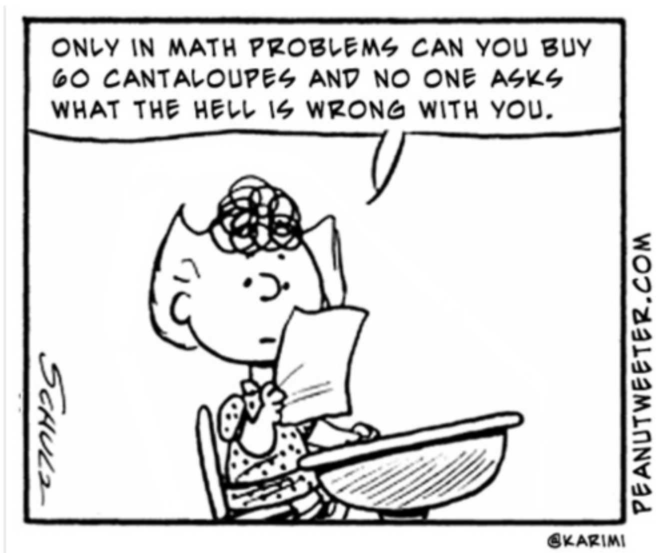
1 Introduction

All of you have different reasons for being in this class. Most of you are here because either you or someone else decided that you need to refresh or acquire some skills that are needed in a course that you may need either for your major or graduation. Ordinarily, that is how the class was treated: an opportunity to remind you of skills and procedures that you have learned and forgotten, or never quite learned. We have different goals for this class. While it's important to develop certain skills, it is more important to know what to do with those skills. In all of the classes that will follow, and not just mathematics ones, as well as in all the interactions with the world around you, it's important to have an ability and skill to think through a problem you encounter, make a plan for solving the problem, execute it and then look back, think through your answer and decide on its reasonableness. It is also important to realize the power mathematics holds in both dealing with the world around us, but also in its own right. We need to get away from the view that mathematics looks like this:



In this section you will find some big ideas that we think are important to keep in mind as you're working through this course. You will also find the learning outcomes that can serve you as a guide to what you should be learning. Each section will contain a list of essential questions you should be able to answer at the end of the unit.

1.1 Big Ideas

1. We can talk about many different instances of a situation at the same time: variability can be described and used productively.
2. The number systems developed from our need to solve various problems. We choose to extend the existing number systems so that the properties of operations are retained.
3. Problems come from various areas and not all of them can be solved, but much can be learned from attempts at solution, successful or not.

1.2 Learning Outcomes

1. Students are willing to engage with problems which are unfamiliar to them and to which the solutions or paths to solutions are not immediately obvious.
2. Students can extract relationships between quantities and describe them in different ways: tables, expressions, graphs, words, and can translate between these representations in order to answer questions most efficiently.
3. Students can answer questions about quantities given relationships between two or more by solving equations, whether it be algebraically, using tables, graphs or approximating.
4. Students understand how different growth patterns influence shape of the graph.
5. Students can recognize linear, exponential and polynomial from verbal descriptions, tables, and graphs.

1.3 Warming up for the semester

Question 1.1 (Brazil) Two mothers and two daughters sleep in the same room. There are only three beds and exactly one person sleeps on each of them, yet all people are accounted for. How is this possible?

Question 1.2 (Ireland) One day three brothers were going past a graveyard. One of them said, "I shall go in so that I may say a prayer for the soul of my brother's son." The second man said the same thing. The third brother said, "I shall not go in. My brother's son is not there." Who is buried in the graveyard?

Question 1.3 (Puerto Rico) Who is the sister of my aunt, who is not my aunt, but is the daughter of my grandparents?

Question 1.4 (Russia) An old man was walking with a boy. The boy was asked, “How is the old man related to you?” The boy replied, “His mother is my mother’s mother-in-law. What relation is that?

Question 1.5 On your calculator:

- Put in first 3 digits of your phone number
- Multiply by 80
- Add 1
- Multiply by 250
- Add the last four digits of your phone number
- Add the last four digits of your phone number
- Subtract 250
- Divide by 2
- What did you get?

Was that surprising? Try to explain why that happened.

Question 1.6 A pot and a lid cost $11 (this was once upon a time). The pot costs $10 more than the lid. How much does each item cost individually?

Question 1.7 What do these questions have to do with mathematics? What do they have to do with algebra? Describe the process you used to solve these questions.

