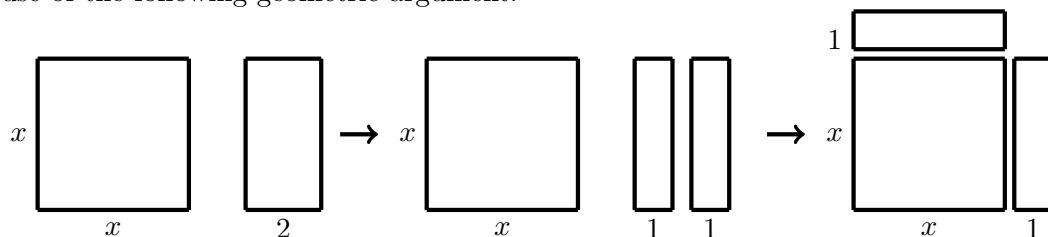
$$f(x) = x^2 + 2x - 3 = 0.$$

The following highlights how to use both the factoring method and the completing the square method. The quadratic expression $x^2 + 2x - 3$ factors as $(x + 3)(x - 1)$. This can be shown by drawing the following picture:



Now we use the zero product property to conclude that $x - 1 = 0$ or $x + 3 = 0$. Solving these two linear equations, gives the solutions to the original quadratic equation; $x = 1$ or $x = -3$. This tells us that the x -intercepts for the graph are $(1, 0)$ and $(-3, 0)$.

To solve the equation $x^2 + 2x - 3 = 0$ using the completing the square method we make use of the following geometric argument:



Which shows that $x^2 + 2x = (x + 1)^2 - 1$. This is used to substitute in the original equation:

$$x^2 + 2x - 3 = (x + 1)^2 - 1 - 3 = (x + 1)^2 - 4 = 0$$

Now we add 4 to both sides:

$$(x + 1)^2 = 4$$

Take the square root of both sides:

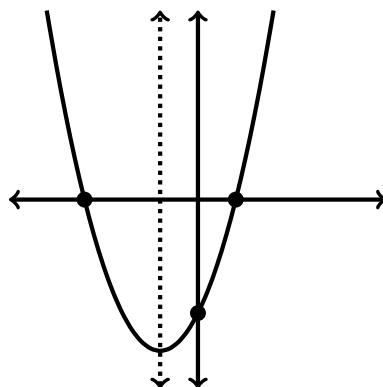
$$x + 1 = \pm 2$$

Subtract one from both sides:

$$x = 1 \pm 2 = 3, -1$$

Axis of symmetry: There are two ways to find the axis of symmetry. The first is to make use of the idea that it is a line of symmetry and therefore must lie half way between the two roots. So we take the average of the roots: $\frac{-3+1}{2} = -1$ and conclude that the axis of symmetry is the line given by $x = -1$. The second way to find the axis of symmetry is by looking at the completed square form: $(x + 1)^2 - 4$. Here we can note that $(x + 1)^2$ will achieve a minimum value at $x = -1$, since any number squared is always nonnegative. This means that the function f achieves its minimum at $x = -1$. The axis of symmetry always passes through the max/min so the line of symmetry is given by the equation $x = -1$.

max/min: We know that f has a local minimum because the parabola opens up. The minimum occurs on the axis of symmetry, and so the minimum is therefore the point $(-1, f(-1)) = (-1, (-1)^2 + 2 \cdot (-1) - 3) = (-1, -4)$.



x -intercepts: $(-3, 0)$ and $(1, 0)$

y -intercepts: $(0, -3)$

axis of symmetry: $x = -1$

minimum at $(-1, -4)$

5.7 Student learning outcomes

1. Students will understand a geometric model for factoring/multiplying.
2. Students will be able to apply a geometric understanding to the process of completing the square.
3. Given a quadratic function in standard form students will be able to graph the function.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.

6 Exponents and Exponential Functions

Essential questions

- Why are exponents useful? How are they used in real world applications?
- Where do rules for exponents come from?
- What does a negative exponent mean?
- How does exponential function compare with polynomial functions, linear in particular?
- How do we undo the exponential function?

6.1 Population growth

Definition 1 An exponent is a convenient way to write repeated multiplication. Given a natural number b the following notation represents a product of b many a 's.

$$a^b = \underbrace{a \cdot a \cdot a \cdot \dots \cdot a}_{b \text{ many } a\text{'s}}$$

Question 6.1 Use exponents to represent the following:

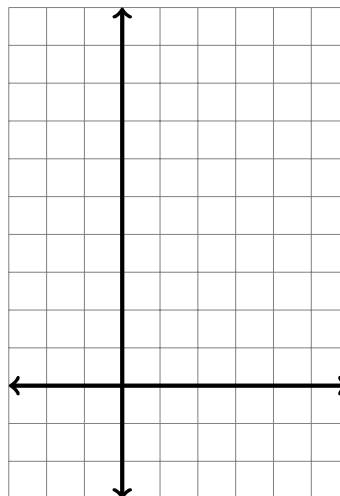
- $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$
- $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$
- $x \cdot x \cdot x$
- $a \cdot a \cdot a \cdot a \cdot a \cdot a$

Question 6.2 A colony of bacteria is being grown in a laboratory. It contains a single bacterium at 12:00 noon (time 0), and the population is doubling every hour.

- How long do you think it would take for the population to exceed 1 million? 2 million? Write down your guesses and compare with other students' guesses.
- Make a table of values showing how this population of bacteria changes as a function of time. Find the population one hour from now, two hours from now, etc. Extend your table until you can answer the questions asked in Question 6.2 a. and graph your points. How close were your guesses?

t	Number of bacteria	
0	1	
1		
2		
3		
4		
5		
6		

- In the third column in Question b. write the population each time as a power of 2 (for example, 4 is 2^2).



- d. What would the population be after x hours? (Write this as a power of 2.)¹
- e. Compare the population after 8 hours with the population after 5 hours.
- a. How much more is the population after 8 hours? (Compare by subtracting.)
 - b. How many times as much is it? (Compare by dividing.)
 - c. Which of your answers is a power of 2? What power of 2 is it?
- f. How many bacteria would there be after three and a half hours?
- g. Why does Question f. demand that we depart from thinking of this as a sequence?
- h. What does $2^{3.5} = 2^{\frac{7}{2}}$ mean? Can you use the graph to estimate this number?
- i. How long exactly do we have to wait to see at least 1,000,000 bacteria?

¹How does this compare to the explicit formula for a geometric sequence?

6.2 Rules for Exponents

Question 6.3 Rewrite the following expressions using just one exponent. To answer the question, think about how many twos would appear after you multiplied everything out.

a. $(2^2)^3$

b. $(2^4)^5$

c. $(2^5)2^7$

d. $(2^9)2^{10}$

Question 6.4 Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $(5^5)^3$

b. $(5^4)^6$

c. $(5^4)5^6$

d. $(5^2)5^{10}$

Question 6.5 Rewrite the following expressions using just one exponent. To answer the question, think about how many twos (or x s) would appear after you multiplied everything out. Think about a and b as positive integers.

a. $(2^a)^b$

b. $(2^a)2^b$

c. $(x^a)^b$

d. $x^a x^b$

Question 6.6 Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $\frac{5^5}{5^2}$

b. $\frac{6^4}{6^3}$

c. $\frac{x^a}{x^b}$

Question 6.7 Evaluate this expression in two different ways: using the rule you just developed and by multiplying everything out:

$$\frac{5^7}{5^8}$$

Question 6.8 For any $x \neq 0$, define x^{-1} to be the number such that $x^{-1}x = 1$. This makes x^{-1} the multiplicative inverse of x .

a. What number is 2^{-1} ?

b. What number is 3^{-1} ?

c. What number is 2^{-2} ?

d. What number is 3^{-2} ?

e. The rule you developed for Question 6.5 Part d. is a rule we want to be true in general. Use that rule and the definition of $2^{-1}2 = 1$ to decide the value of 2^0 .

x	2^x
5	32
4	16
3	8
2	4
1	2
0	1
-1	
-2	
-3	
-4	

f. This is the table you filled out recently. Use the patterns apparent in the table to decide why this definition makes sense:

Question 6.9 Let's go back to the question: "What does $2^{3.5} = 2^{\frac{7}{2}}$ mean?"

- a. Calculate $(2^{\frac{7}{2}})^2$. Assume the rules for from 6.5 apply.
- b. Explain what $\sqrt{2^7}$ means.
- c. Combine Parts a. and b. to make sense of $2^{\frac{7}{2}}$

Question 6.10 A colony of bacteria is being grown in a laboratory. It contains a single bacterium at 12:00 noon (time 0), and the population is doubling every hour. How many bacteria are there after 3.5 hours?

Question 6.11 Let us redo this for $a^{\frac{1}{2}}$:

- a. Calculate $(a^{\frac{1}{2}})^2$
- b. Explain what \sqrt{a} means.
- c. Combine Parts a. and b. to make sense of $a^{\frac{1}{2}}$:

Question 6.12 Think about how $f(x) = x^{\frac{1}{2}}$ is the inverse function of $g: [0, \infty) \rightarrow \mathbb{R}$ defined by $g(x) = x^2$.

- a. Why is the domain of g limited to $[0, \infty)$?
- b. What would be the inverse function of $h: \mathbb{R} \rightarrow \mathbb{R}$ given by $h(x) = x^3$?
- c. What would be the inverse function of $l: [0, \infty) \rightarrow \mathbb{R}$ given by $l(x) = x^4$?
- d. What would be the inverse function of $p: [0, \infty) \rightarrow \mathbb{R}$ given by $p(x) = x^n$?

Question 6.13 What does $2^{\frac{7}{5}}$ mean?

- Calculate $(2^{\frac{7}{5}})^5$. Assume the rules for from 6.5 apply.
- Explain what $\sqrt[5]{2^7}$ means.
- Combine Parts a. and b. to make sense of $2^{\frac{7}{5}}$.

Question 6.14 Use Question 6.13 to come up with a good definition of $5^{\frac{m}{n}}$.

Question 6.15 In the following exercises, we will write the expression in a simplified version, which means that every power will be written using only positive exponents.

a. $6w^5(2w^{-2})$

b. $(3a^{-2}b^{-4})^2$

c. $\frac{2^{-3}r^{-2}(r^{-1})^{-2}}{r(r^3)^{-3}}$

d. $\left(\frac{3q}{4p^2}\right)^2\left(\frac{2p}{5q}\right)^{-2}$

6.3 Graphs of Exponential Functions

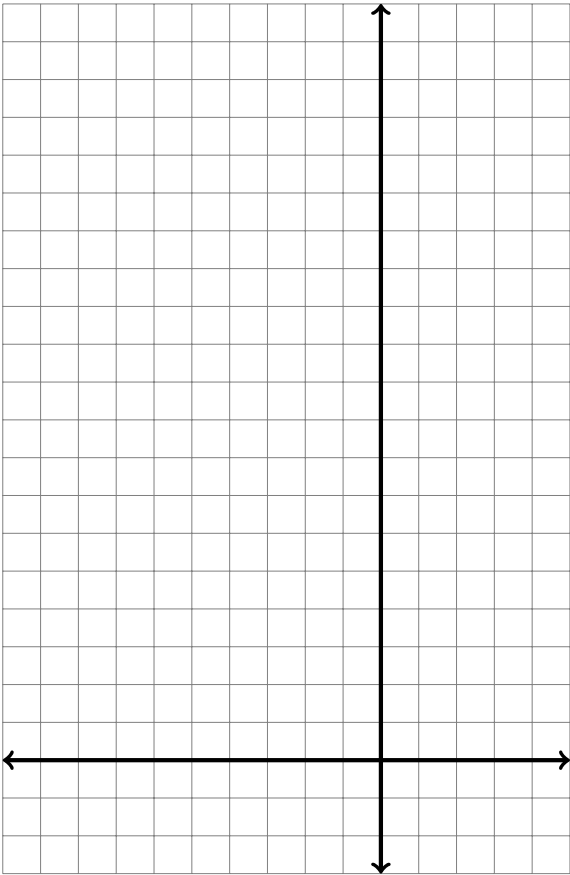
Question 6.16 Let’s make some predictions.

	Similarities	Differences
$f(x) = 2^x$ & $g(x) = 5^x$		
$f(x) = 2^x$ & $h(x) = (\frac{1}{2})^x$		
$h(x) = (\frac{1}{2})^x$ & $k(x) = (\frac{1}{5})^x$		

Question 6.17 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by the rule $f(x) = 2^x$.

- a. Fill out the table:
- b. Sketch the graph for f :

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

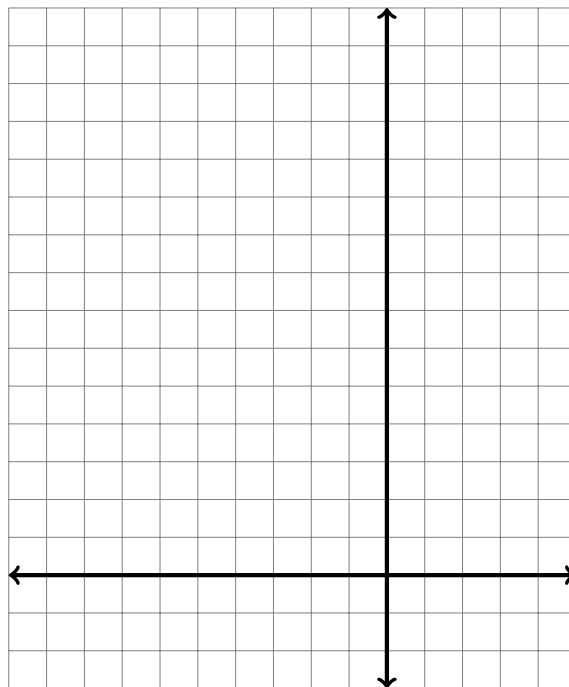


Question 6.18 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by the rule $f(x) = (\frac{1}{2})^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

b. Sketch the graph for f .

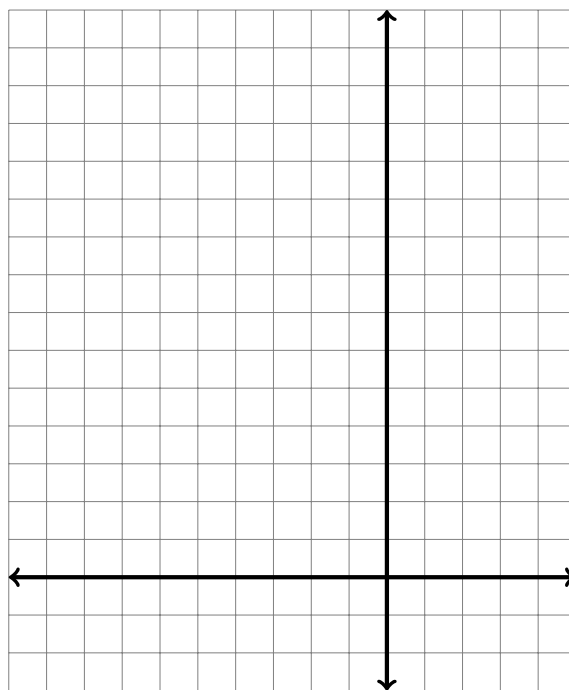


Question 6.19 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by the rule $f(x) = 5^x$.

a. Fill out the table:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	
1	
2	
3	
4	
5	

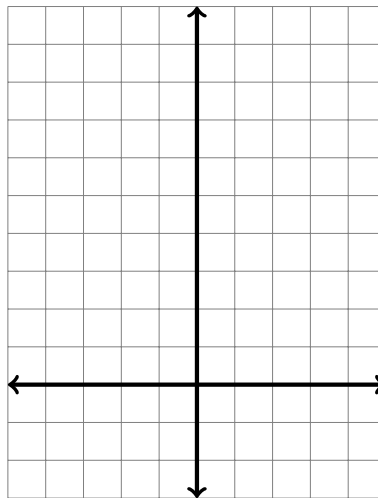
b. Sketch the graph for f .



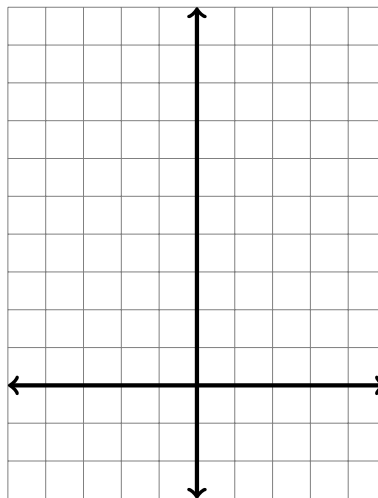
Question 6.20 Look at the graphs you drew in Questions 6.17, 6.18, and 6.19.

a. All three graphs share a common point. Which point is this?

b. Let $a > 1$. Use Questions 6.17 and 6.19 to help you sketch a graph of $f(x) = a^x$. Articulate why this is the general shape.



c. Let $0 < b < 1$. Use Question 6.18 to help you sketch a graph of $g(x) = b^x$.



d. From looking at the graphs are the functions $f(x) = a^x$ and $g(x) = b^x$ invertible? Explain.

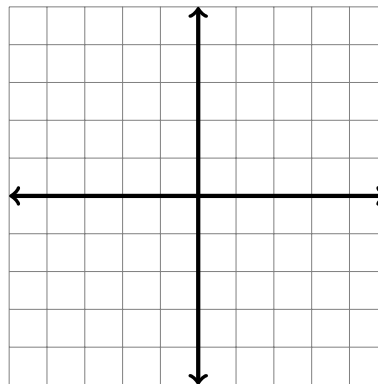
e. Why does it not make sense to talk about functions of the form $h(x) = c^x$ when $c < 0$?

6.4 Inverse function

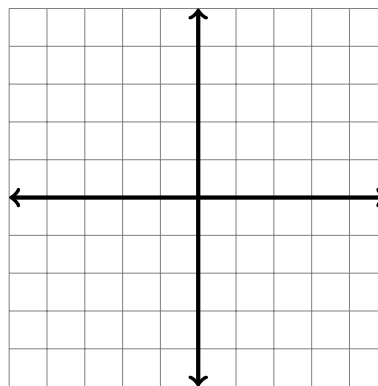
You have already discovered that exponential functions are invertible. Before we think about their inverse functions, let's solve a few problems as a warm up.

Question 6.21 Graph each of the functions. Make a table! Make a table for the inverse relation. Then graph the inverse relation. Decide if these functions have inverse functions.

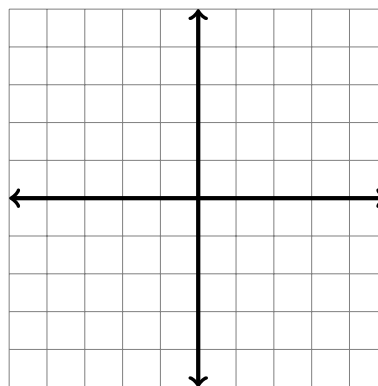
a. $f(x) = 4x + 5$



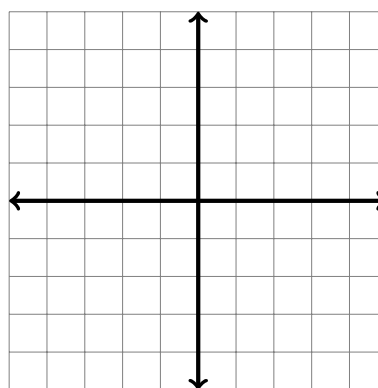
b. $g(x) = (x + 5)(x - 4)$



c. $h(x) = x^3 + 4$



d. $f_2(x) = 2^x$



Question 6.22 Let $f_2 : \mathbb{R} \rightarrow \mathbb{R}$ be defined by $f_2(x) = 2^x$.

- a. For what value of x does $f_2(x) = 4$? f. For what value of x does $f_2(x) = -5$?

- b. For what value of x does $f_2(x) = 16$?

With this knowledge fill out the following table:

x	$f_2^{-1}(x)$
4	
16	
128	
$\frac{1}{2}$	
$\frac{1}{4}$	
-5	
1	
2	
8	

- c. For what value of x does $f_2(x) = 128$?

- d. For what value of x does $f_2(x) = \frac{1}{2}$?

- e. For what value of x does $f_2(x) = \frac{1}{4}$?

Question 6.23 In the spirit of the previous question, let's use the following notation: $f_b(x) = b^x$. In other words, f_b denotes the exponential function with base b . Evaluate the following:

a. $f_4^{-1}(16)$

b. $f_3^{-1}(81)$

c. $f_5^{-1}(125)$

d. $f_{\frac{1}{2}}^{-1}(4)$

e. $f_{\frac{2}{3}}^{-1}\left(\frac{9}{4}\right)$

6.5 Solving Exponential and Logarithmic Equations

An exponential equation is an equation of the form: $y = ab^x$. If you know a, b , and x , it is easy to calculate y , but sometimes you need to find one of the other three variables. Let's consider the three examples below.

Question 6.24 Solve for a :

- a. You want to know how much someone needs to deposit in an account so that after seven years the amount in the account is \$287.17. The interest rate 2%, compounded annually. Write and solve the equation.

- b. Solve $y = ab^x$ for a .

Question 6.25 Solve for b :

- a. You want to know the yearly decay rate of a chemical that is decaying exponentially. At time 0, there was 300 grams of the substance. 10 years later there was 221 grams left. Write and solve the equation.

- b. Solve $y = ab^x$ for b .

Question 6.26 Solve for x :

- a. You want to know how long it will take for a bacteria population to triple, if the hourly growth rate is 160%.

- b. Solve $y = ab^x$ for x .

Question 6.27 Solve the following equations:

a. $2^x = 16$

b. $5^x = 125$

c. $3 \cdot 2^x = 24$

d. $2 \cdot 5^{x-2} + 1 = 51$

Question 6.28 Solve the following equations:

a. $\log_3 27 = x$

b. $\log_4 x = -2$

c. $2\log_3 x = 4$

d. $3\log_4 x + 1 = 7$

6.6 Applications

Question 6.29 In 1975, the population of the world was about 4.01 billion and was growing at a rate of about 2% per year. People used these facts to project what the population would be in the future.

- a. Complete the following table, giving projections of the world's population from 1976 to 1980, assuming that the growth rate remained at 2% per year.

Year	Calculation	Projection (billions)
1976	$4.01 + (.02)4.01$	4.09

- b. Find the ratio of the projected population from year to year. Does the ratio increase, decrease, or stay the same?
- c. There is a number that can be used to multiply one year's projection to calculate the next. What is that number?
- d. Use repeated multiplication to project the world's population in 1990 from the 1975 number, assuming the same growth rate.
- e. Compare your result to the previous problem with the actual estimate of the population made in 1990, which was about 5.33 billion.
- (a) Did your projection over-estimate or under-estimate the 1990 population?
- (b) Was the population growth rate between 1975 and 1990 more or less than 2%? Explain.
- f. Write an algebraic expression for $f(x)$ which predicts the population of the world x years after 1975.

- g. At a growth rate of 2% a year, how long does it take for the world's population to double? We call this *doubling time*.

- h. Complete the following table:

x Years passed since 1975	n number of doubling times	f Projection (billions)
0	0	4.01

- i. Give an algebraic expression for the function f as a function of the number of doubling times n .
- j. Give an algebraic expression for the function n as a function of years x passed since 1975.
- k. Find the composition $f \circ n$ and explain what it represents in terms of the population projection.

a. What kind of function do you expect will model the decay of carbon 14? Explain what evidence you have for your claim.

- b. Write an algebraic expression (rule) for the function that models the decay of carbon-14.

6.7 Summary

Much like linear functions were generalizations of arithmetic sequences, the exponential functions are generalizations of geometric sequences. If we are observing equal sized intervals of inputs we will notice that there is a constant ratio of corresponding outputs. We can think of exponential functions as having a constant multiplicative "rate of change".

Definition 2 An exponential function is every function $f : \mathbb{R} \rightarrow \mathbb{R}$ of the form

$$f(x) = ab^x$$

where $a \neq 0$, and $b \neq 1$.

You might wonder how it is that the domain of the exponential function defined above is the set of real numbers. Indeed, if we think about exponents only as repeated multiplication, then this function can only be defined on the set of natural numbers. We need to make sense of what it might mean to raise a number to a negative power. Or rational power. Or irrational power. In order to do this, we investigate exponentiation and develop some properties that are obvious if we write out the given exponentiation in terms of multiplication. For this purpose, let us suppose that c is any real number, and that a and b are natural numbers.

$$c^a c^b = \underbrace{(c \cdot c \cdot c \cdot \dots \cdot c)}_{a \text{ many } c\text{'s}} \underbrace{(c \cdot c \cdot c \cdot \dots \cdot c)}_{b \text{ many } c\text{'s}} = \underbrace{c \cdot c \cdot c \cdot \dots \cdot c \cdot c \cdot c \cdot \dots \cdot c}_{a+b \text{ many } c\text{'s}} = c^{a+b}$$

In the similar way we obtain the familiar properties of exponents:

$$\begin{aligned} c^a c^b &= c^{a+b} \\ (c^a)^b &= c^{ab} \\ \frac{c^a}{c^b} &= c^{a-b} \end{aligned}$$

If we consider the last property when $b = a + 1$ we'll notice that we have

$$\frac{c^a}{c^{a+1}} = c^{a-(a+1)} = c^{-1}$$

At the same time, we note that the denominator on the left hand side has one more factor of c , so

$$\frac{c^a}{c^{a+1}} = \frac{1}{c}$$

Since if two things are equal to the same thing, then they themselves must be equal, it seems reasonable to define

$$c^{-1} = \frac{1}{c},$$

for every real number c other than 0 (because division by 0 is undefined). We have now extended the domain of our exponential function to the whole set of integers because, for example:

$$f(-4) = ab^{-4} = a(b^4)^{-1} = a \cdot \frac{1}{b^4}$$

We would like to further allow the exponents to be fractions, so we think about what might happen if we did have a fractional power, say $\frac{1}{3}$. If we use the second property we listed above, then we can conclude that

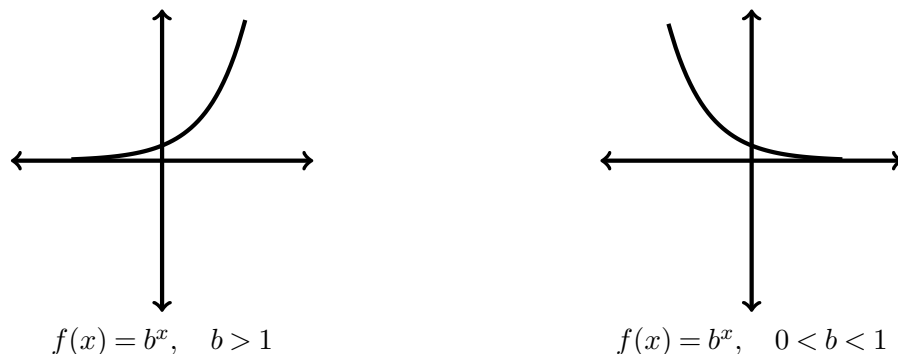
$$\left(b^{\frac{1}{3}}\right)^3 = b^{\frac{1}{3} \cdot 3} = b^1 = b.$$

We see that $b^{\frac{1}{3}}$ is the number which cubed gives us b . We know one such number already: $\sqrt[3]{b}$, and so conclude that these two numbers must be the same! In general then, it makes sense to define:

$$b^{\frac{m}{n}} = \sqrt[n]{b^m}.$$

To complete the definition of exponential function on irrational numbers, we'll postpone this story to little later when we have few other tools and understandings at our disposal. We will only say that a exponents thus defined will extend the properties we've had so far.

If we restrict ourselves to positive coefficients a , depending on whether b is smaller or larger than 1, the exponential function is either increasing or decreasing, respectively. (If $a < 0$, then the situation is reversed. You should be able to explain why that is the case.) For a brief period, let's think about the case when $a = 1$. Now, the graph of every exponential function $f(x) = b^x$ will pass through the point $(0, 1)$ and will have x -axis as the horizontal asymptote (a line which the graph will never intersect, but will get closer and closer to). It makes perfect sense that the graphs of exponential functions have no x -intercepts, since no power of b will be 0 or smaller than 0.



By looking at the graphs you should be able to tell that every exponential function has an inverse.

Definition 3 $\log_b : \mathbb{R}_+ \rightarrow \mathbb{R}$ is the inverse function of the exponential function with base b .

We can interpret that in following way:

$$\log_b(b^x) = x \quad \text{and} \quad b^{\log_b x} = x.$$

Or, we can think about the fact that the inverse function contains pairs whose inputs and outputs are reversed those of the original function, so the question

What is $\log_b a$?

is the same as wondering how to fill out the following table:

x	$\log_b(x)$
a	

But, we know how to do this!

It is the same as filling out the following table:

x	b^x
	a

Which answers the question:

To which power must we raise b to get a ?

Or, in symbols:

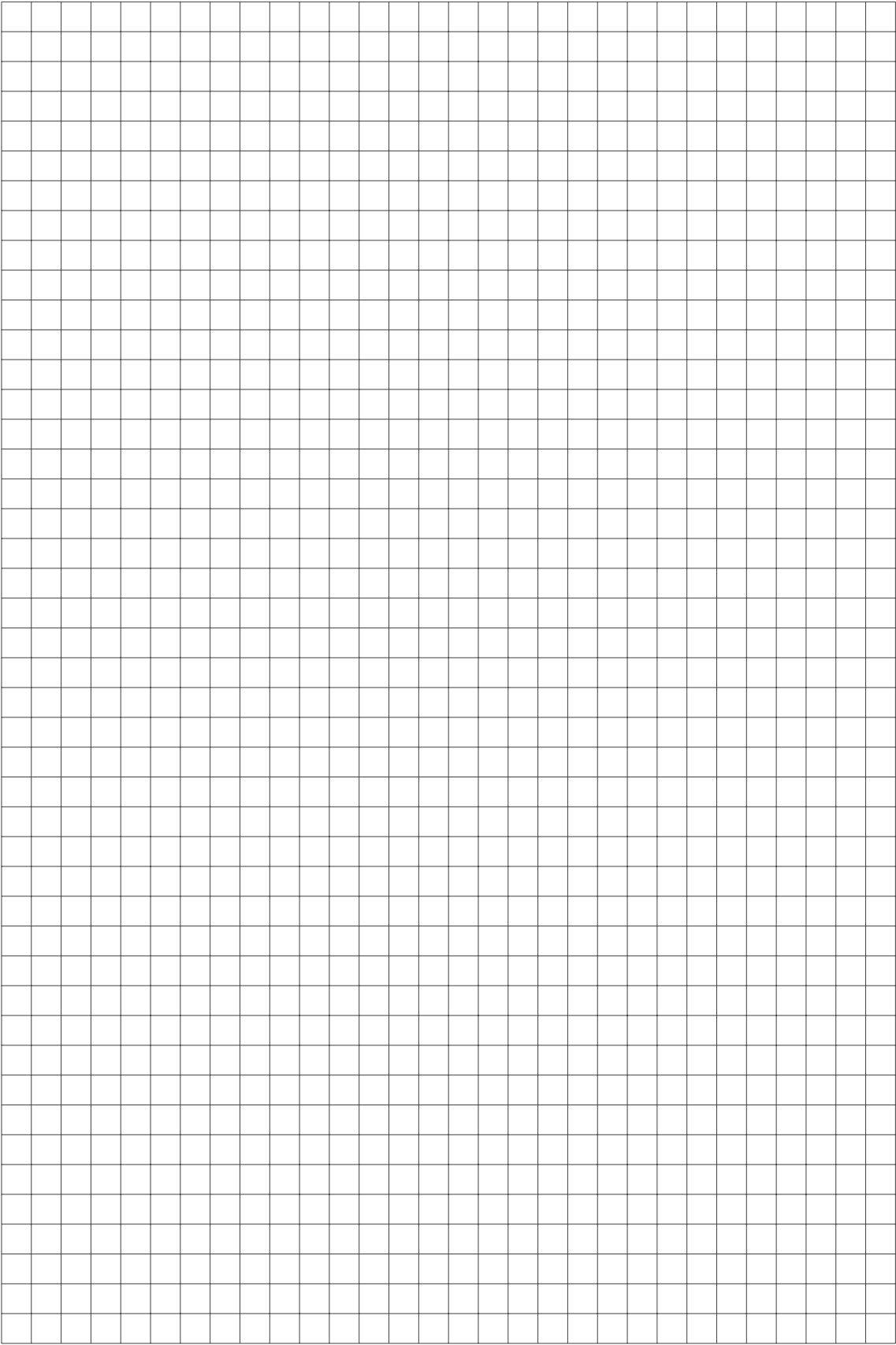
$$\text{Saying } \log_b a = c \quad \text{is the same as saying } b^c = a.$$

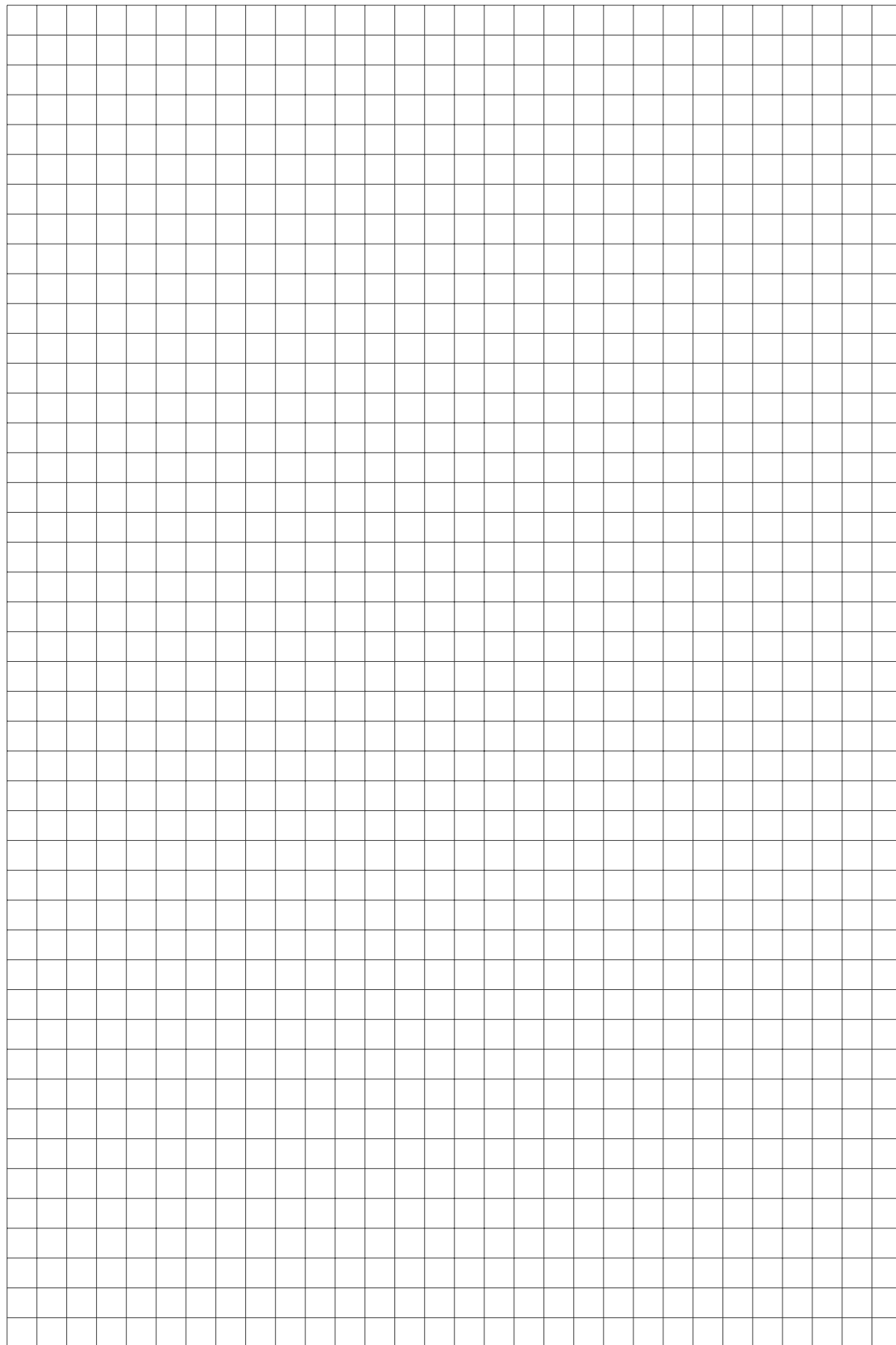
6.8 Student learning outcomes

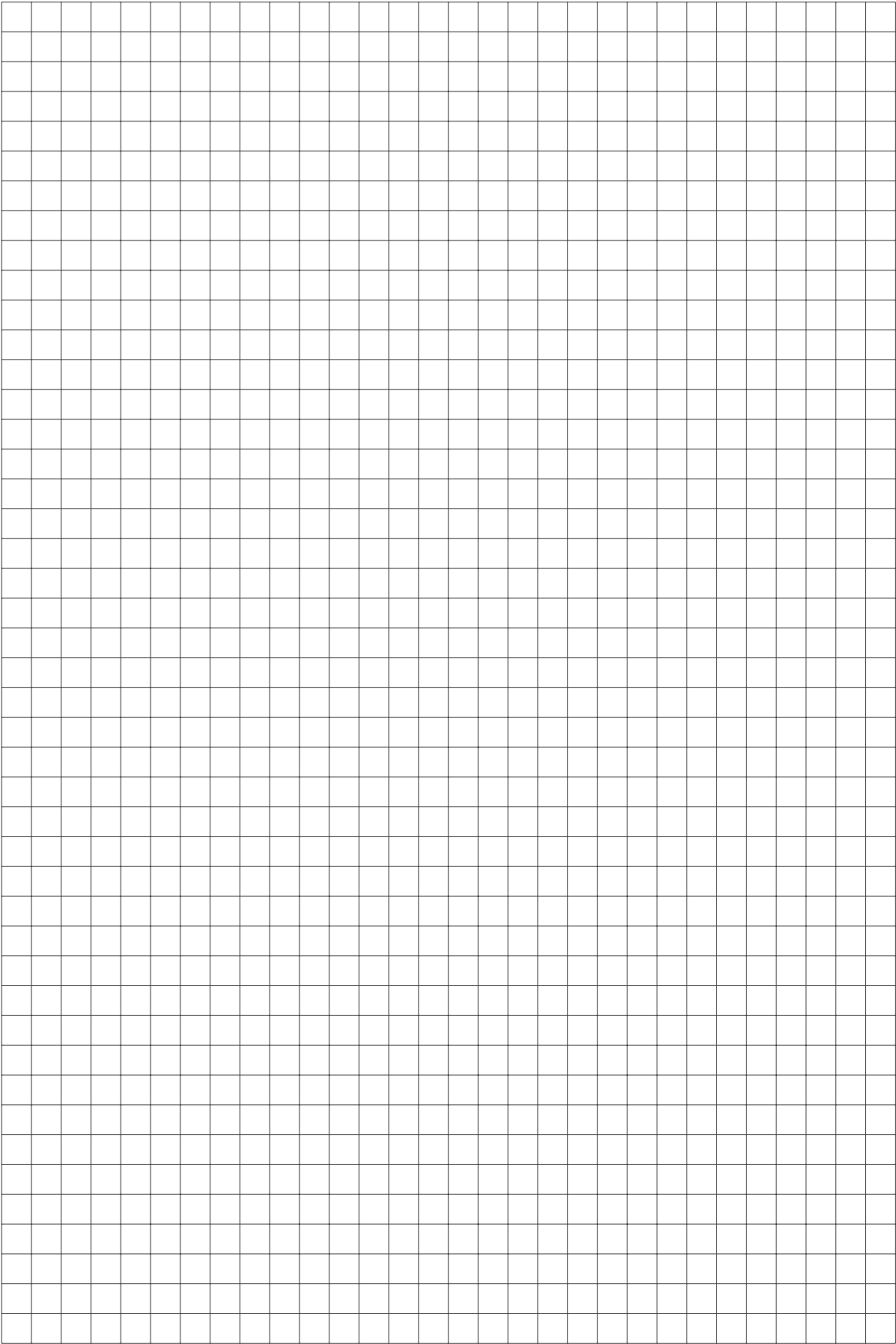
1. Students will be able to recognize exponential functions from graphs, equations, tables and verbal descriptions.
2. Students will understand that the exponential function increases or decreases faster than any linear (or polynomial) function.
3. Students will be able to explain how the rules of exponents arise and apply them to simplify various expressions.
4. Students will be able to use a table and graph to calculate the inverse of exponential functions.

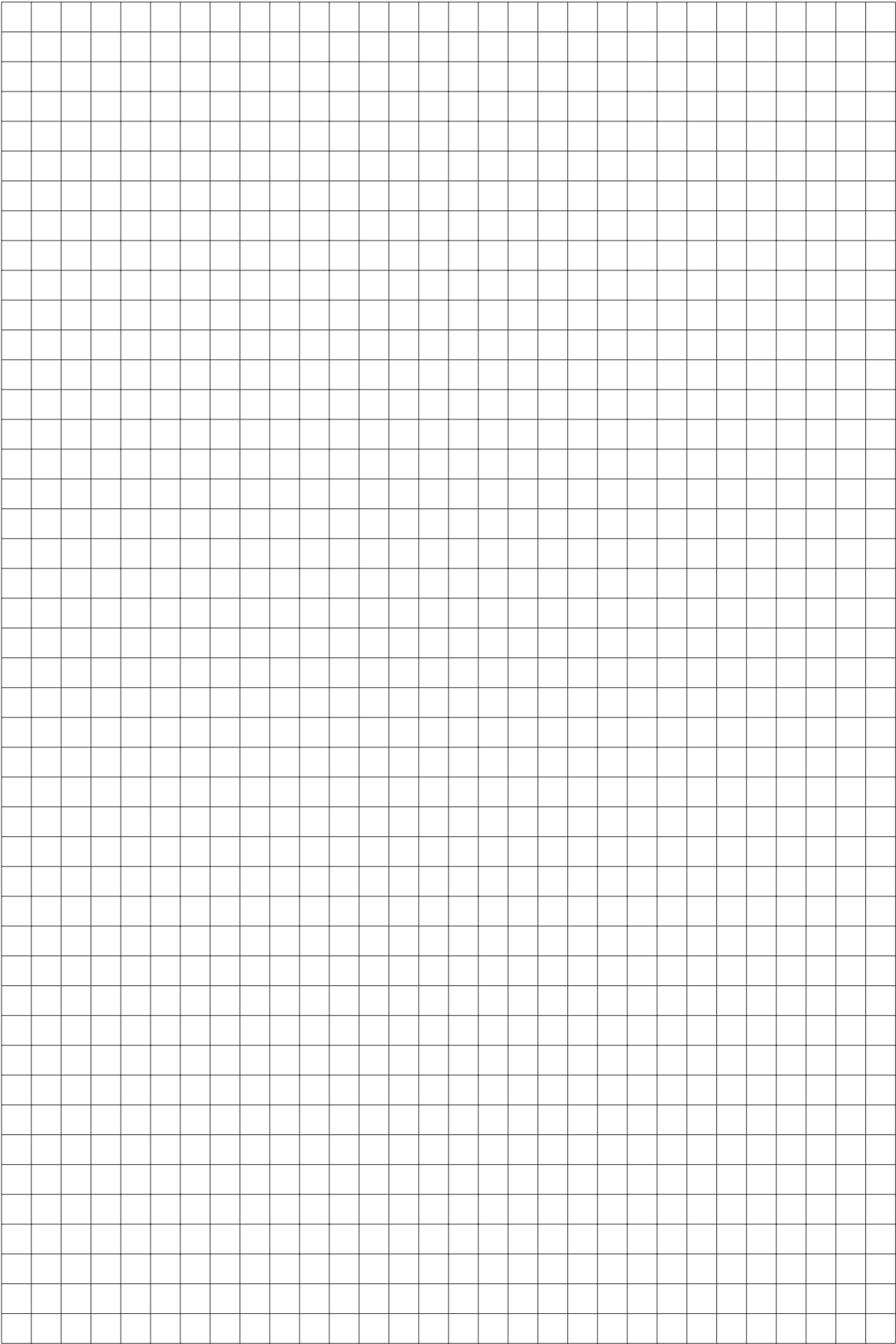
Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.