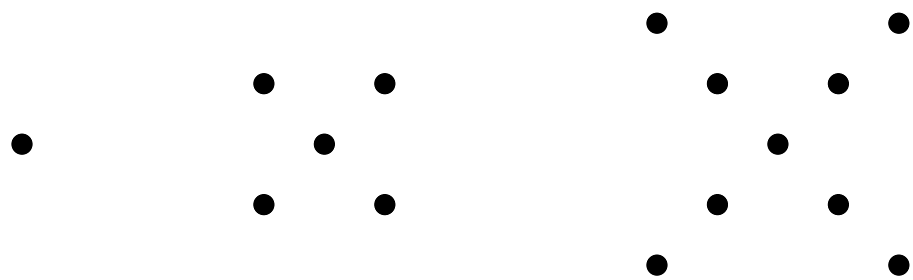
2 Sequences

Essential questions

1. How do we describe a pattern?
2. How can patterns be used to make predictions?
3. What are some ways to represent, describe, and analyze patterns?

2.1 Visual Patterns

Question 2.1 Look at the pattern below and answer the questions:



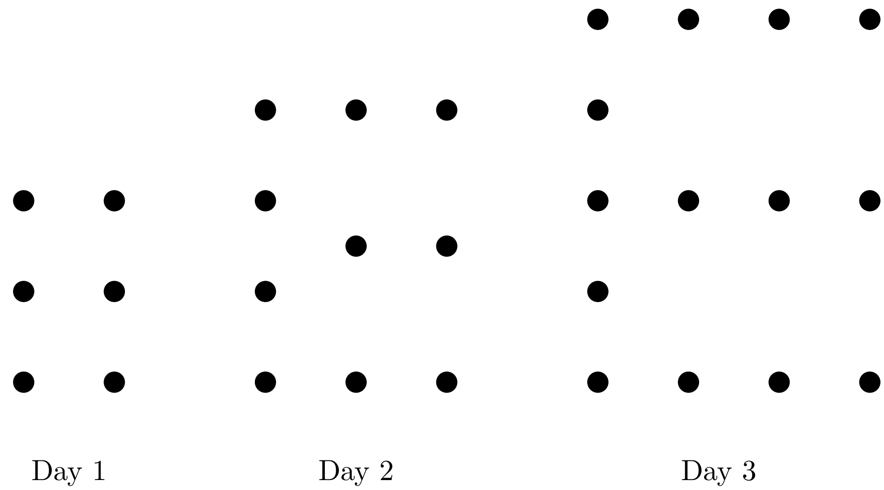
Day 1 Day 2 Day 3

a. Describe the pattern that you see in the sequence of figures above.

b. Assuming the sequence continues in the same way, how many dots are there on the fourth day? On the fifth day? On the tenth day?

c. How many dots are there on the 100^{th} day?

Question 2.2 Look at the pattern below and answer the questions:



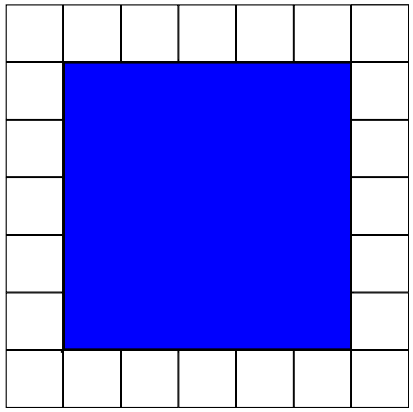
a Describe the pattern that you see in the sequence of figures above.

b Assuming the sequence continues in the same way, how many dots are there on the fourth day? On the fifth day? On the tenth day?

c How many dots are there on the 100^{th} day?

2.2 Tiling a Pool

Question 2.3 The summer season is nearly over and the owner of the local pool club is thinking of what all needs to be done once the pool closes. One of the common things in need of repair are the tiles around the perimeter of the pool. In the picture below a 5 foot square pool has been tiled with 24 square tiles (1 foot by 1 foot).



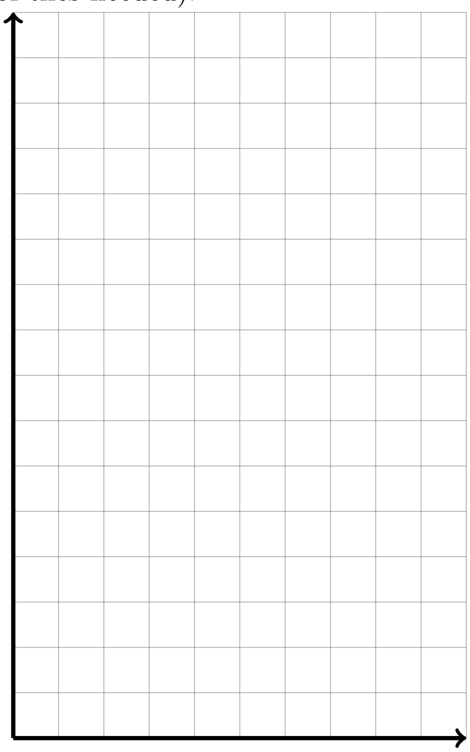
a. Make sketches to help you figure out how many tiles are needed for the borders of square pools with sides of length 1, 2, 3, 4, 6, 10 feet without counting. Record your results in a table.