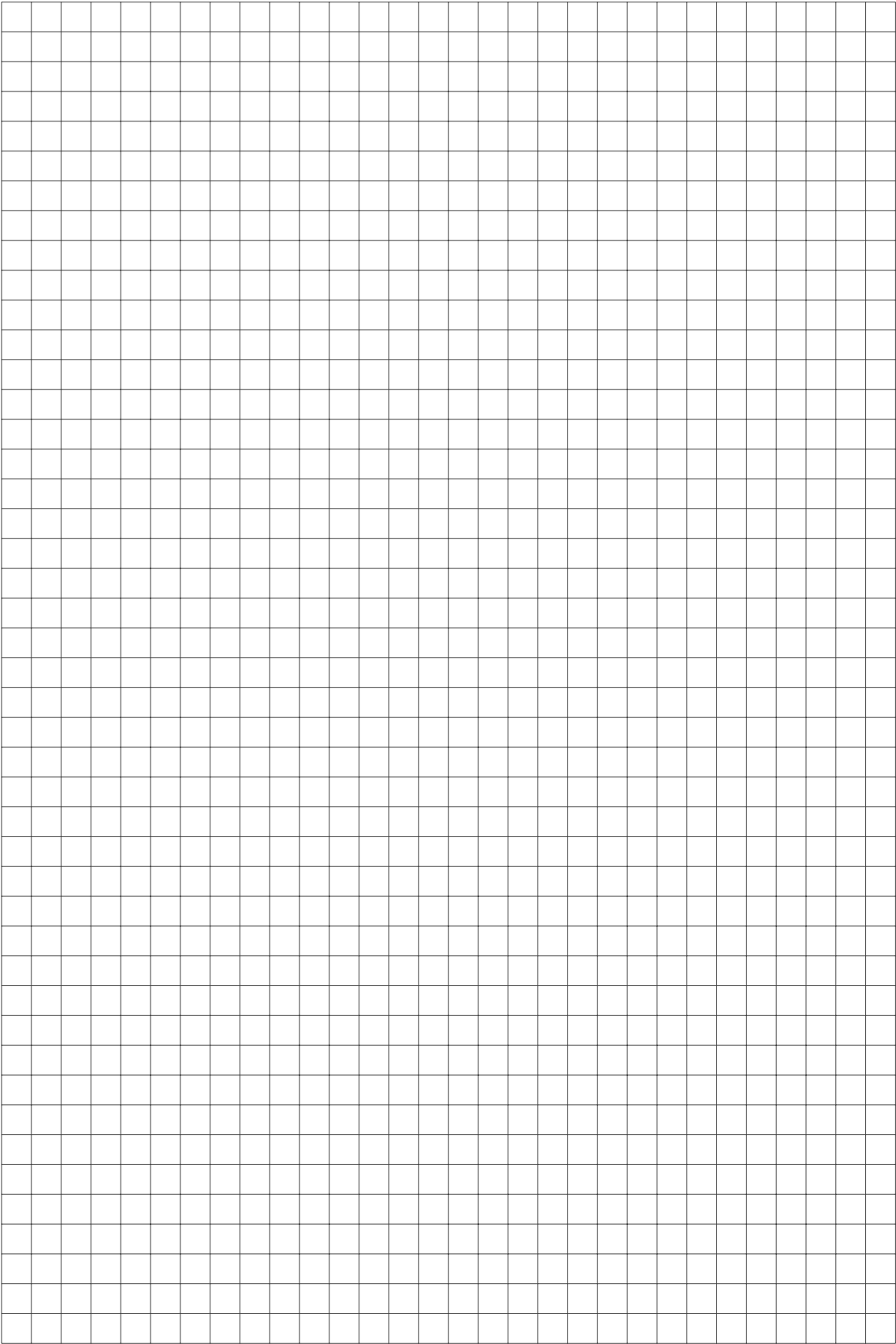
7 Notes

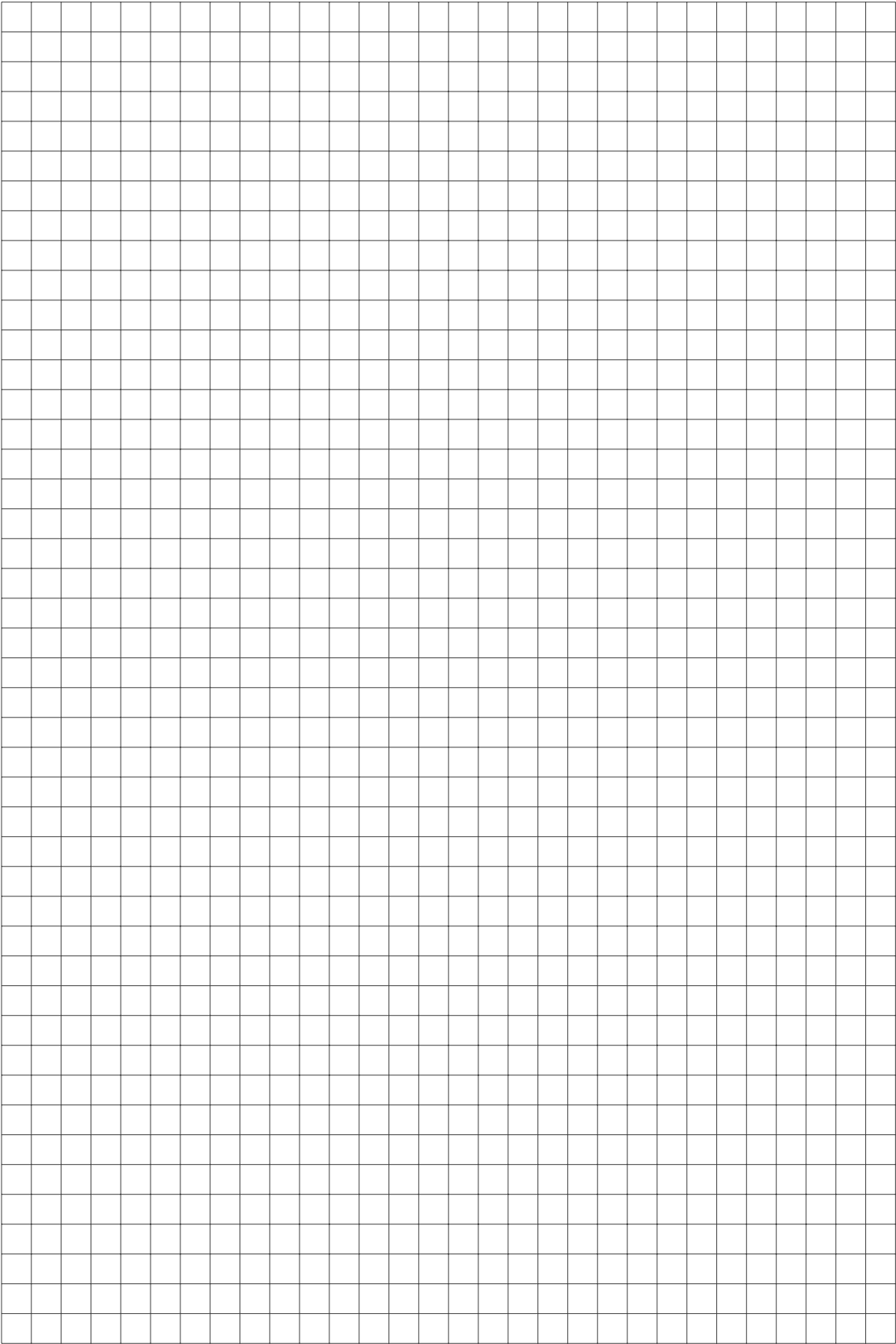






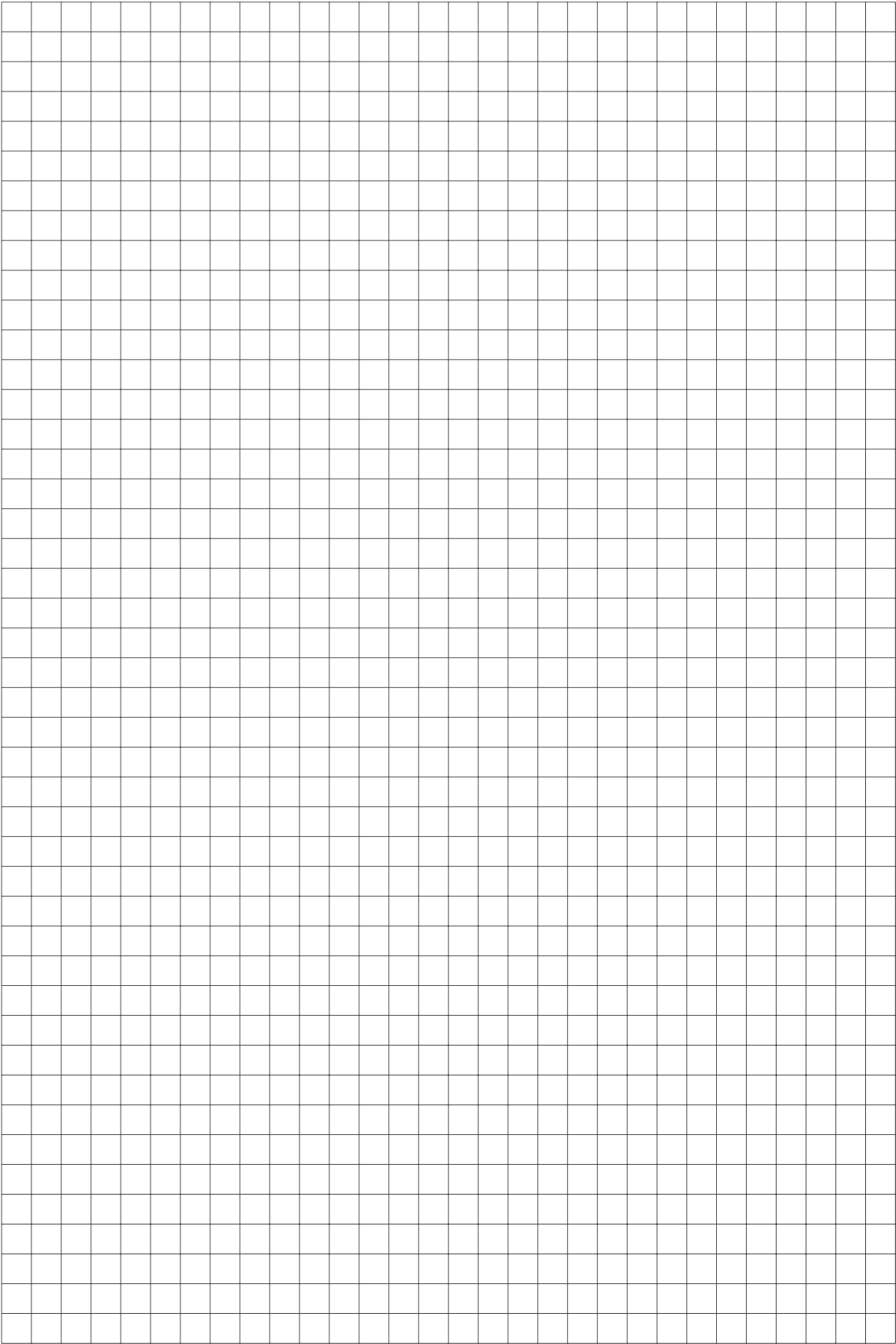




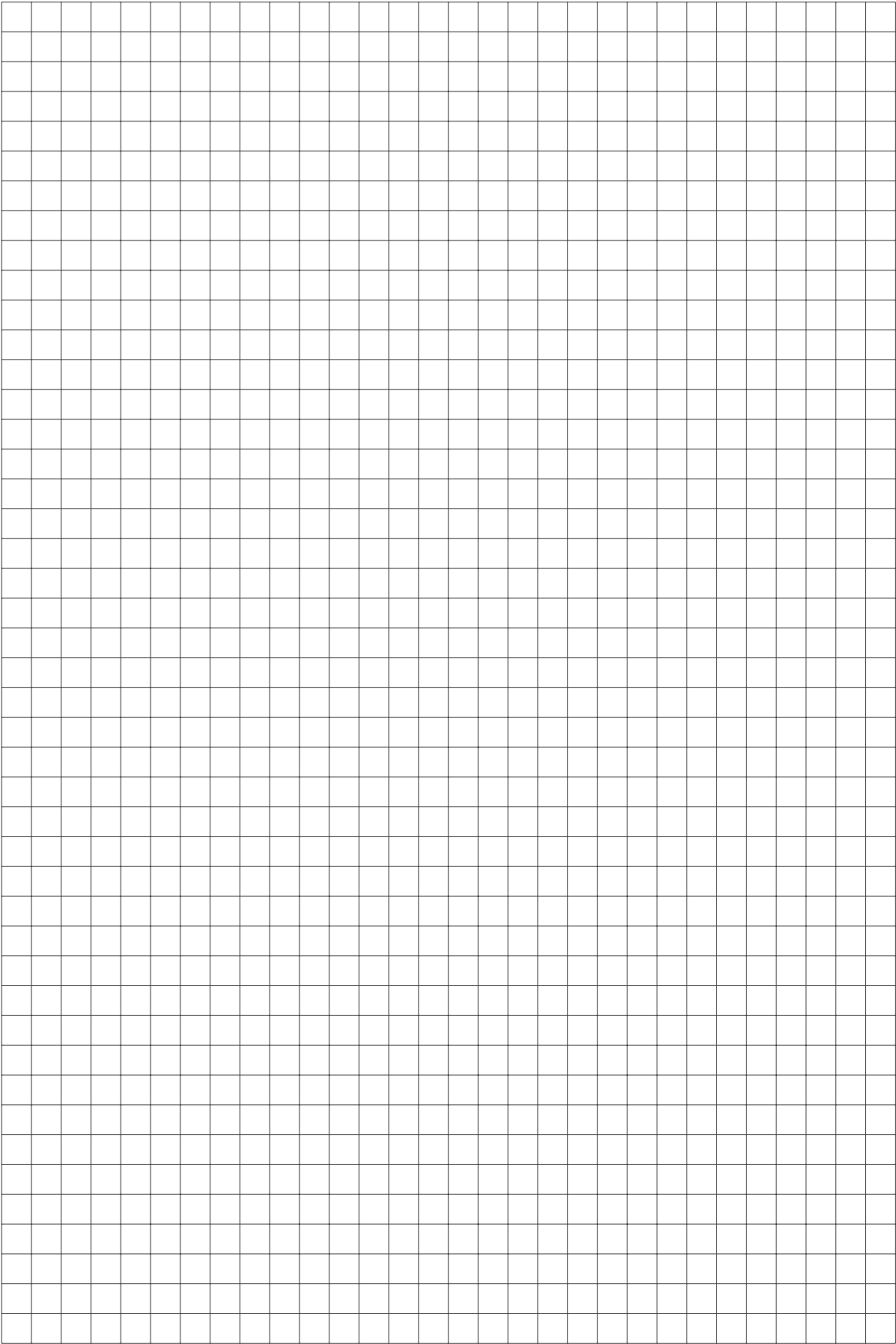




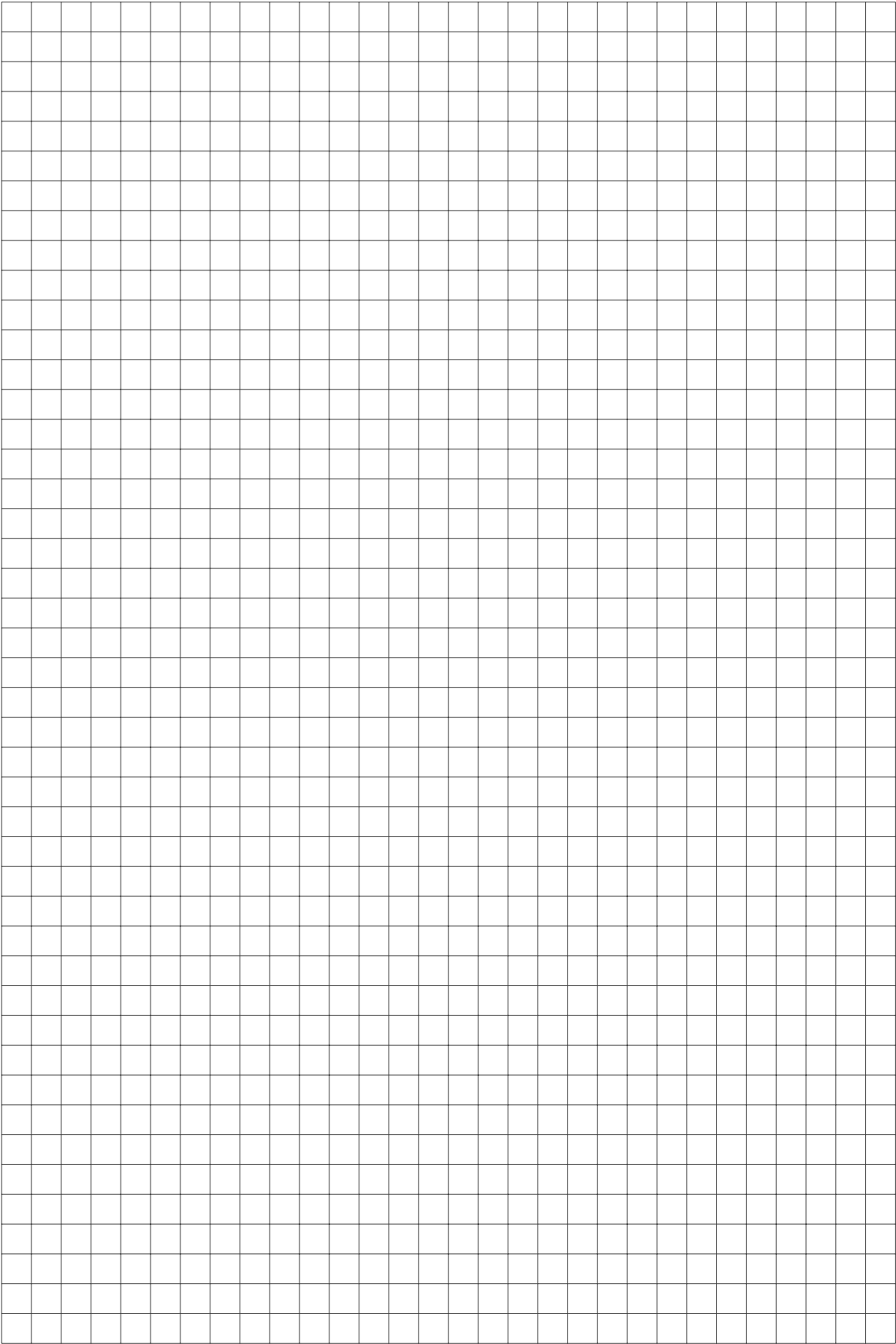










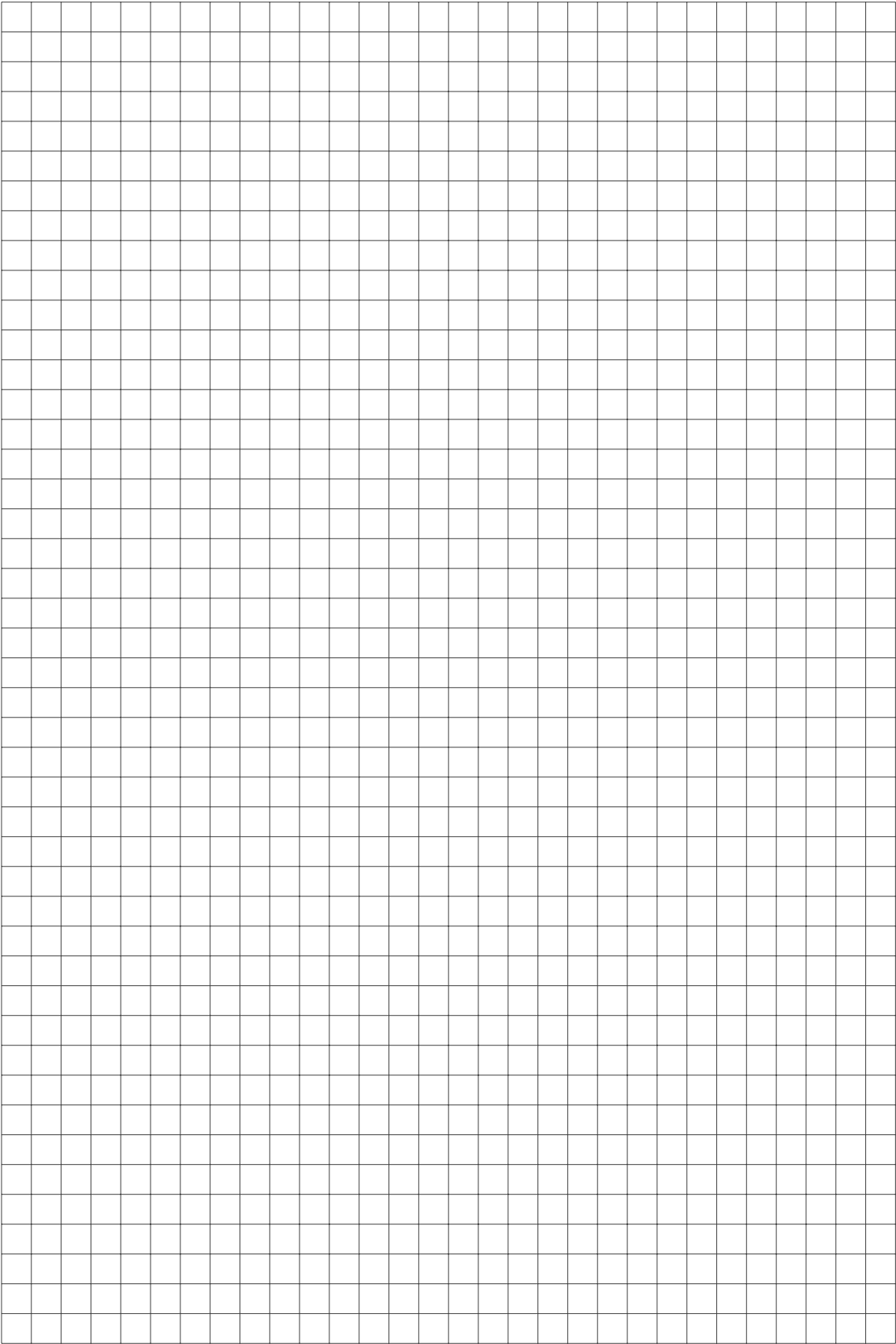


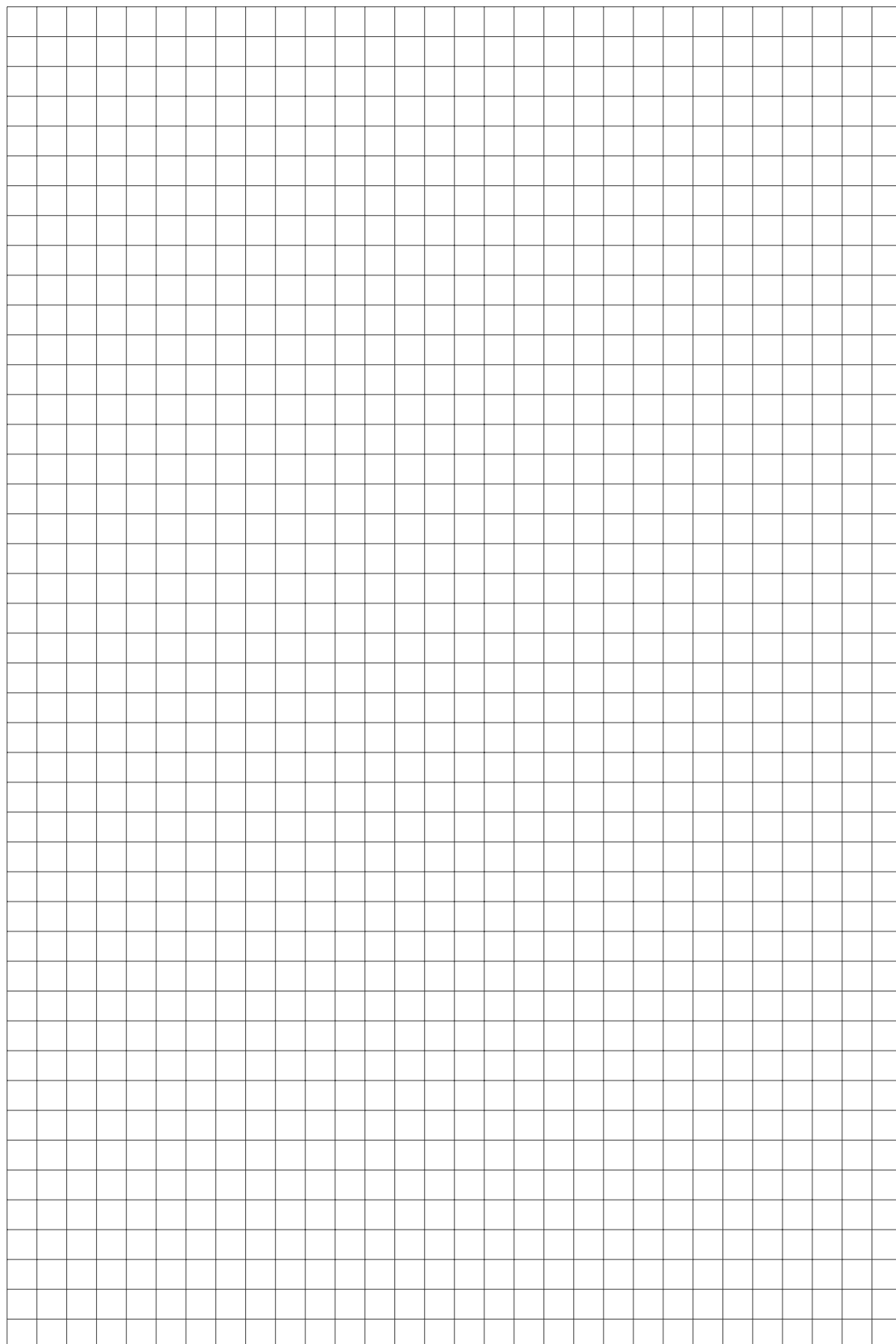


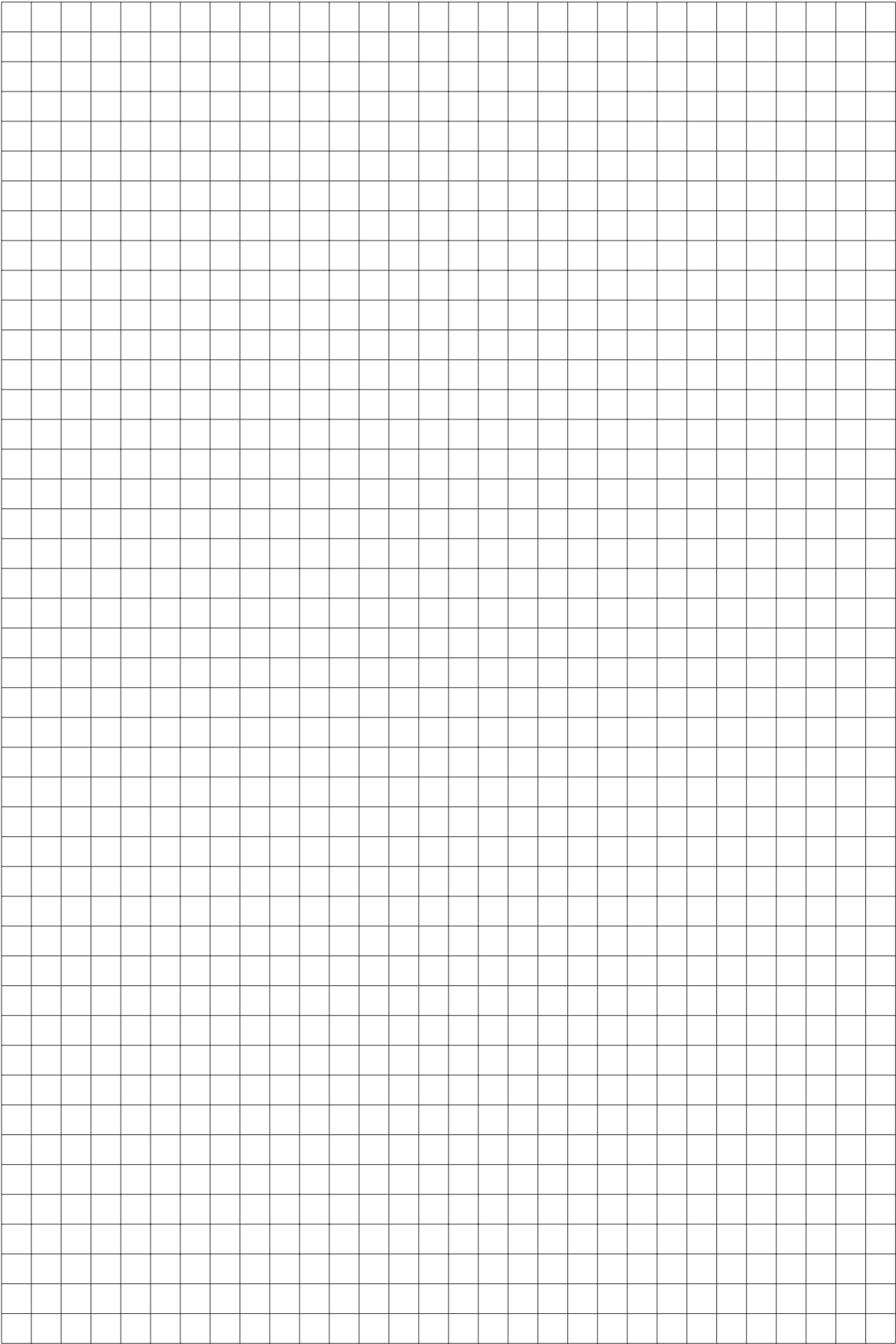


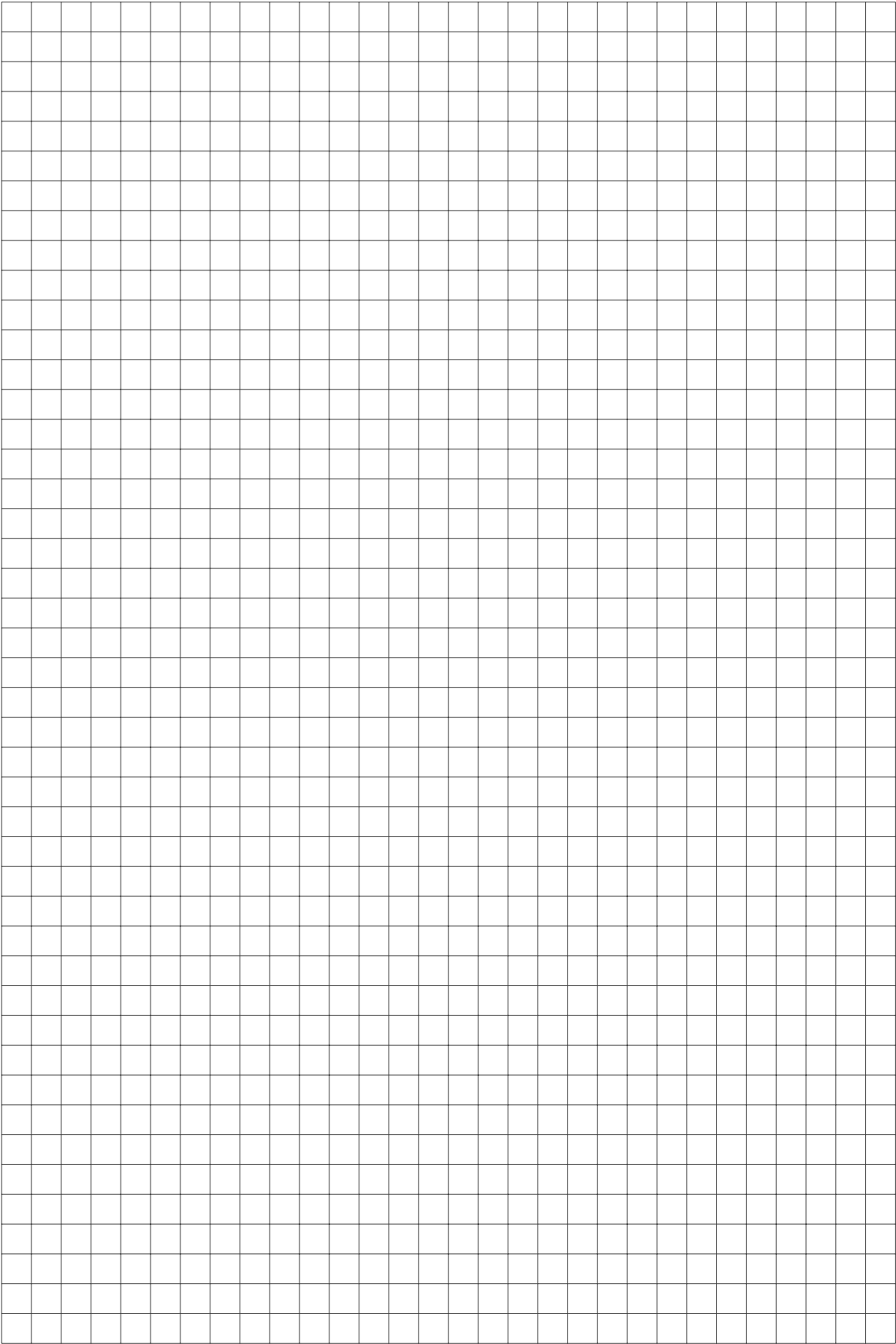












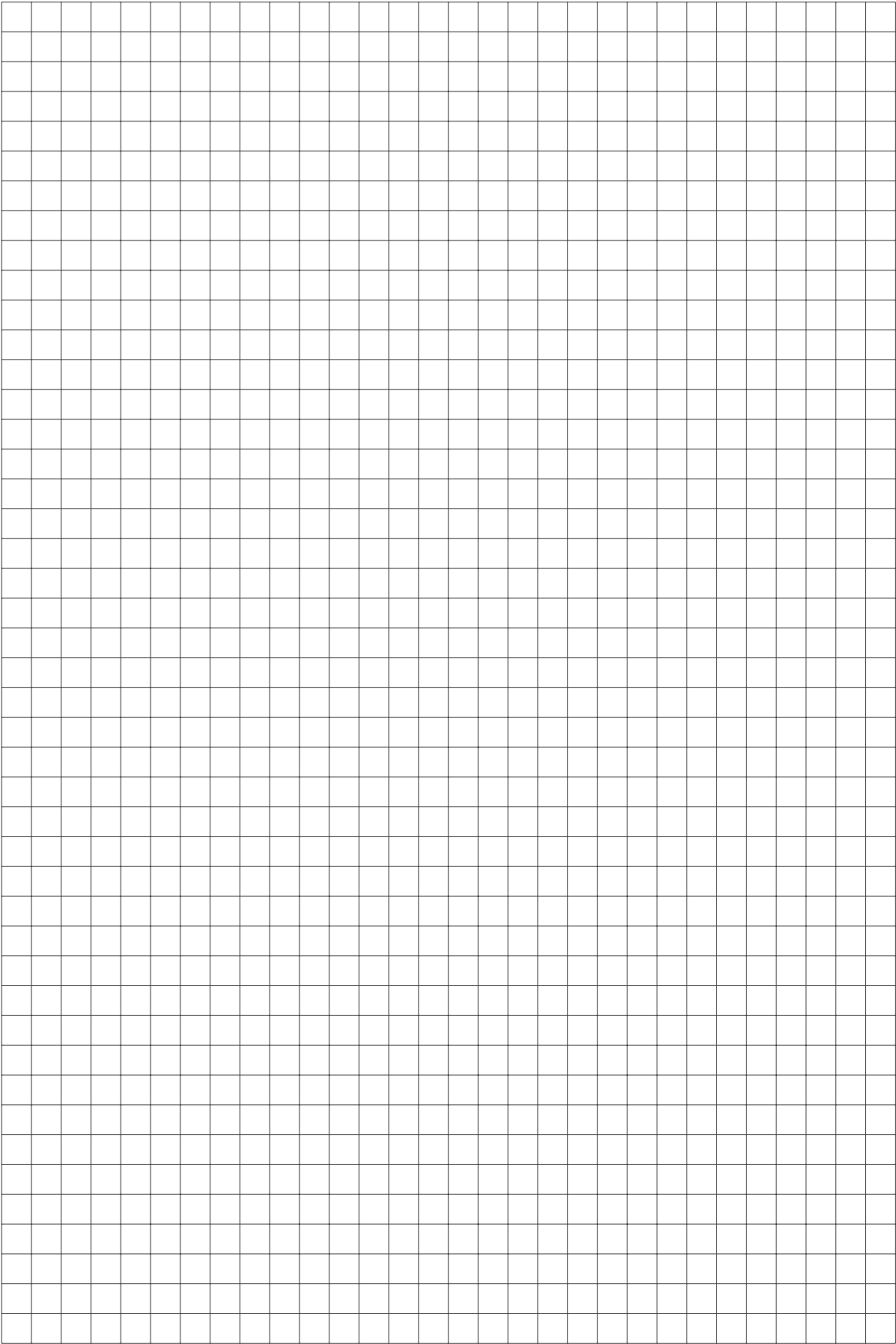




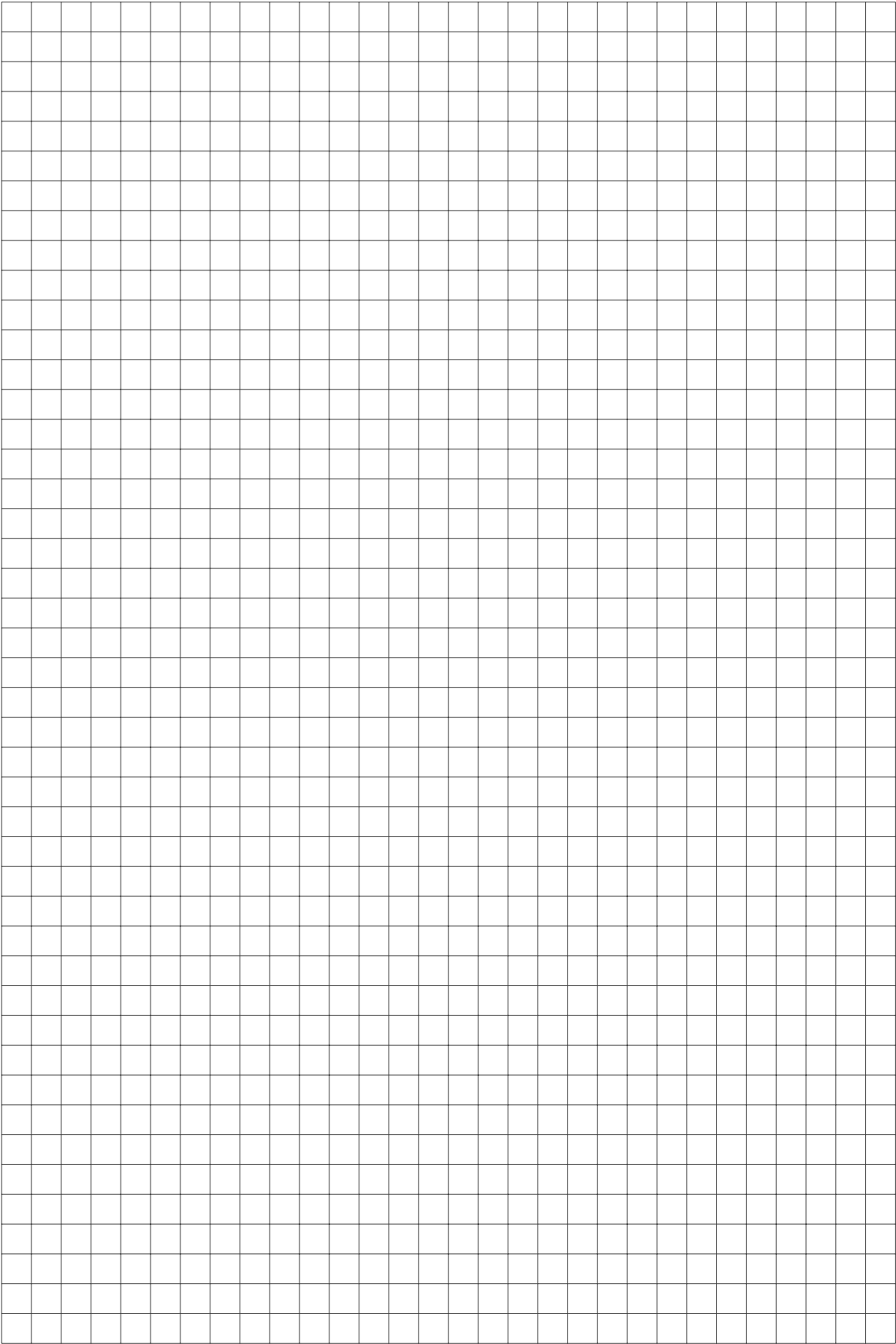


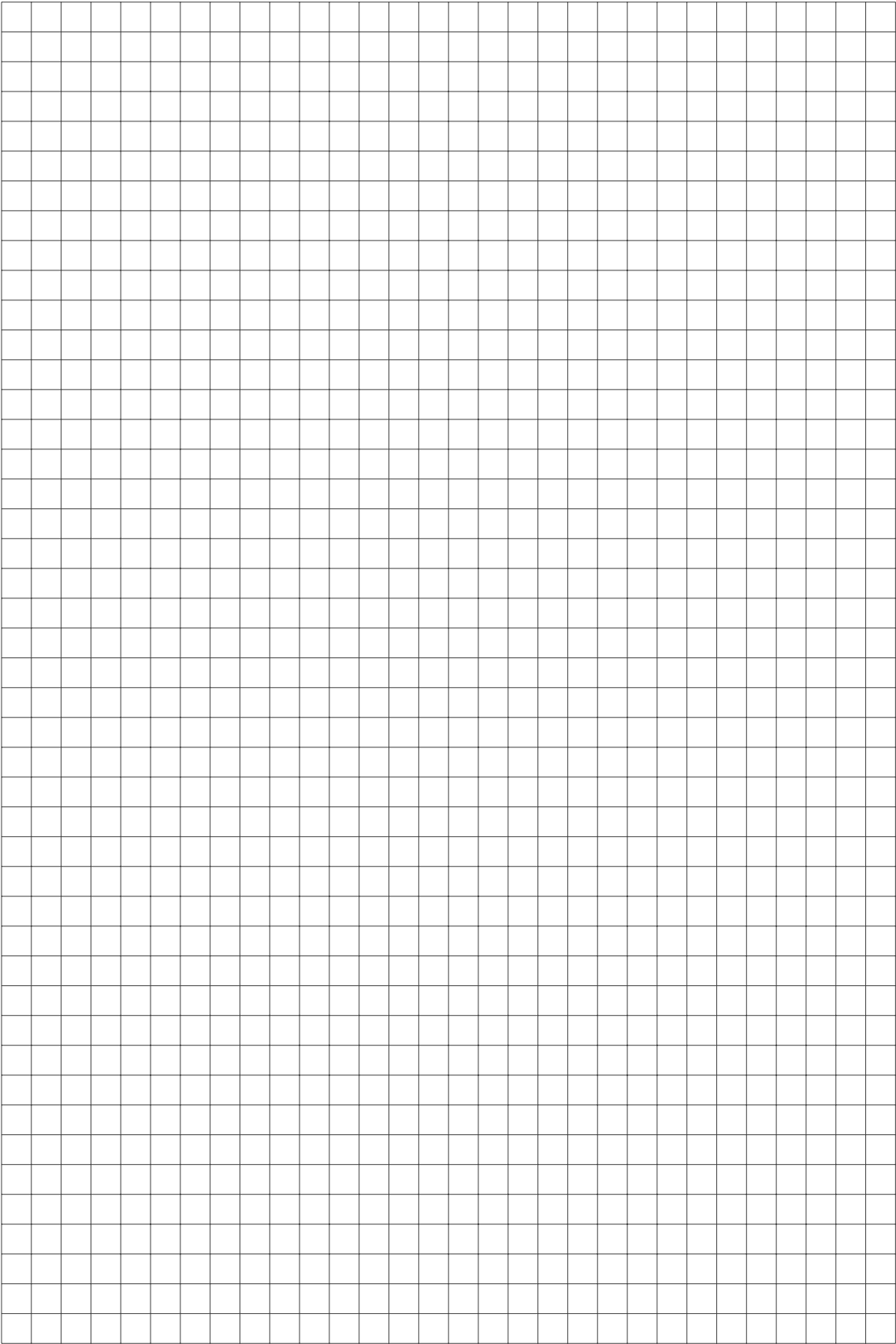


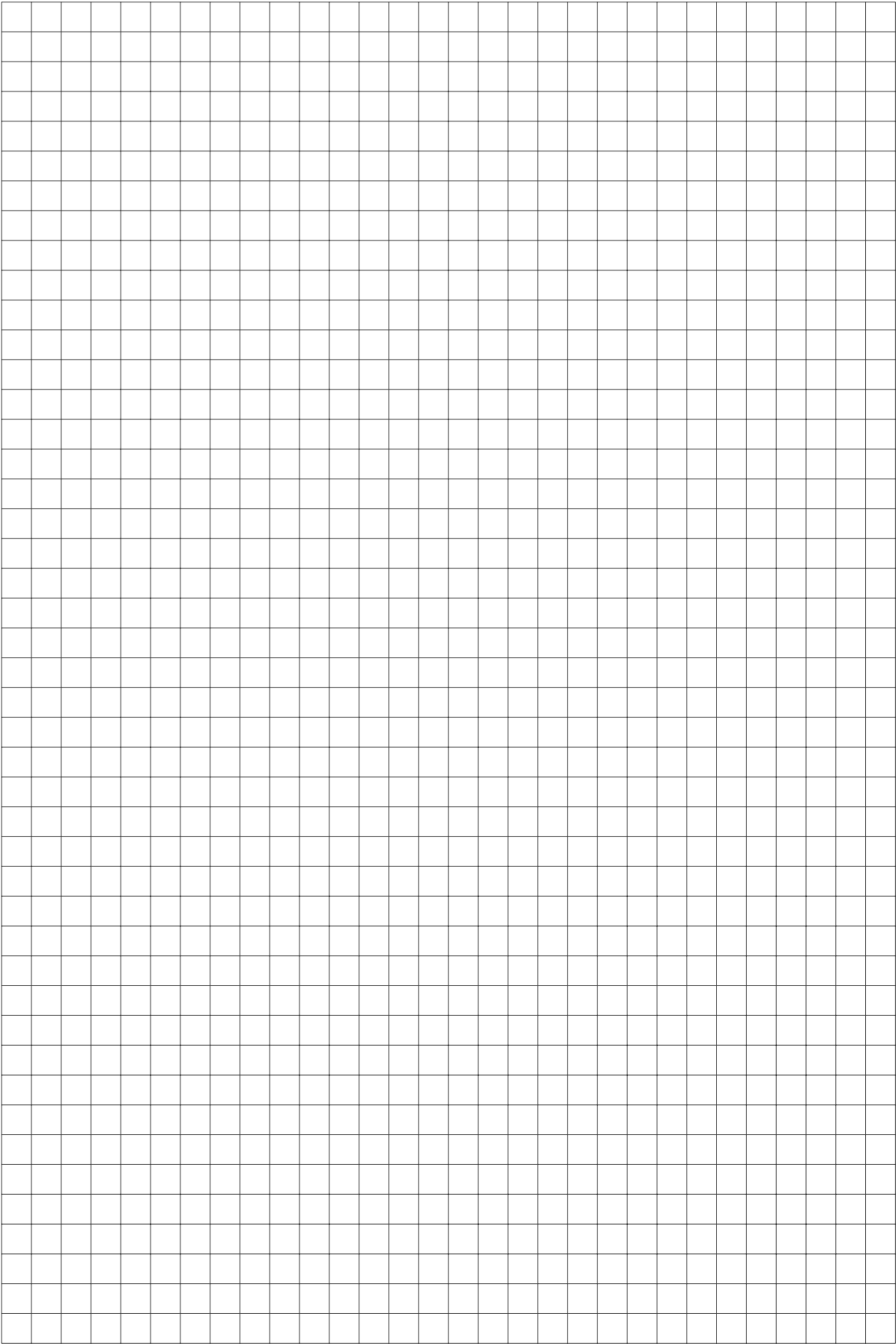




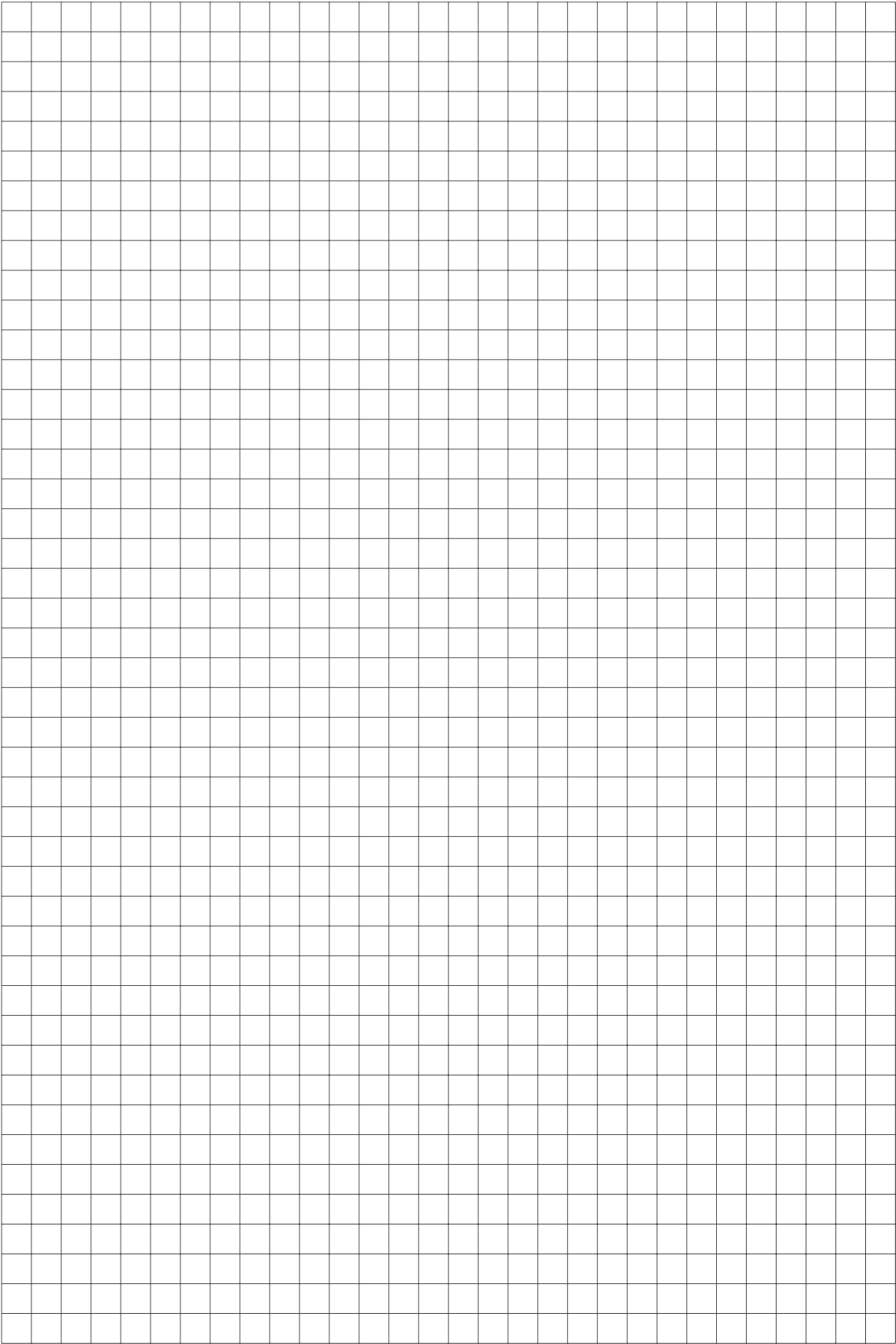




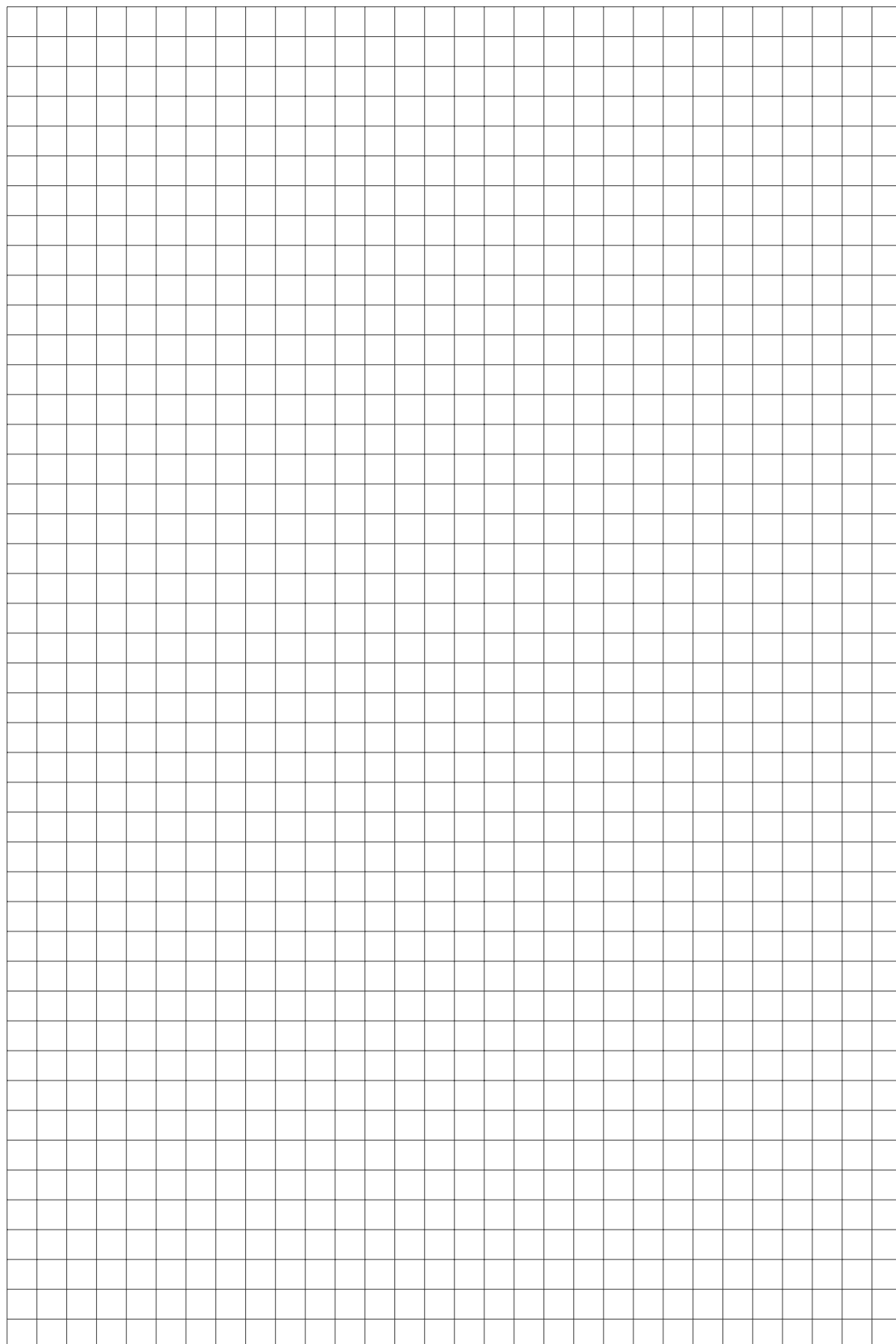




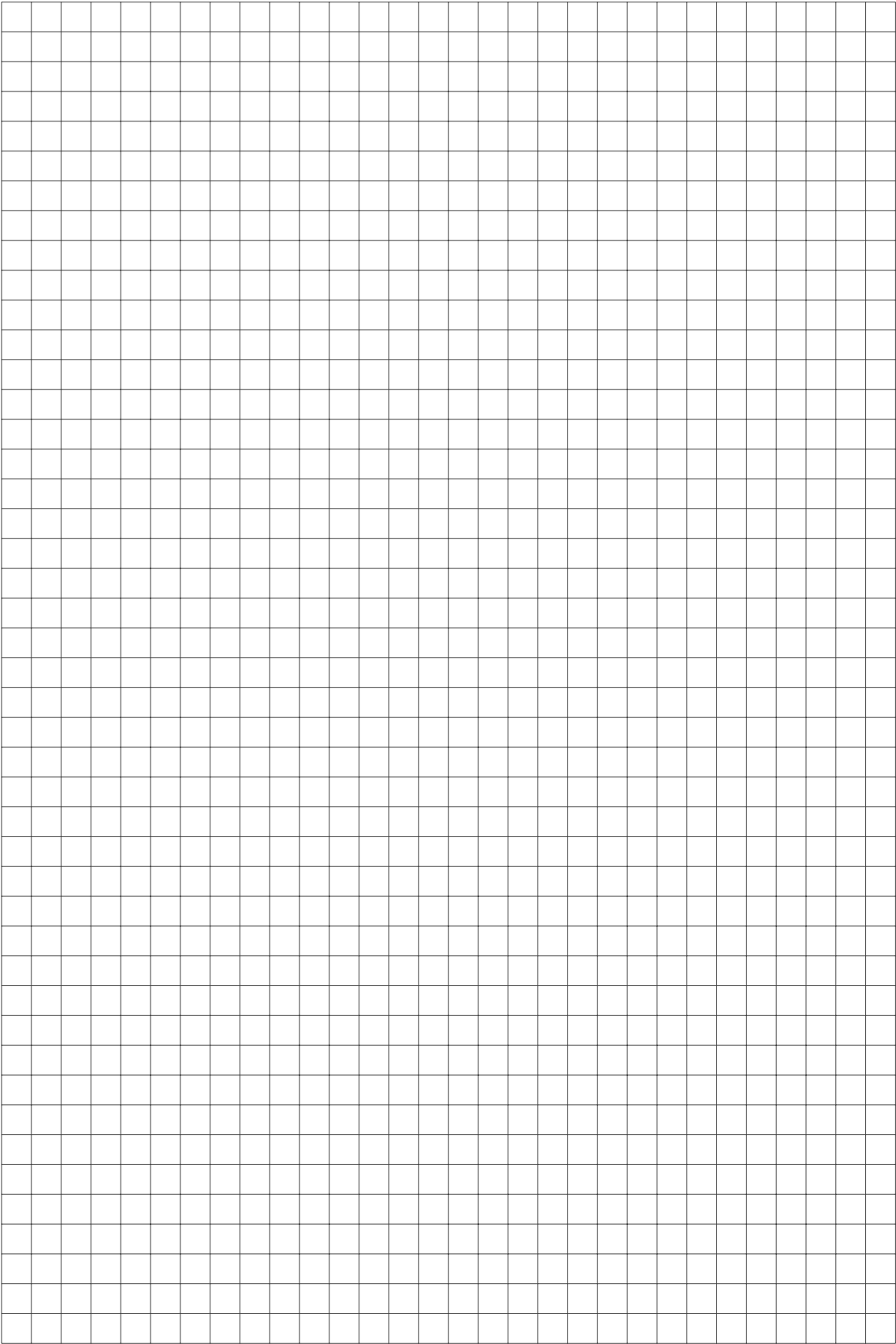


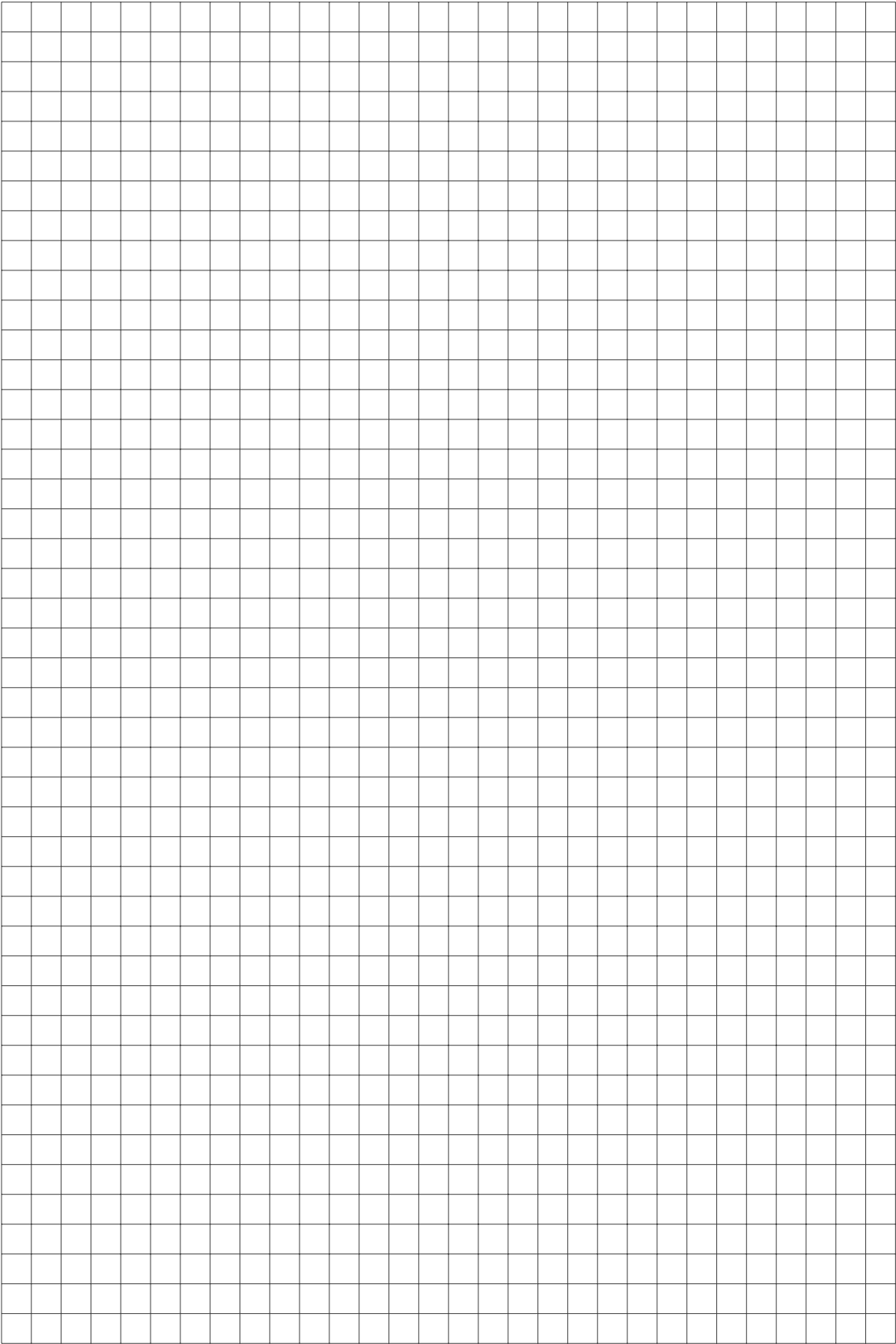






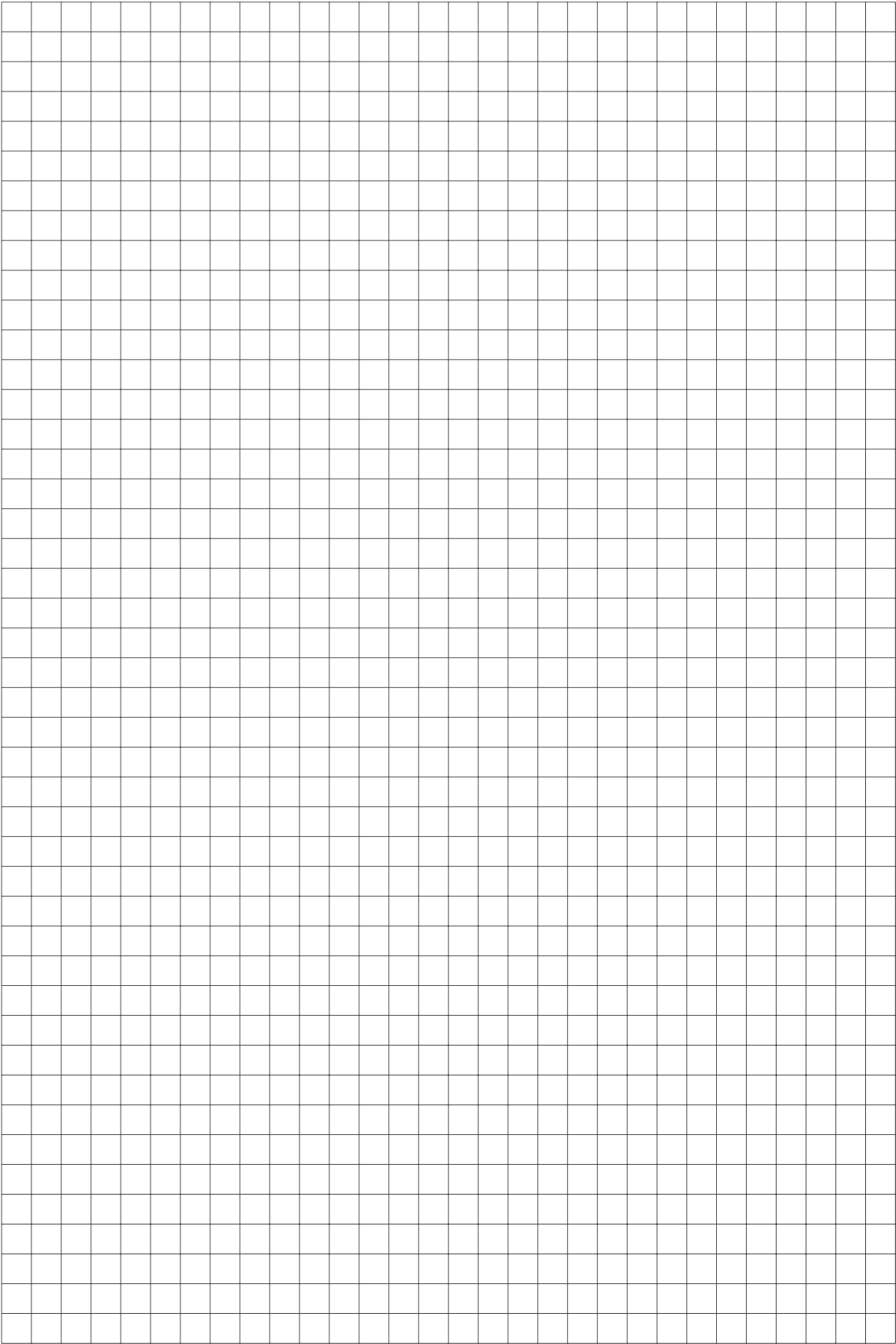








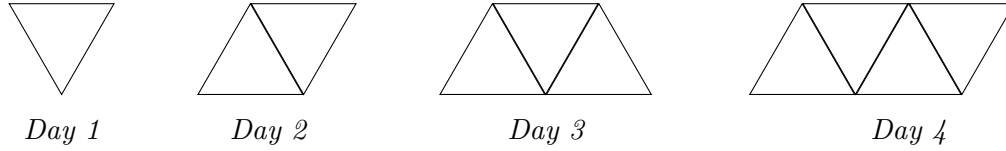




8 Handouts

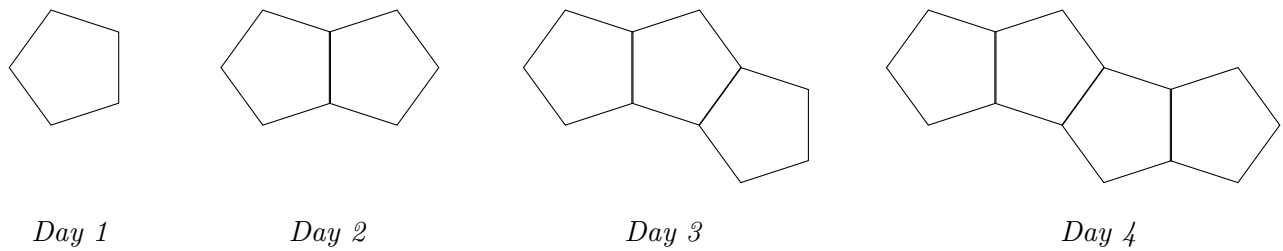
8.1 Visual Patterns

Exercise 1 Look at the pattern below and answer the questions:



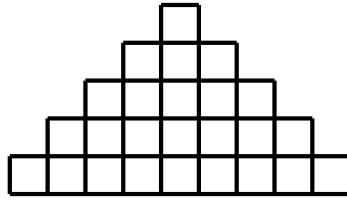
- Describe the pattern that you see in the sequence of figures above.
- Draw the figure that would appear on the fifth day.
- The figure on the second day requires 5 line segments. How many line segments are needed on the fifth day?
- How many line segments are needed on the 25th day?

Exercise 2 Look at the pattern below and answer the questions:



- Describe the pattern that you see in the sequence of figures above.
- Assuming the pattern continues in the same way, draw the figure that occurs on the fifth day.
- The side length of each pentagon is 1. What is the perimeter of the figure on fifth day?
- What is the perimeter of the figure on day 30?

Question 8.1 Look at the figure below and answer the questions that follow.



- a. How many squares are in the top row?
- b. How many squares are in the second row?
- c. How many squares are in the fourth row?
- d. If the figure were extended indefinitely forever, how many squares would be in the n^{th} row?
- e. How many unit squares are in the first row? (a unit square is the smallest one in the picture)
- f. How many unit squares are in the first two rows?
- g. How many unit squares are in the first n rows?
- h. What is the sum of the first n odd numbers?

8.2 Tiling a Pool

Exercise 3 *A cafeteria in a school has square tables where students can eat lunch in groups of four. If six students want to eat lunch at the same table, then they can push two tables together to accommodate their group; even larger groups can be handled by joining together more tables in a straight line.*

- a. Draw diagrams representing this sequence.
- b. Construct a sequence that models this situation and fill in a table for the first several members of the sequence.
- c. If possible, describe both recursive and explicit rules for the sequence.
- d. Draw a graph representing this situation.
- e. Find the value of the 32nd member of the sequence. What does the 32nd term in the sequence tell us?
- f. Is there a member of the sequence whose value is 324? How do you know?

Exercise 4 For each arithmetic sequence below, find the common difference, and write the n th term in terms of n :

a. $2, 7, 12, 17, 22, \dots$

b. $2 + 1 \cdot 5, 2 + 2 \cdot 5, 2 + 3 \cdot 5, \dots$

c. $y + 1 \cdot 5, y + 2 \cdot 5, y + 3 \cdot 5, \dots$

d. $y + 1 \cdot x, y + 2 \cdot x, y + 3 \cdot x, \dots$

Exercise 5 The following sequences are arithmetic. Find the missing terms.

a. $7, \square, 15, \dots$

b. $10, \square, \square, -14, \dots$

c. $15, \square, \square, \square, \square, 15, \dots$

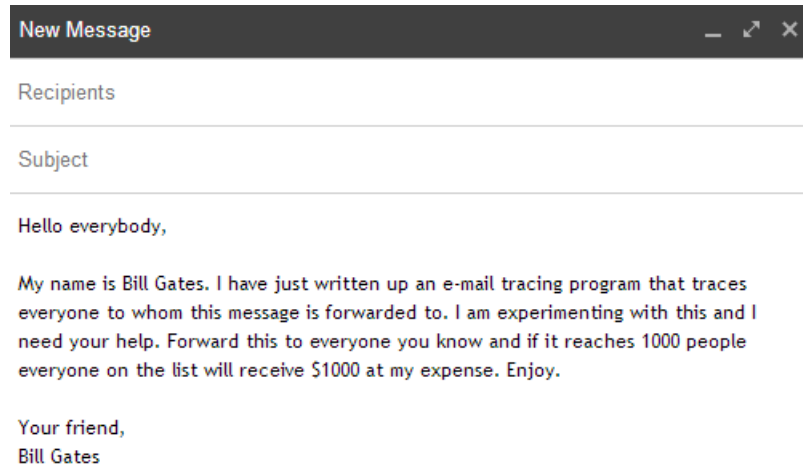
Exercise 6 Here is a picture of a candy machine at the Gateway Mall (400 W 100 S). Each time a customer inserts a quarter, 13 candies come out of the machine. The machine holds 14 pounds of candy. Each pound of m&m's contains 220 individual candies.



- a. How many candies are in the machine when it is full?
- b. How many candies are in the machine after 1 customer? How many are in the machine after 2 customers?