b. Write an equation for the number of tiles N needed to form a border for a square pool with sides of length s feet. How do you see this equation in the table? How do you see the equation in your pictures?

c. Try to write at least one more equation for N. How would you convince someone that your expressions for the number of tiles are equivalent?

d. Use your work to decide how many tiles you would need for a square pool whose sides are 127 feet long. What about a square pool whose sides are 128 feet long?

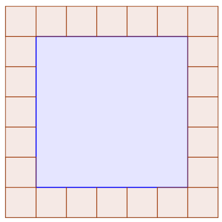
e. Graph the relationship you observed between s (the side length) and N (the number of tiles needed).



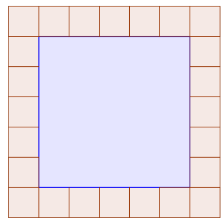
s side length	N number of tiles
1	
2	
3	
4	
5	24
6	
10	

f. Relate the growth pattern in each of the representations of the pattern (table, equation, graph).

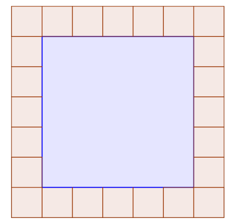
Question 2.4 Some students calculated the number of tiles differently. Show their solution in the diagram and explain their thinking.



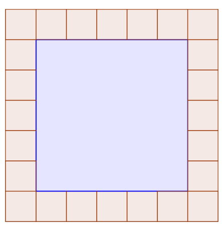
$4(s+1)$



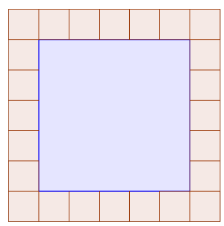
$4s+4$



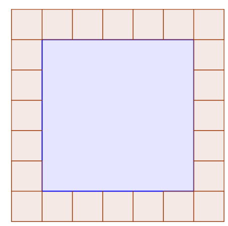
$2s+2(s+2)$



$4(s+2)-4$



$4(s+2\cdot\frac{1}{2})$



$(s+2)^2-s^2$

How can you convince someone that all of the expressions are equivalent? Use both properties of operations on whole numbers as well as the diagrams.

Definition 1 *An infinite list of numbers is called a* ***sequence****. Sequences are written in the form*

$$a_1, a_2, a_3, \ldots$$

a_n *is called the* **nth** ***term of the sequence****.*

The third term in the sequence listed above is a_3. The "3" refers to the position of the member within the sequence and a_3 refers to the number that is in that position.
For example, if we look at the sequence $\{b_n\}$ whose members are listed below:

$$17, 13, 9, 5, 1, -3, -7, ...$$

We can tell that $b_1 = 17$, $b_6 = -3$. What is b_4? What is the 10th term of this sequence?[1]

Question 2.5 If $p_1, p_2, p_3, \ldots$ is a sequence such that

$$p_n = \#\text{tiles around a square pool of side length } n,$$

a. What is the value of p_5?

b. What is the value of p_{15}?

c. What is the value of p_n?

d. What is the relationship between p_n and p_{n+1}.

Definition 2 *A sequence $a_1, a_2, a_3, \ldots$ is an* ***arithmetic sequence*** *if there is a number d such that you obtain any member of the sequence by adding d to the member that came before it. Symbolically, we'd write that:*

$$a_n = a_{n-1} + d.$$

Question 2.6 Is the sequence $p_1, p_2, p_3, \ldots$ (from Question 2.5) an arithmetic sequence?

[1] You would write that: $b_{10} = \quad$.

2.3 Geometric Sequences

Question 2.7 Social media has created a way to quickly share information (articles, videos, jokes, ...). Gangnam Style is a YouTube video that became popular in July 2012. On September 6^{th}, the video had 100,000,000 views. On December 21^{st} the video was the first video in history to have over 1,000,000,000 views. If Gangnam style was released on July 15, how many days did it take to for the video to hit 100,000,000 views? How many days did it take for the video to breach 1,000,000,000 views?