- c. The amount of candies in the machine after n customers can be modeled using an arithmetic sequence c_n . Write down the explicit formula for c_n . What is the relationship between c_n and c_{n+1} ?
- d. When does our model stop making real world sense?
- e. To avoid theft, the owners of the machine don't want to let too much money collect in the machine, so they take all the money out when they think the machine has about \$30 in it. The tricky part is that the store owners can't tell how much money is actually in the machine without opening it up, so they choose when to remove the money by judging how many candies are left in the machine. About how full should the machine look when they take the money out? How do you know?

8.3 Geometric Sequences

Exercise 10 *The following message began circulating on the Internet around 21 November 1997:*



The first step in the chain has Bill Gates sending the email out to his 8 closest friends. Assume that everyone that receives the email forwards it to 10 people. So the second step in the chain has 8 friends each sending out 10 emails, therefore a total of 80 people receive the email in the second step.

- a. How many people receive the email in the third step?
- b. How many people receive the email in the fourth step?
- c. Let's model how many people receive the email on the n -th day using a geometric sequence e_n . Write an explicit formula for e_n .
- d. How is e_n related to e_{n-1} ?
- e. After the second step a total of 88 people received the email. How many total people received the email after the third step?
- f. How many steps does it take the chain letter to reach 1000 recipients?
- g. If after the fifth step Bill Gates realizes it is getting out of control and puts an end to the chain letter, how much money is he on the hook for?

Exercise 11 *The following sequence is a geometric sequence: 2, 10, 50, 250, ...*

- a. What is the common ratio?
- b. What is the n -th term?
- c. What is the recursive definition for this sequence?
- d. Graph the first 5 terms.

Exercise 12 *For each geometric sequence below, find the common ratio, write the formula for n -th term, and find 12th term of the sequence:*

a. 2, 6, 18, 54, ...

b. $2 \cdot 3, 2 \cdot 9, 2 \cdot 27, \dots$

c. 25, 5, 1, $\frac{1}{5}, \dots$

d. $y \cdot x, y \cdot x^2, y \cdot x^3, \dots$

Exercise 13 *A culture of bacteria doubles every 2 hours. If there are 500 bacteria at the beginning, how many bacteria will there be after 24 hours?*

Exercise 14 *Each sequence below is either arithmetic or geometric. First decide if the sequence is arithmetic or if it is geometric, find the next two terms of the sequence, find 83 term of the sequence, then give the explicit formula for each sequence:*

a. $54, 18, 6, \dots$

b. $2 \cdot 3, 2 \cdot 3 + 4, 2 \cdot 3 + 8, \dots$

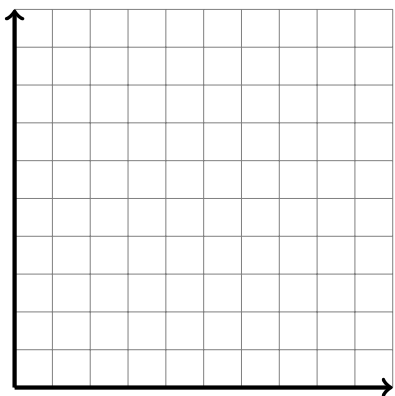
c. $-3, -4, -5, -6, -7, -8, \dots$

d. $25, -5, 1, -\frac{1}{5}, \dots$

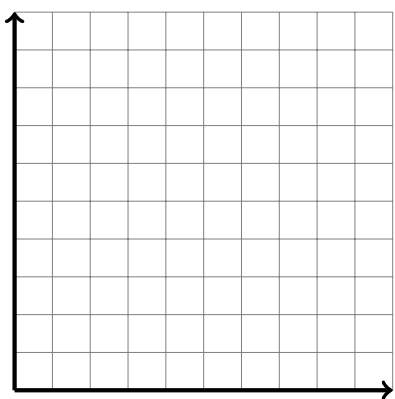
e. $ab, a^2b^3, a^3b^5, \dots$

Exercise 15 *The following sequences are neither geometric or arithmetic. Graph the terms given for each sequence and describe how the graph shows that the sequences are not arithmetic.*

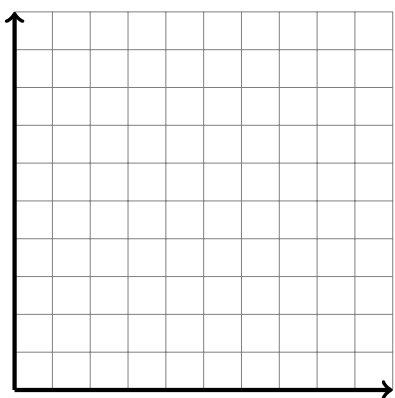
a. $5, 8, 13, 20, 29, 40, 53, 68, \dots$



b. $7, 13, 23, 37, 55, \dots$



c. $-2, 7, 22, 43, 70, \dots$



Exercise 16 *The terms $140, a, \frac{45}{28}$ are the first, second and third terms, respectively, of a geometric sequence. If a is positive, what is the value of a ?*

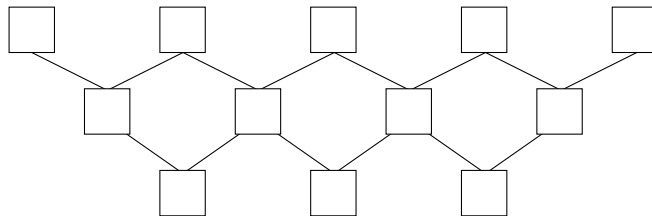
8.4 Counting High-Fives

Exercise 17 Let $\{t_1, t_2, \dots\}$ be a sequence where the n -th term is given by $4n^2 - n + 4$.

1. Fill in the following table:

n	t_n
1	
2	
3	
4	
5	
6	
7	

2. Calculate a few second differences of this sequence. To help you organize your thoughts, use the following diagram:

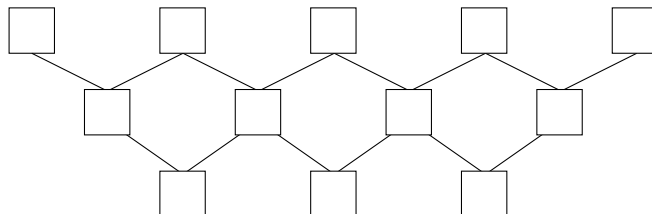


Exercise 18 Let $\{q_1, q_2, \dots\}$ be a sequence where the n -th term is given by $3n^2 - n + 1$.

1. Fill in the following table:

n	q_n
1	
2	
3	
4	
5	
6	
7	

2. Calculate a few second differences of the hand shake sequence. To help you organize your thoughts, use the following diagram:



Exercise 19 Let S_n be the sum of the first n positive integers, for example $S_5 = 1 + 2 + 3 + 4 + 5$.

- a. Fill in the following table:
- b. Is this sequence similar to another sequence we have studied? Use this information to write down an explicit formula for S_n .

n	S_n
1	
2	
3	
4	
5	
6	
7	

The following captures a beautiful argument attributed to Gauß to compute S_{100} .

$$\begin{aligned}
 S_{100} &= 1 + 2 + 3 + \dots + 98 + 99 + 100 \\
 &= (1 + 100) + (2 + 99) + (3 + 98) + \dots + (50 + 51) \\
 &= \underbrace{101 + 101 + 101 + \dots + 101}_{\text{There are fifty 101's}} = 50 \cdot 101 = 5050
 \end{aligned}$$

- c. Which algebraic rules are being used when we write the second equality?
- d. Use this argument to verify our formula for S_n .

Exercise 20 Let s_1, s_2, s_3, \dots be the sequence such that s_n is the sum of the first n even numbers, for example: $s_5 = 2 + 4 + 6 + 8 + 10 = 30$.

- a. Fill in the following table

n	1	2	3	4	7	n
s_n	2					

- b. Is s_n a geometric or arithmetic sequence?
- c. What is the relationship between s_n and s_{n-1} ?
- d. Write down the algebraic expression for s_n .

8.5 More Sequences

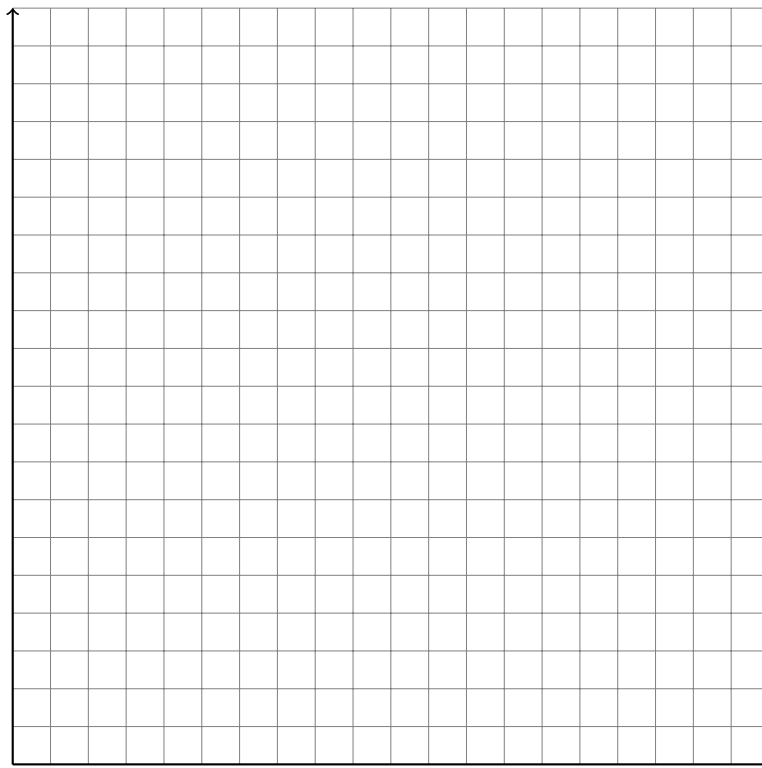
Exercise 21 *Each spring, a fishing pond is restocked with fish. That is, the population decreases each year due to natural causes, but at the end of each year, more fish are added. Here's what you need to know.*

- *There are currently 3000 fish in the pond.*
- *Due to fishing, natural death, and other causes, the population decreases by 20% each year.*
- *At the end of each year, 1000 new fish are added to the pond.*

Using the information provided we will try to understand what happens to this population of fish in the long term.

- a. Fill in the table that will contain the amount of fish in the pond during the first 15 years. b. Graph this data.

Year	Number of fish in pond
1	3400
2	
3	
4	
5	
6	
7	
8	
9	
10	
11	
12	
13	
14	
15	



- c. Explain what seems to be happening to the fish.

- d. Let f_n be the number of fish in the pond after n years. What is the relationship between f_n and f_{n-1} ?

- e. Predict approximately how many fish will be in the pond after 40 years.

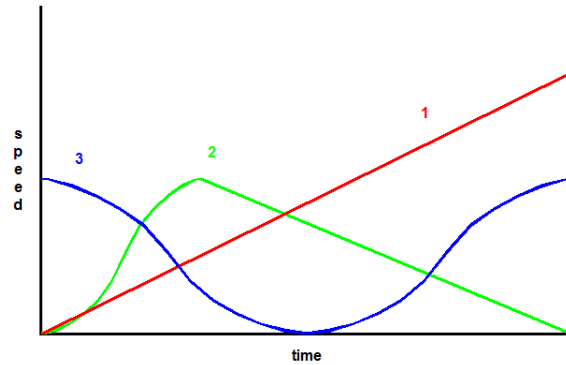
Exercise 22 *Suppose you put 2 cents in a jar today and each day thereafter you triple the amount you put in the previous day. How much would you put in on the 17th day? How big must your jar be?*

Exercise 23 *Is there a sequence that can be claimed to be both arithmetic and geometric?*

Exercise 24 *Miguel was asked to consider the pattern 0, 5, 10, 15, ... and list the next term. Miguel said 24. Can you figure out why Miguel chose that instead of 20?*

8.6 Describing Relationships

Exercise 25 *Imagine you are driving to school. You come to a stop sign and stop, then move on. Which of the graphs depicted below best represents the speed of your car?*



A car stops at a stop sign, then moves on.

Which graph? 1 2 3

- a. Write couple of sentences explaining why the graph you chose describes the situation above.

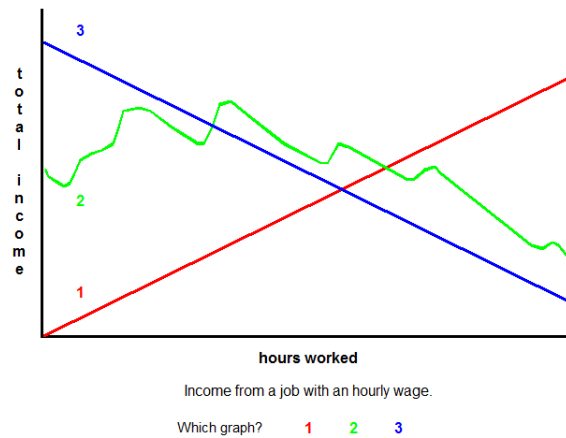
- b. Write couple of sentences explaining what situation you would observe if the car's speed had been described by the other graphs.

- c. Put a scale on your graph. On the horizontal axis place 0-60 second, and on the vertical axis place 0-35 miles per hour.
 - How fast was the car moving after 30 seconds? 45 ? 60?

 - At what time was the car going 20 miles per hour? How many seconds had passed when you were going 0 miles per hour?

 - Complete the following pairs so each belongs to the graph: (,0), (10,), (25,), (,25).

Exercise 26 *If you work an hourly wage job, which graph would best depict your total wage?*



- a. Write couple of sentences explaining why the graph you chose describes the situation above.

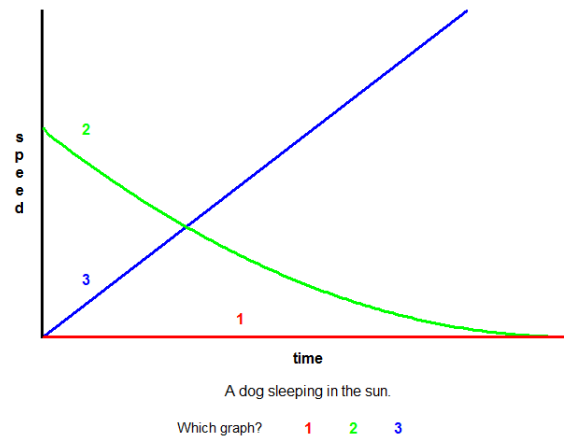
- b. Write couple of sentences explaining what your wages would be like if the other two graphs were the correct graphs.

- c. Put a scale on your graph. On the horizontal axis place 0-8 hours, and on the vertical axis place 0-100 dollars.
 - How much money did you make after 2 hours? After 3 hours?

 - For you to make 50 dollars how long did you have to work?

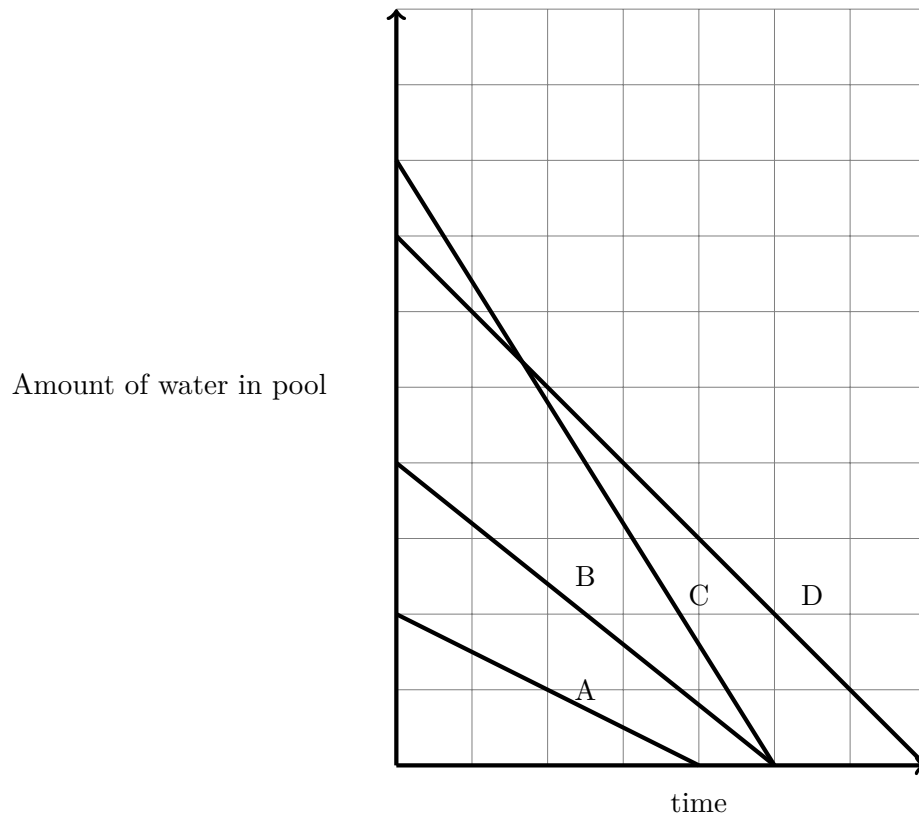
 - How much do you make per hour?

Exercise 27 *Your dog spends a full day sleeping in the sun (oh, to be a dog!). Which graph best depicts your dog's movement?*



- Write couple of sentences explaining why the graph you chose describes the situation above.
- Write couple of sentences explaining your dog's activities if the other two graphs were correct graphs.
- Put a scale on your graph. On the horizontal axis place 0-12 hours, and on the vertical axis place 0-10 miles per hour.
 - How fast was your dog moving after 3 hours? After 6 hours?
 - What was the time when your dog was moving 3 miles per hour? 0 miles per hour?

Exercise 28 Each graph below represents the amount of water in pool as it is being drained by a pump.



Carefully explain the answers to each question using the scale on the graph.

- Which pool had the most water to begin with?
- Which pool was empty first?
- Which pump pumps the most water per minute?

8.7 Relations and Functions

Exercise 29 Do the following sets of ordered pairs represent functions? Explain.

- $\{(1,0), (0,1), (3, 2), (5,0)\}$
- $\{(-5,4), (4,4), (7.6, 3.87), (9, 213)\}$
- $\{(1,0), (2,3), (1,6), (5,8)\}$
- $\{(\text{happy, purple}), (\text{bumble bee, -6.78}), (\text{grumpy, potato salad}), (\text{\&, 0.001dog}), (\text{\#, })\}$

Exercise 30 Do the following tables represent functions? Explain.

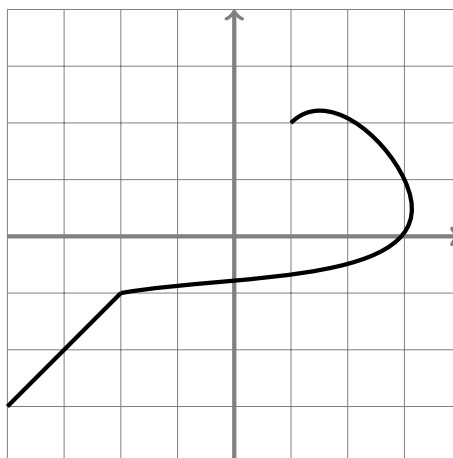
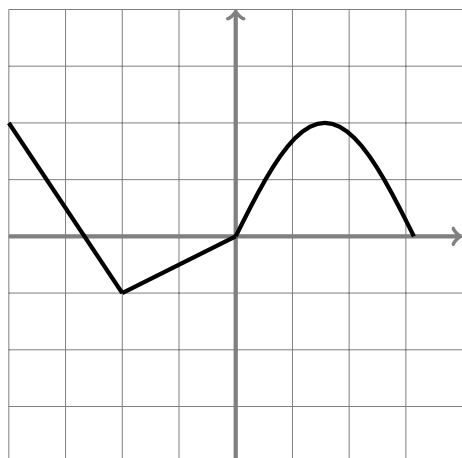
x	$a(x)$
1	-1
2	3
3	7
1	-1
4	11

x	$b(x)$
A	A
B	C
C	D
D	E
E	A

x	$c(x)$
1	-1
2	3
4	11
-2	-5
2	2

x	$d(x)$
1	5
2	5
3	5
4	5
12	5

Exercise 31 Are the following graphs, graphs of some function? Explain.



Explanation.

8.8 Ways to represent a function

Exercise 32 *Is it always possible to represent a function by an equation? Explain.*

Exercise 33 *We have a function $h : \mathbb{R} \rightarrow \mathbb{R}$ given by an algebraic rule $h(x) = -x^2 + 2x - 1$.*

a. Complete the following table:

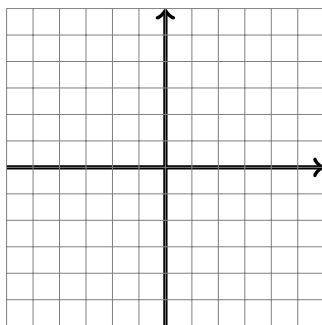
x	$h(x)$
-2	
0	
$\frac{1}{2}$	
2	
2.5	

b. Does the point $(-1, -4)$ lie on the graph of h ? How do you know?

c. Does the point $(0, -1)$ lie on the graph of h ? How do you know?

d. Find a few more ordered pairs for h .

e. Graph h .



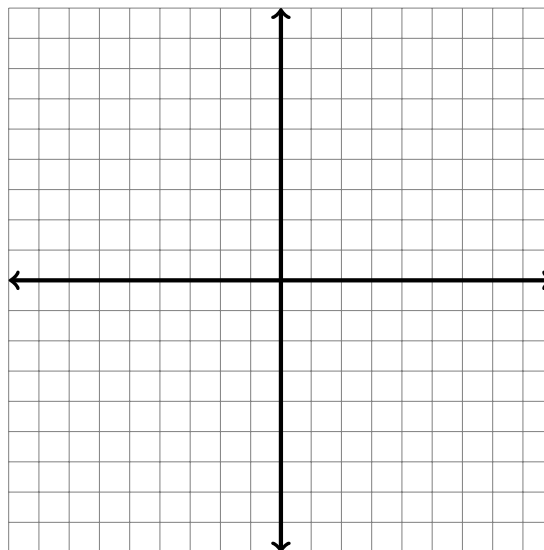
f. Use the graph to approximate the input x such that $h(x) = 0$.

Exercise 34 *A spherical balloon becomes bigger and bigger as it is filled with more and more air, although it always retains its spherical shape. It starts out at $t = 0$ as a balloon with a volume of 10 cubic centimeters. With every second, 30 cubic centimeters more of air is pumped into the balloon. Make a table, graph and find the expression for the function that gives volume as a function of time (in other words, the time is the input and the volume is the output).*

Exercise 35 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = -3x + 1$.

- Construct a table for $f : \mathbb{R} \rightarrow \mathbb{R}$.
- Plot the entries from your table on a graph.

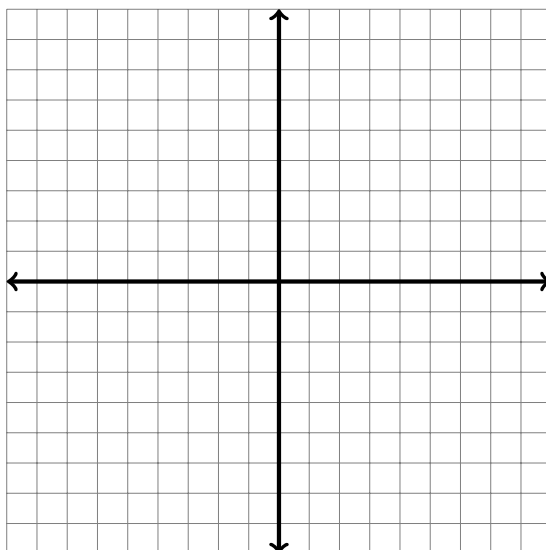
x	$f(x)$



Exercise 36 Graph the function $f : \mathbb{R} - \{1\} \rightarrow \mathbb{R}$ given by the algebraic rule

$$f(x) = \frac{x-2}{x-1}.$$

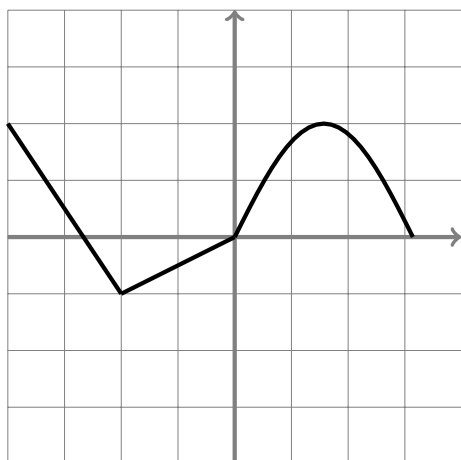
Explain briefly why the domain of f isn't the whole \mathbb{R} .



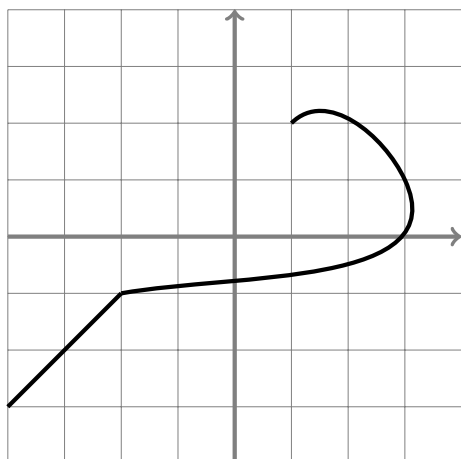
Exercise 37 An x -intercept is a point where a graph of a function crosses the x -axis. How many x -intercepts does $y = -\frac{1}{2}x + 3$ have? How many x -intercepts does $y = x(x - 2)$ have?

Exercise 38 A y -intercept is a point where a graph of a function crosses the y -axis. No function can have more than one y -intercept. Explain why not.

Exercise 39 For each of the relations below make a table of values which includes every integer input that can be read from the graphs.



x	$f(x)$

[illegible]

8.9 Combining Functions

Exercise 40 In each question below f and g are functions defined by an algebraic rule. For each pair of functions find a simplified formula for the required combination of functions, then evaluate those functions at 1 and -1 :

a. $f(x) = x^2 + 1$; $g(x) = x - 3$.

(a) $(f + g)(x) =$

$(f + g)(1) =$

$(f + g)(-1) =$

(b) $(f - g)(x) =$

$(f - g)(1) =$

$(f - g)(-1) =$

(c) $(f \cdot g)(x) =$

$(f \cdot g)(1) =$

$(f \cdot g)(-1) =$

(d) $(f \div g)(x) =$

$(f \div g)(1) =$

$(f \div g)(-1) =$

(e) $(f \circ g)(x) =$

$(f \circ g)(1) =$

$(f \circ g)(-1) =$

(f) $(g \circ f)(x) =$

$(g \circ f)(1) =$

$(g \circ f)(-1) =$

b. $f(x) = 10 - x$; $g(x) = \sqrt{x}$.

(a) $(f + g)(x) =$

$(f + g)(1) =$

$(f + g)(-1) =$

(b) $(f - g)(x) =$

$(f - g)(1) =$

$(f - g)(-1) =$

(c) $(f \cdot g)(x) =$

$(f \cdot g)(1) =$

$(f \cdot g)(-1) =$

(d) $(f \div g)(x) =$

$(f \div g)(1) =$

$(f \div g)(-1) =$

(e) $(f \circ g)(x) =$

$(f \circ g)(1) =$

$(f \circ g)(-1) =$

(f) $(g \circ f)(x) =$

$(g \circ f)(1) =$

$(g \circ f)(-1) =$

c. $f(x) = \sqrt{4x}$; $g(x) = \frac{1}{x}$.

(a) $(f+g)(x) =$

$(f+g)(1) =$

$(f+g)(-1) =$

(b) $(f-g)(x) =$

$(f-g)(1) =$

$(f-g)(-1) =$

(c) $(f \cdot g)(x) =$

$(f \cdot g)(1) =$

$(f \cdot g)(-1) =$

(d) $(f \div g)(x) =$

$(f \div g)(1) =$

$(f \div g)(-1) =$

(e) $(f \circ g)(x) =$

$(f \circ g)(1) =$

$(f \circ g)(-1) =$

(f) $(g \circ f)(x) =$

$(g \circ f)(1) =$

$(g \circ f)(-1) =$

d. $f(x) = 5x + 1$; $g(x) = 2x^2 - 7$.

(a) $(f+g)(x) =$

$(f+g)(1) =$

$(f+g)(-1) =$

(b) $(f-g)(x) =$

$(f-g)(1) =$

$(f-g)(-1) =$

(c) $(f \cdot g)(x) =$

$(f \cdot g)(1) =$

$(f \cdot g)(-1) =$

(d) $(f \div g)(x) =$

$(f \div g)(1) =$

$(f \div g)(-1) =$

(e) $(f \circ g)(x) =$

$(f \circ g)(1) =$

$(f \circ g)(-1) =$

(f) $(g \circ f)(x) =$

$(g \circ f)(1) =$

$(g \circ f)(-1) =$

Exercise 41 *I was at the store and over heard the following conversation:*

Customer: I have a 20% off coupon. I would like you to apply it after the 8% sales tax to maximize my savings.

Cashier: No, you want me to apply it before the tax. That way the tax is applied to a smaller price. Who was correct? Use the language of function composition as you discuss.

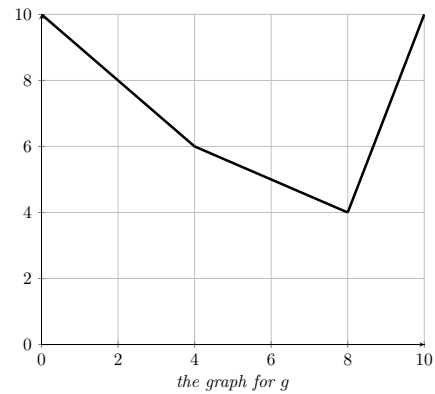
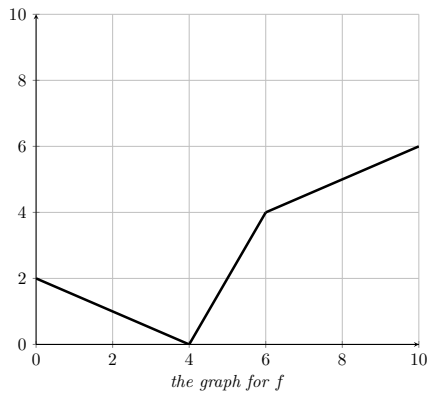
Exercise 42 *Find $f(x)$ and $g(x)$ such that $h(x) = (f \circ g)(x)$. Answers may vary.*

a. $h(x) = (3x - 5)^4$.

b. $h(x) = \sqrt{9x + 1}$.

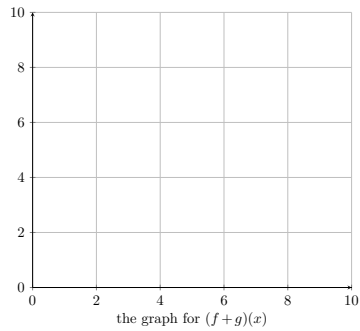
c. $h(x) = \frac{6}{5x-2}$

Exercise 43 Two functions are given by their graphs:

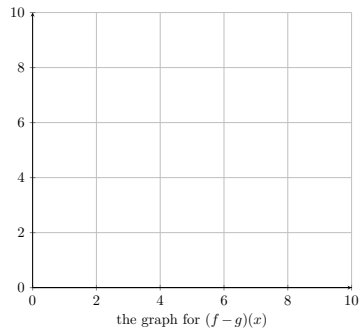


Find a table and graph each of the following functions:

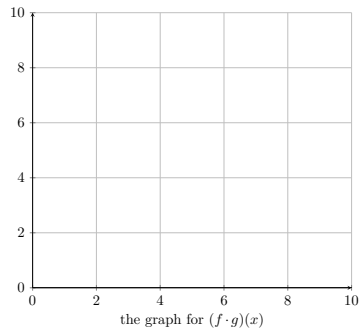
x	$(f + g)(x)$



x	$(f - g)(x)$



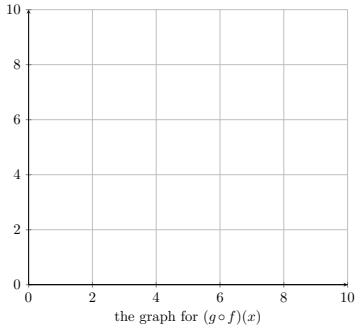
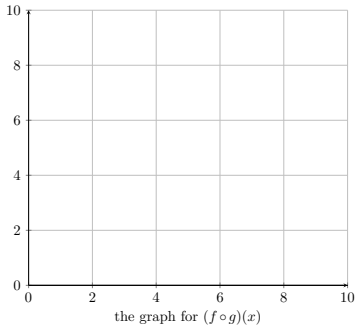
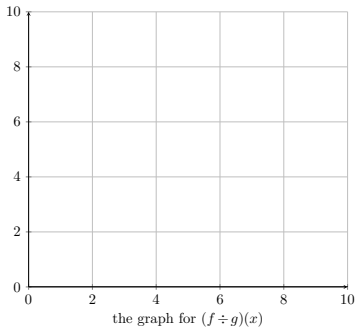
x	$(f \cdot g)(x)$



x	$(f \div g)(x)$

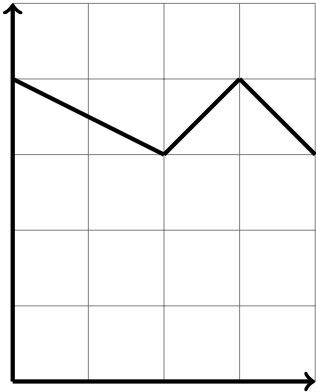
x	$(f \circ g)(x)$

x	$(g \circ f)(x)$



Exercise 44 Below is the portion of the graph for the function $f : \mathbb{R} \rightarrow \mathbb{R}$.

- a. What is $f(2)$?
- b. What is $(f \circ f \circ f)(3)$?



Exercise 45 You work forty hours a week at a furniture store. You receive a \$220 weekly salary, plus a 3% commission on sales over \$5000. Assume that you sell enough this week to get the commission. Given the functions $f(x) = 0.03x$ and $g(x) = x - 5000$, which of $(f \circ g)(x)$ and $(g \circ f)(x)$ represents your commission?

Exercise 46 In 2010, the Deepwater Horizon oil explosion spilled millions of gallons of oil in the Gulf of Mexico. The oil slick takes the shape of a circle. Suppose that the radius of the circle was increasing at rate .5 miles per day: $r(t) = .5t$.

- a. Let $A : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ be a function that describes the area of the spill in square miles t days after the spill occurred. Write a rule for $A(t)$.
- b. Evaluate $A(30)$ and explain what $A(30)$ means.

Exercise 47 According to the U.S. Energy Information Administration, a barrel of crude oil produces approximately 20 gallons of gasoline. EPA mileage estimates indicate a 2011 Ford Focus averages 28 miles per gallon of gasoline.

- a. Write an expression for $g(x)$, the number of gallons of gasoline produced by x barrels of crude oil.
- b. Write an expression for $m(g)$, the number of miles on average that a 2011 Ford Focus can drive on g gallons of gasoline.
- c. Write an expression for $m(g(x))$. What does $m(g(x))$ represent in terms of the context?
- d. One estimate (from www.oilvoice.com) claimed that the 2010 Deepwater Horizon disaster in the Gulf of Mexico spilled 4.9 million barrels of crude oil. How many miles of Ford Focus driving would this spilled oil fuel?

8.10 Inverse Functions

Exercise 48 $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by the algebraic rule $f(x) = 3x + 4$. If $g : \mathbb{R} \rightarrow \mathbb{R}$ is a function with $g(7) = 0$, could f and g be inverse functions? Explain.

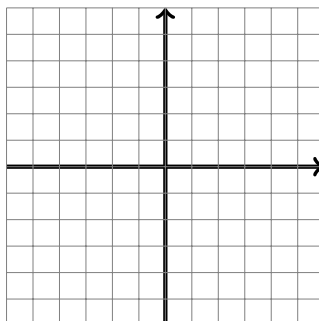
Exercise 49 How can you tell if a function is invertible by looking at its graph? Explain your answer.

Exercise 50 Two functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ are given by algebraic rules $f(x) = \frac{3x+4}{2x-1}$ and $g(x) = \frac{x+4}{2x-3}$. Are these two functions inverses? Explain.

Exercise 51 Given two functions f and g does there always exist a function $f \circ g$? Explain. Give examples or counterexamples.

Exercise 52 $f : \mathbb{R} \rightarrow \mathbb{R}$ is given by the rule $f(x) = 4x^4$.

a. Graph f



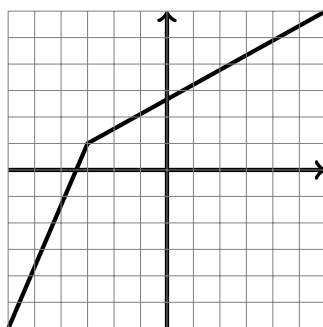
b. Is f invertible? Explain how you know.

Exercise 53 The following chart summarizes some values of two functions f and g .

x	$f(x)$	$g(x)$
1	3	4
2	5	9
3	4	9
4	1	4

- What is the value of $f(4)$?
- Evaluate $g(3)$.
- Compute $(f \circ g)(1)$.
- Could $g(x)$ be invertible? Explain your answer.
- Is $f(x)$ invertible?
- If $f(x)$ were invertible, what would the value of $f^{-1}(3)$ have to be?

Exercise 54 The following is a graph of a function f .



- Is $f(x)$ invertible? How do you know?
- What is the value of $f^{-1}(0)$?
- Sketch a graph for $f^{-1}(x)$.

8.11 Interpolating a Discrete Set of Data

Exercise 55 You recently put \$1000 in a bank account and earn 5% interest per year. Let g be a function such that your balance after t years be given by $g(t)$.

- a. Fill in the following table:

t	$g(t)$
0	
1	
2	
3	

- b. Is your balance g represented by a linear function? Why or why not?

- c. Let $f : [0, \infty) \rightarrow \mathbb{R}$ be a linear function given by $f(t) = 1000 + 50t$. Fill in the following table:

t	$f(t)$
0	
1	
2	
3	

- d. For some amount of time $g(t) - f(t)$ is small. For how many years would you be willing to use f to estimate g ?

Exercise 56 Rocky Mountain Power provides electricity to Salt Lake City. If you are a residential costumer of Rocky Mountain Power you pay \$5.00 Basic Charge every month as well as \$1.75 City Franchise Tax and a \$1.28 Utah State Tax and then you are charged \$0.0888540 per kwh used.

- a. Let $f(x)$ be the amount you pay using x - kwh. Notice that f is a linear function. What is the slope? What is the y -intercept?
- b. If in September your electric bill is \$32.39 how many kwh did you use?
- c. If in September your electric bill is \$32.39 what is your average cost per day?

- d. If in your monthly budget you have allotted \$40.00 for the electric bill, how many kwh can you use?
- e. You recently purchased a new TV you upgraded from a 32 inch TV to a 60 inch TV. Assuming average viewing, a 32 inch TV uses 60 kwh per year and 60 inch TV uses 165 kwh per year. How much would you expect your monthly bill to go up?
- f. When you enroll in Blue Sky, Rocky Mountain Power purchases certified renewable energy certificates from regional renewable energy facilities on your behalf. This guarantees that electricity from wind facilities is delivered to the regional power system. Electricity from renewable energy facilities replaces and reduces the need for electricity generated from non-renewable sources like fossil fuels, creating measurable environmental benefits. If you buy one block of renewable energy it costs \$1.95. Would buying a block of renewable energy effect the slope of the cost function or the y -intercept? (write m or b)

Exercise 57 *If a computer program has a loop in it, the length of time it takes the computer to run the program varies linearly with the number of times it must go through the loop. Suppose a computer takes 8 seconds to run a given program when it goes through the loop 100 times, and 62 seconds when it loops 1000 times.*

- a. Write the particular equation expressing seconds in terms of loops.
- b. Predict the length of time needed to loop 30 times; 10,000 times.
- c. Suppose the computer takes 23 seconds to run the program. How many times does it go through the loop?
- d. How long does it take the computer to run the rest of the program, excluding the loop? What part of the mathematical model tells you this?
- e. How long does it take the computer to go through the loop once? What part of the mathematical model tells you this?
- f. Plot the graph of this function.

8.12 Slope

Exercise 58 Below is a table for a function $f : \mathbb{R} \rightarrow \mathbb{R}$. Could f be a linear function? Why or why not?

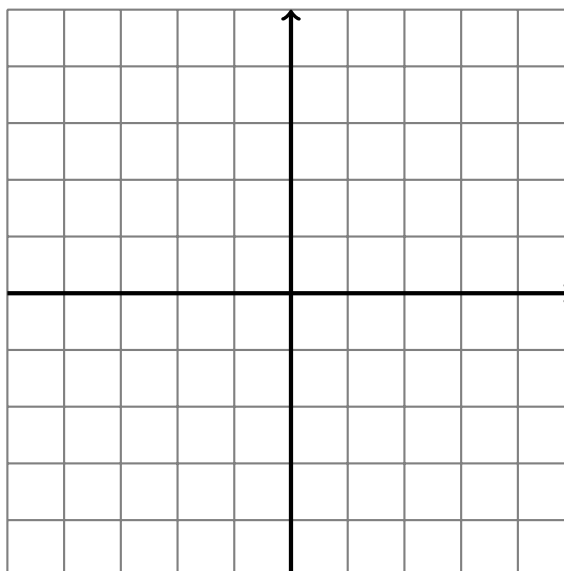
x	$f(x)$
3	14
4	17
5	21

Exercise 59 Find the slope between the following pairs of points:

a. $(-1, 3)$ and $(2, 7)$

b. $(2, 6)$ and $(5, 10)$

c. $(2, -4)$ and $(5, 0)$

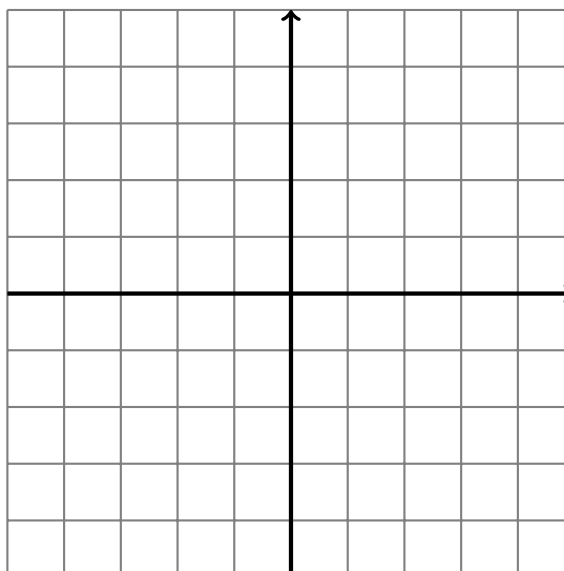


Once finished, graph all the pairs and the line segments between them. What do you notice?

Exercise 60 Draw the line segments between the following pairs of points:

a. $(0, 0)$ and $(2, 7)$

b. $(0, 0)$ and $(-7, 2)$



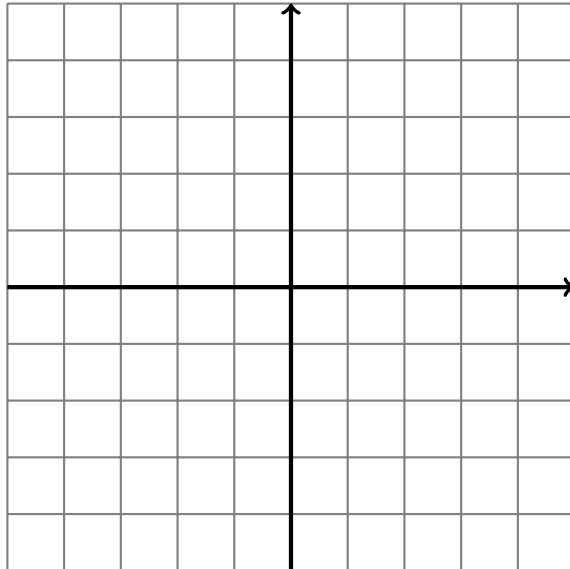
What is the slope of each of the line segments?

Exercise 61 Draw the line segments between the following pairs of points:

a. $(-1, 3)$ and $(2, 7)$

b. $(1, -3)$ and $(-3, 0)$

What is the slope of each of the line segments?



Exercise 62 Below is a table for an arithmetic sequence:

x	$f(x)$
1	-5
2	-2
3	1

Can this table represent a linear function as well?

What is the explicit formula for the n^{th} term of this sequence?

8.13 Lines

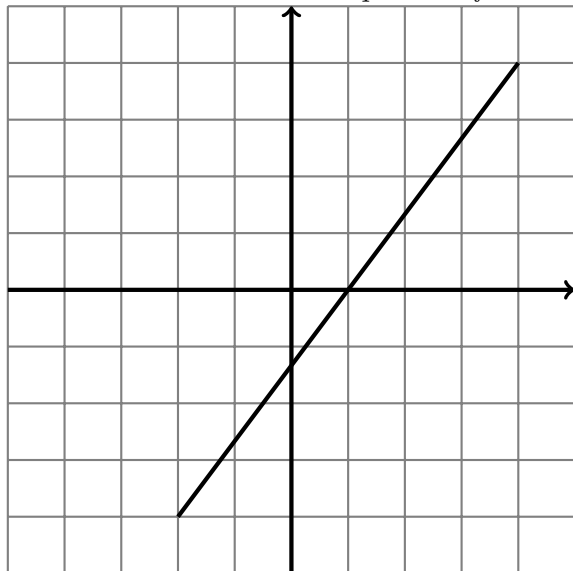
Exercise 63 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by the algebraic rule $f(x) = 4x + 5$.

- What is the x -intercept?
- What is the y -intercept?
- What is the slope?
- For what value of x does $f(x) = 17$?

Exercise 64 Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by the algebraic rule $g(x) = -2x + 4$.

- What is the x -intercept?
- What is the y -intercept?
- For what value of x does $g(x) = 12$?

Exercise 65 What is the equation of the line graphed below:

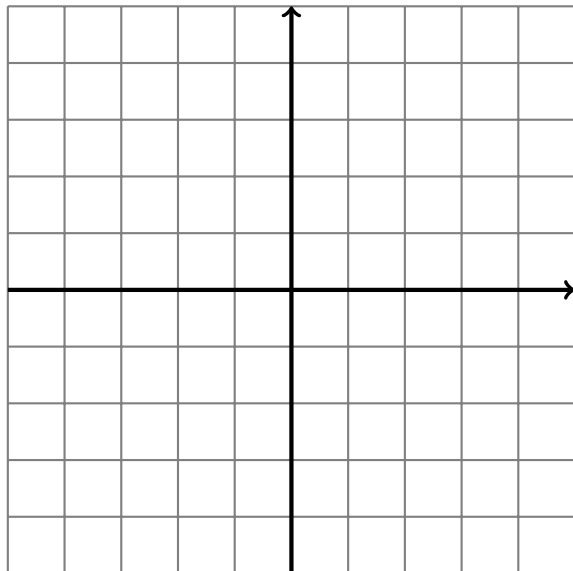


Exercise 66 What is the equation for the line that goes through the points $(3, 4)$ and $(2, 5)$? Report the slope and the y -intercept.

Exercise 67 Is the point $(2, 1)$ on the line given by $3y + 2x = 7$? How do you know?

Exercise 68 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a function given by $f(x) = 4x + 3$.

- Is the graph of f is a line? How do you know?
- What is the x -intercept?
- What is the y -intercept?
- What is the slope?
- Sketch the graph for f . Which quadrant does the graph not pass through?



Exercise 69 Which of the following functions from \mathbb{R} to \mathbb{R} are linear?

$$f(x) = x\sqrt{2} - \frac{1}{2} \quad \text{Yes} \quad \text{No}$$

$$2x + \frac{y}{4} = 2 \quad \text{Yes} \quad \text{No}$$

$$y = \frac{2}{x} + 2 \quad \text{Yes} \quad \text{No}$$

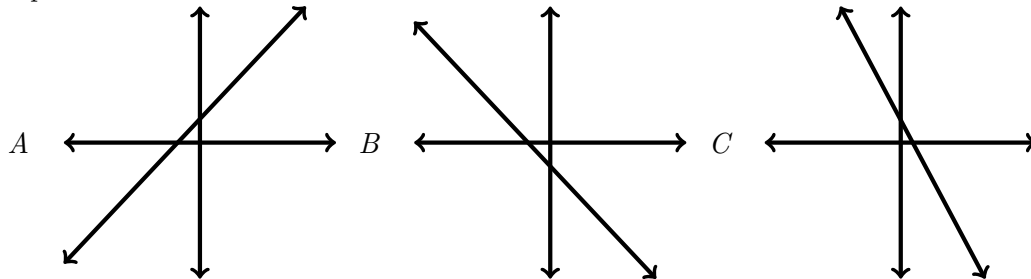
$$y - x + 1 = 0 \quad \text{Yes} \quad \text{No}$$

Exercise 70 Does the point $(8,1)$ lie on the line $y = -\frac{1}{2}x + 3$? Show how you arrived at the answer.

Exercise 71 Find a pair (x,y) that satisfies the equation $y = -\frac{1}{2}x + 3$.

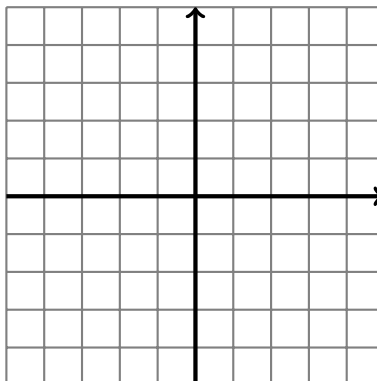
Exercise 72 Does the pair you found in the previous question lie on the line $y = -\frac{1}{2}x + 3$? Explain.

Exercise 73 Which of the following graphs could possibly be a graph of $y = -\pi x + \sqrt{73}$? Explain.



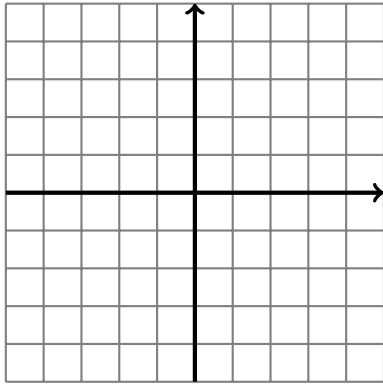
Exercise 74 Given the equation $y = 3x - 7$:

- a. Evaluate y if $x = -1$, $x = 2$, and $x = 5$.



- b. Show by graphing that the points lie in a straight line.

Exercise 75 Plot the graph of $y + 3 = -2(x + 1)$. Then transform the equation to slope-intercept form. Transform the equation to $Ax + By = C$ form, where A , B , and C are integers.

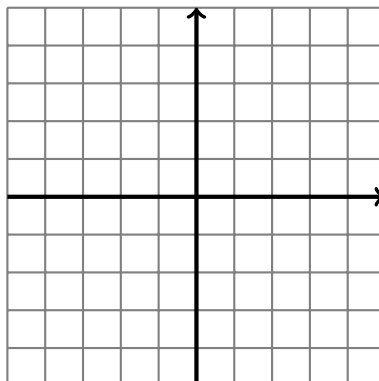


Exercise 76 Find the particular equation of the line described.

- a. Contains $(2, -7)$ and $(5, 3)$.
- b. Contains $(-4, 1)$ and is parallel to the graph of $2x - 9y = 47$.
- c. Contains $(3, -8)$ and is perpendicular to the graph of $y = 0.2x + 11$.
- d. Has x -intercept of 5 and y -intercept of -6.
- e. Is vertical, and contains $(-13, 8)$.
- f. Is horizontal, and contains $(22, \pi)$.
- g. Has the x -axis as its graph.
- h. Has a slope that is infinitely large, and contains $(5, 7)$.

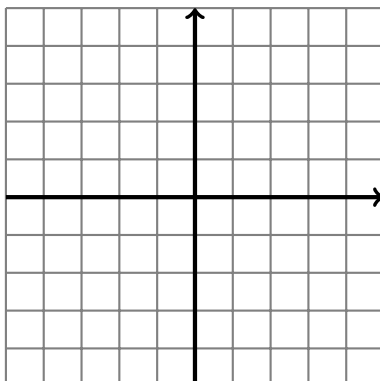
Exercise 77 Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be functions given by the following algebraic rule $f(x) = 4x + 5$ and $g(x) = -2x + 4$.

- Is there a value of x such that $f(x) = g(x)$?
- How many values of x does $f(x) = g(x)$?
- What does this value of x tell us about the graphs of f and g ?



Exercise 78 Solve the following two equations at the same time, that is find a pair of numbers: x and y which make both equations true

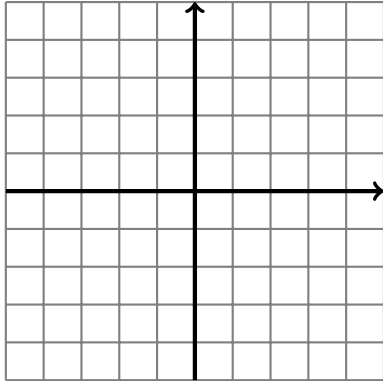
$$\begin{aligned} y &= 3x + 5 \\ 2x - y &= 4. \end{aligned}$$



Exercise 79 The following table records the prices for Horizon Organic Fat-Free Milk at Target (1110 S 300 W).

Gallons	Price
.25	\$2.49
.5	\$3.99
1	\$6.99

For the purposes of this problem, you may round the prices to the nearest tenth, or you can use exact values.



- Does (Gallons, Price) lay on a line? How do you know?
- Write the equation of the line that passes through at least two of the points given.
- Use the equation you developed to determine the price of 10 gallons of milk.
- If you paid \$14.50 for milk, according to your linear function how many gallons of milk did you buy?

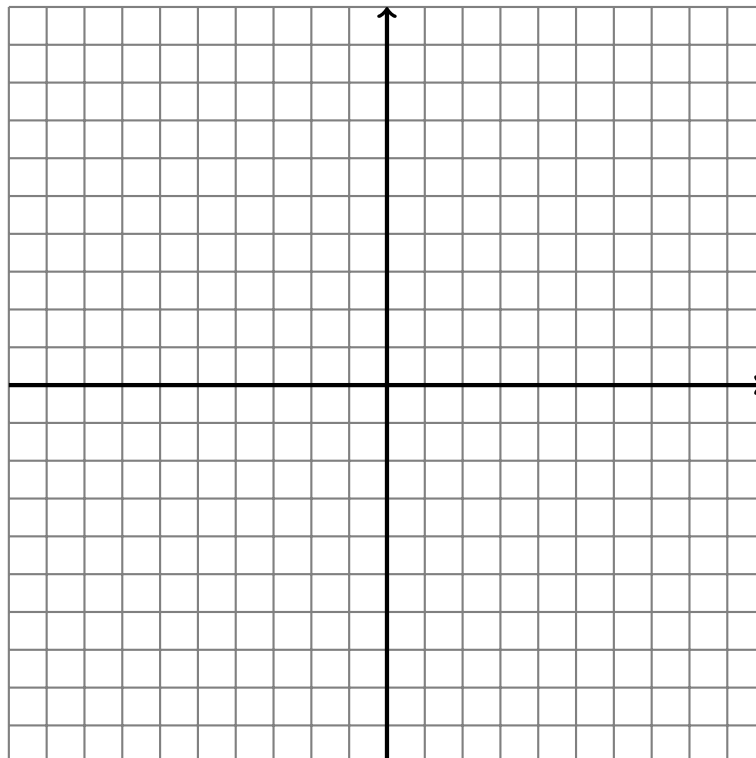
8.14 Graphs of Quadratic Functions

Exercise 80 If $y = ax^2 + bx + c$, where a, b, c are constants and x, y are variables answer the following questions.

- What does the graph look like?
- When is the parabola opening up?
- Opening down?
- When does the graph have a maximum value?
- Minimum value?
- Will it always have x -intercepts?
- Will it always have a y -intercept?

Exercise 81 Graph the following functions:

- $f(x) = (3x - 1)(x + \sqrt{12})$
- $g(x) = (x - 3)(x + 3) - 2$
- $h(x) = x^2 - 2x$
- $m(y) = y^2 - 16$
- $p(c) = c^2 - 12 - c$



Exercise 82 Each table represents either a linear, quadratic or “neither” function. Identify whether each table is linear, quadratic, or neither, and then **write a sentence** explaining how you know.

1. Table A

x	$f(x)$
-1	1.5
0	2
1	3
2	5

Table A is (circle one):

Linear · **Quadratic** ·
Neither

because:

2. Table B

x	$f(x)$
-1	4
0	3
1	4
2	7

Table B is (circle one):

Linear · **Quadratic** ·
Neither

because:

3. Table C

x	$f(x)$
-1	3
0	1
1	-1
2	-3

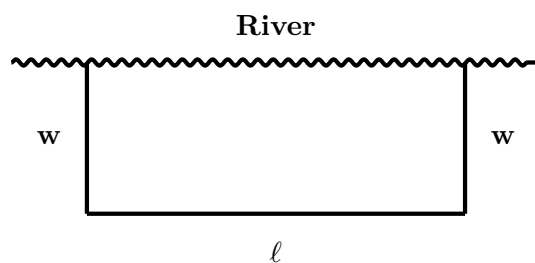
Table C is (circle one):

Linear · **Quadratic** ·
Neither

because:

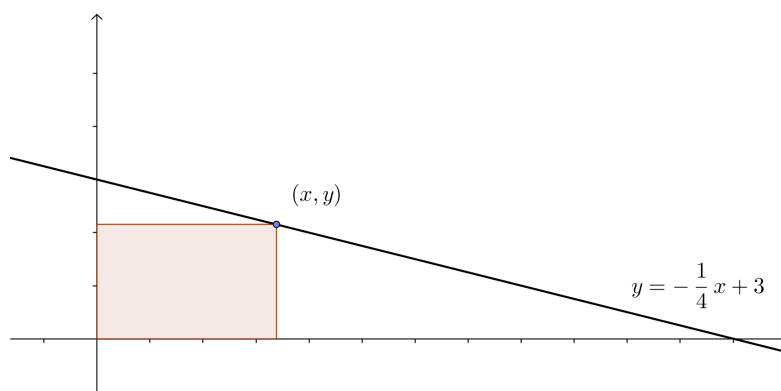
Exercise 83 Find the x -intercepts, and the y -intercept of the graph whose equation is $y = x(x - 7)$.

Exercise 84 *What if we have 50 feet of fencing but we can build Ellie's pen next to a river so that we only need to enclose 3 sides as in the picture below.*



- Make a table for values of ℓ and w .
- Graph w as a function of ℓ .
- Write the width w as function of the length ℓ .
- Make a table of ℓ and Area.
- Write area as a function of ℓ .
- What choice of ℓ maximizes the area?

Exercise 85 A point from the first quadrant is chosen on the line $y = -\frac{1}{4}x + 3$. Lines are drawn from this point parallel to the axes so that a rectangle is formed under the given line (as in the figure).



- a. Among all such rectangles, what are the dimensions of the one with the maximum area?

- b. What is the maximum area?

8.15 The Zero Product Property

Exercise 86 *What does it mean to “solve” an equation?*

Exercise 87 *How do you draw a picture to show the factorization of a quadratic expression? What do the factors tell you geometrically? Where is the expression in standard form in your picture?*

Exercise 88 *Factor the following quadratic expressions (given to you in standard form; your job is to rewrite the expressions in factored form) by drawing a picture:*

a. $x^2 + 8x + 7$

b. $x^2 + 8x + 15$

c. $x^2 + 8x + 16$

d. $x^2 + 4x + 3$

e. $2x^2 + 4x + 2$

Exercise 89 *In this question you will think about the differences and similarities between expression, equation, and function.*

- a. Factor $x^2 + 12x + 36$.

- b. Solve $x^2 + 12x + 36 = 0$.

- c. Graph the function $f(x) = x^2 + 12x + 36$. What are the x -intercepts? What are the y -intercepts? What is the axis of symmetry? Does f have a minimum or a maximum? What is the vertex?

- d. Factor $x^2 + 8x + 15$.

- e. Solve $x^2 + 8x + 15 = 0$ for x .

- f. Graph $f(x) = x^2 + 8x + 15$. What are the x -intercepts? What are the y -intercepts? What is the axis of symmetry? Does f have a minimum or a maximum? What is the vertex?

Exercise 90 *If $a \cdot b = 0$, what can you say about a or b ? This question will assess your ability to use the zero product property.*

a. Solve the equation $(3x+3)(4x+5) = 0$ for x . How did you accomplish this task?

b. Could you solve $x+4$?

c. Could you solve $(x+4)(x-4)(x+1) = 0$?

d. Multiply: $(x+4)(x-4)(x+1)$. What is the degree of the product?

e. If you graph $f(x) = (x+4)(x-4)(x+1)$, what would the x -intercepts be? What would the y -intercepts be?

Exercise 91 *Write a quadratic equation, in the form $ax^2 + bx + c = 0$, whose roots are 2 and 5.*

Exercise 92 *Solve the following equations:*

1. $(3x - 1)(x + \sqrt{12}) = 0$

2. $(x - 3)(x + 3) = 2$

3. $x^2 - 2x = 0$

4. $y^2 - 16 = 0$

5. $c^2 - 12 = c$

6. $\frac{x}{9} = \frac{4}{x}$

8.16 Completing the Square

Exercise 93 *Solve the following quadratic equations:*

a. $x^2 - x - 6 = 0$

d. $x^2 + 29x + 100 = 0$

b. $x^2 - 7x + 12 = 0$

e. $x^2 + 8x = -7$

c. $x^2 + 16x - 17 = 0$

f. $x^2 + 8x + 15 = 0$

g. $x^2 = -8x - 16$

j. $x^2 + 4x + 3 = 0$

h. $x^2 + 3x + 1 = 0$

k. $2x^2 + 4x = -2$

i. $2x^2 + 3x + 1 = 0$

l. $(2x + 3)(x - 1) = 0$

m. $(x+3)^2 = 2$

p. $x^2 + 3x + 2 = 0$

n. $(x+3)^2 = 2x - 7$

q. $x^2 - 3x + 2 = 0$

o. $-9(x-5)(x+2) = 0$

Exercise 94 *Solve the following quadratic equation using the completed square form:*

$$9x^2 + 4x + 2 = 0$$

Exercise 95 *The square of a number exceeds 5 times the number by 24. Find the number(s).*

Exercise 96 *Looking back at the entire chapter on quadratic functions, answer the following essential questions:*

- a. What are the x -intercepts?
- b. For a general quadratic function $g : \mathbb{R} \rightarrow \mathbb{R}$ how do you find the x -intercepts?
- c. What are the y -intercepts?
- d. For a general quadratic function $g : \mathbb{R} \rightarrow \mathbb{R}$ how do you find the y -intercepts?
- e. Let g be a general quadratic function. How many y -intercepts can g have? How many x -intercepts can g have?
- f. What is the axis of symmetry? How did you know this?
- g. In general what is symmetry?
- h. What does the sign of the x^2 term in $g(x)$ tell you about its graph?
- i. What is the maximum or minimum value?
- j. How can we find the zeros of a quadratic function?
- k. How do we calculate the max or min of a quadratic function?

Notes:

8.17 Rules for Exponents

Exercise 97 Rewrite the following using one exponent. Justify your answers.

1. $a^5 a^6$

8. $3y^3 \cdot 2y^2$

2. $(a^4)^5$

9. $(-3t^4)^3$

3. $(5^5)^2 5^3$

10. $q^3 p^2 \cdot (p^2 q^3)^4$

4. $-5y^2 y^3$

11. $(-5pr)(r^2 p^3)^3$

5. $3z \cdot z^3$

12. $a \cdot a^2 \cdot (-a)^4$

6. $3t^2 \cdot (-2t^6)$

7. $(vw^3)^5 \cdot v^2$

13. $(t^3 r^2)^3$

Exercise 98 Rewrite the following expressions using one exponent:

1. $(a^5 a^{-3})^3$

2. $(a^{-2} a^3)^2$

3. $(\frac{x}{x^2})^3$

Exercise 99 Explain how the exponent $^{-1}$ means different things when we think about a^{-1} and f^{-1} , where a is a real number, and f is a function.

Exercise 100 Rewrite the following using positive exponents. Justify your answers.

1. 5^{-1}

8. $\left(\frac{3x}{4y}\right)^2$

2. 7^{-3}

9. $\frac{5x^2y}{10x^4y^2}$

3. $\frac{1}{4^{-2}}$

10. $\frac{7t^2}{tr^3} \cdot \frac{2tr^5}{3r}$

4. $\frac{3^4}{3^5}$

11. $\frac{(-4xy)^3}{8xy^2}$

5. 8^{-2}

12. $\left(\frac{7}{8}\right)^0$

6. $\frac{27m^5n^6}{9mn^3}$

13. $\frac{5}{xy^{-1}}$

7. $\frac{28x^2y^3}{2xy^2}$

14. $\frac{5}{5^{-2}}$

Exercise 101 Give an example to show how the Quotient of Powers rule can explain why $b^0 = 1$.

Exercise 102 Give an example to show how the Quotient of Powers rule can explain why $b^{-2} = \frac{1}{b^2}$.

Exercise 103 *The Earth is about $93 \cdot 10^6$ miles from the sun. Light travels at about $1.86 \cdot 10^5$ miles/second. About how long does it take light from the sun to reach the Earth? Show how one or more of the laws of exponents is useful in solving this problem.*

Exercise 104 *In 1980, the population of the U.S. was about $227 \cdot 10^6$. If the land of the U.S. was about $35 \cdot 10^6$ square miles, what was the average number of people per square mile of land? Show how one or more of the laws of exponents is useful in solving this problem.*

Exercise 105 *For which figure is the ratio of the volume to surface area greater: a sphere or a cube? (Volume of a sphere: $\frac{4}{3}\pi r^3$, where r = radius; Surface area of a sphere: $4\pi r^2$)*

Notes:

8.18 Graphs of Exponential Functions

Exercise 106 What can you say about the shape of the graph of $f(x) = ab^x$ if $a > 0$ and $0 < b < 1$? As b increases in value from 0 toward 1 how does the graph change?

Exercise 107 What can you say about the shape of the graph of $f(x) = ab^x$ if $a > 0$ and $b > 1$? As b increases in value how does the graph change?

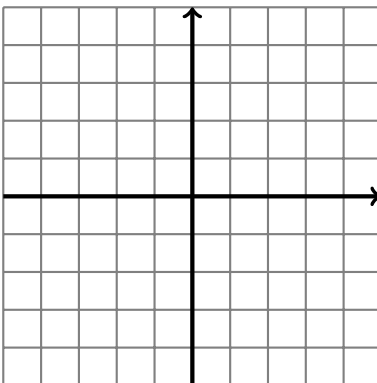
Exercise 108 Suppose $b = 1$ in the equation $f(x) = ab^x$, where $a > 0$. What does the graph of this function look like? In what way(s) is this graph fundamentally different from the graphs you sketched in the previous two problems? Answer the same question for when $b = 0$.

Exercise 109 Now suppose $b = -2$ and $a = 1$, so the equation is $f(x) = (-2)^x$.

- a. Without using the graphing or table features of your calculator complete the following table.¹

x	-2	-1	0	1	2	3	4
$f(x)$							

- b. Plot the points you found in Part a and connect them with a smooth curve so as to indicate what the graph of $f(x) = (-2)^x$ might look like.



- c. Describe how the graph in Part b differs from all the other graphs you've sketched.
- d. Explain why we do not use negative numbers for b when we speak of exponential functions $f(x) = ab^x$.

¹Make sure to enclose -2 in parentheses when doing your calculations.

Exercise 110 Determine the value of a and b in $f(x) = ab^x$ if the following facts are known about f :

a. $(0,3)$ and $(2,5)$ are on the graph of the function.

b. $(1,2)$ and $(3,10)$ are on the graph of the function.

c. $(2,3)$ and $(8,1)$ are on the graph of the function.

d. $(-3,8)$ and $(1,1)$ are on the graph of the function.

e. $f(0) = 5$ and $f(3) = 20$.

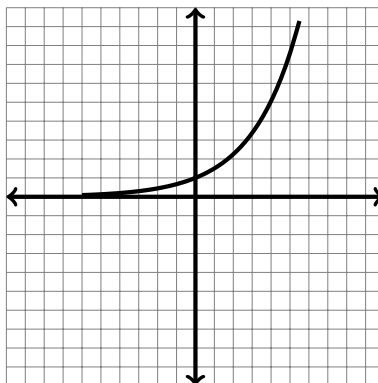
Exercise 111 The graph of $y = f(x)$ is shown below. Use this graph to quickly sketch a reasonable graph for each of the following functions:

a. $y = f(x) + 2$

b. $y = f(-x)$

c. $y = -f(x)$

d. $y = -f(x) + 2$



Notes:

8.19 Inverse Functions

Exercise 112 Evaluate each of the following logarithms:

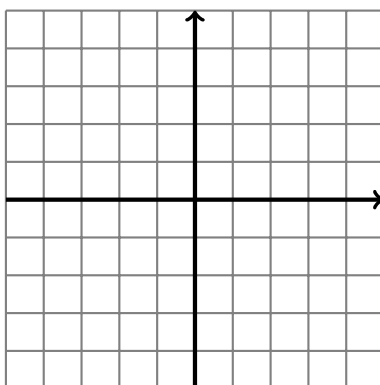
a. $\log_{10} 1000$

b. $\log_{10} \frac{1}{100}$

c. $\log_3 \frac{1}{3}$

d. $\log_3 \sqrt{3}$

Exercise 113 Graph the inverse function of $f : \mathbb{R} \rightarrow \mathbb{R}$ given by $f(x) = 5^x$. Make sure that you clearly state the domain and the target of f^{-1} .



Exercise 114 Explain why there is no function given by the rule $f(x) = \log_{-3}(x)$.

Exercise 115 A function $g : \mathbb{R} \rightarrow \mathbb{R}$ is given by the rule $g(x) = 15 \cdot \left(\frac{1}{2}\right)^x - 3$. Decide whether g has an inverse function. If it does, what is the algebraic expression for the inverse?

Exercise 116 Find the expressions for the inverse functions of the following functions:

a. $a(x) = 3 \log_{10} x - 1$

b. $b(x) = \log_{10} x^2 - 2$

c. $c(x) = \frac{1}{4} \cdot 3^{2x-1}$

d. $d(x) = \frac{1500}{2+3^{x-1}}$

8.20 Solving Exponential and Logarithmic Equations

Exercise 117 *Solve the following exponential equations:*

a. $4^{x-1} = 16$

b. $7^{2x+1} = 7^{3x-1}$

c. $3^{2x-1} = 27^x$

d. $5^{3x-8} = 25^{2x}$

Exercise 118 *Solve the following exponential equations:*

a. $5^x = 9$

b. $3^{2x+1} = 15$

c. $\left(\frac{1}{2}\right)^x = 3$

d. $e^x = 30$

e. $12 \cdot 5^{0.1x} = 30$

f. $3^{x-1} - 4 = 7$

Exercise 119 *Solve the following logarithmic equations:*

a. $\log_4 x = 2$

b. $\log_4(x+2)x = 2$

c. $3\log_7(x-1) = 6$

d. $\frac{2}{3}\log_4(x-1) + 3 = 6$

Exercise 120 *Looking back at the entire chapter on exponential functions, answer the following essential questions:*

a. Why are exponents useful? How are they used in real world applications?

b. What does a negative exponent mean? What does a fractional exponent mean?

c. Where do rules for exponents come from?

8.21 Applications

Exercise 121 *Disease can spread quickly. Suppose the spread of a direct contact disease in a small town is modeled by the function:*

$$P(t) = \frac{10000}{1 + 2^{3-t}}$$

where $P(t)$ is the total number of people infected after t days.

- a. Estimate the initial number of people infected with the disease. Explain how you found your answer.
- b. How many people will be infected after 1,2,3,4, and 5 days? (Fill out a table)
- c. What is the maximum number of people who can become infected? How do you know?
- d. The town officials must inform the citizens when 5000 people become infected. Which day will the town officials make an announcement?

Exercise 122 *As soon as you drive a new car off the dealer's lot, the car is worth less than what you paid for it. This is called depreciation. Chances are that you will sell it for less than the price that you paid for it. Some cars depreciate more than others, but most cars do depreciate. On the other hand, some older cars actually increase in value. This is called appreciation. Suppose you have a choice between buying a 1999 Mazda Miata for \$19,800 which depreciates at 22% a year, or a 1996 Honda Civic EX for \$16,500 which only depreciates at 18% a year. In how many years will their values be the same? Should you instead buy a 1967 Ford Mustang for \$4,000 that is appreciating at 10% per year? Which car will have the greatest value in 4 years? In 5 years?*

Exercise 123 *Suppose that the number of bacteria per square millimeter in a culture in your biology lab is increasing exponentially with time. On Tuesday there are 2000 bacteria per square millimeter. On Thursday, the number has increased to 4500 per square millimeter.*

- a. Derive the particular equation.
- b. Predict the number of bacteria per square millimeter that will be in the culture on Tuesday next week.
- c. Predict the time when the number of bacteria per square millimeter reaches 10,000.
- d. Draw the graph of the function.

Exercise 124 *If you invest \$1000 at 4% interest and it is compounded continuously, your balance is given by the function $b(t) = 1000(2.718)^{.04t}$ (where t is the number of years you have invested your money).*

- a. How much money do you have after 1 year?
- b. How much money do you have after 4.5 years? What does the fractional exponent mean?
- c. How long would you have to wait to have \$10,000?