Question 2.8 To model the sensation of "viral videos", assume that on day one there was one view, that every new view corresponds to a new person seeing the video and on average a new viewer shows the video to 2 new people.

a. How many times was the video viewed on day 2?

b. How many times was the video viewed on day 3?

c. How many times was the video viewed on day 5?

d. How many times was the video viewed on day n?

e. Let $v_1, v_2, v_3, \dots$ be a sequence such that

$$v_n = (\text{the number of times the video is viewed on the } n^{th} \text{ day}).$$

Write down an algebraic relationship between v_n and v_{n+1}.

f. How many times is the video viewed in the first two days?

g. How many times is the video viewed in the first three days?

h. How many times is the video viewed in the first five days?

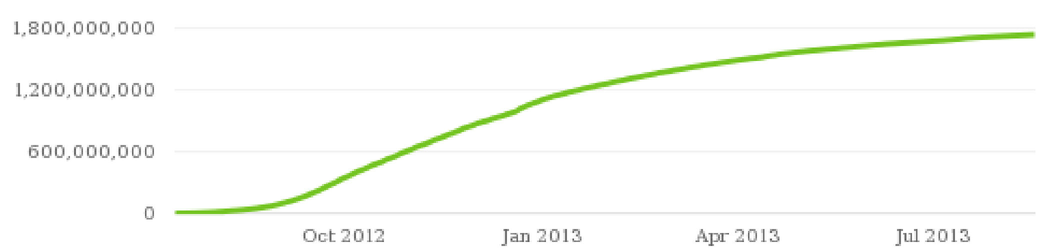
i. How many times is the video viewed in the first n days?

j. Let $t_1, t_2, t_3, \ldots$ be a sequence such that

$$t_n = (\text{total number of views from day 1 to day } n).$$

Write down an algebraic relationship between t_n and t_{n+1}.

Question 2.9 The graph below is data from YouTube about the actual number of views of Gangnam Style. Does our model accurately describe the behavior of the viral video phenomena? What do you think some limitations of our model are?



Question 2.10 A ball is dropped from a height of 10 feet. The ball bounces to 80% of its previous height with each bounce.

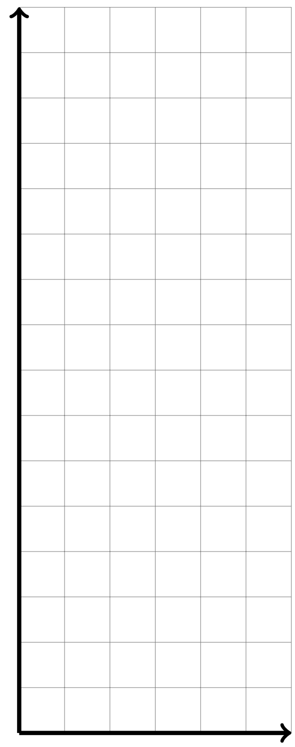
a. How high does the ball bounce after the first bounce?

b. How high does the ball bounce after the third bounce?

c. How high does the ball bounce after the n^{th} bounce?

d. Let $b_1, b_2, b_3, \ldots$ be a sequence where b_n is the height the ball bounces after the n^{th} bounce. What is the relationship between b_n and b_{n+1}?

e. Record (n, b_n) in a table and graph the relationship between n and b_n.



n	b_n
1	8
2	
3	
4	
5	
6	
10	

f. The sequence b_n models the height of a ball bouncing. How many times does the model predict the ball will bounce? Is this realistic?

Question 2.11 Assume you invest \$1,000 in a savings account that pays 5% a year.

a. How much money will you have after one year?

b. How much money will you have after two years?

c. How much money will you have after 50 years?

d. How much money will you have after n years?

n	m_n
1	
2	
3	
4	
5	
6	
10	

e. Let $m_1, m_2, m_3, \ldots$ be the sequence such that $m_n =$ dollars in account after n years. What is the relationship between m_n and m_{n+1}?

Question 2.12 The height of a ball bouncing, the number of viral video daily views, and the amount of money in the bank account are all examples geometric sequences. Describe similarities and differences among these three examples.

2.4 Counting High-Fives

After a sporting event, the opposing teams often line up and exchange high-fives. Afterward, members of the same team exchange high-fives. In this problem, you will explore the total number of high-fives that take place at the end of a game.

- Every player exchanges exactly one high-five with every other player.
- When two players exchange a high-five, it counts as one exchange, not two.

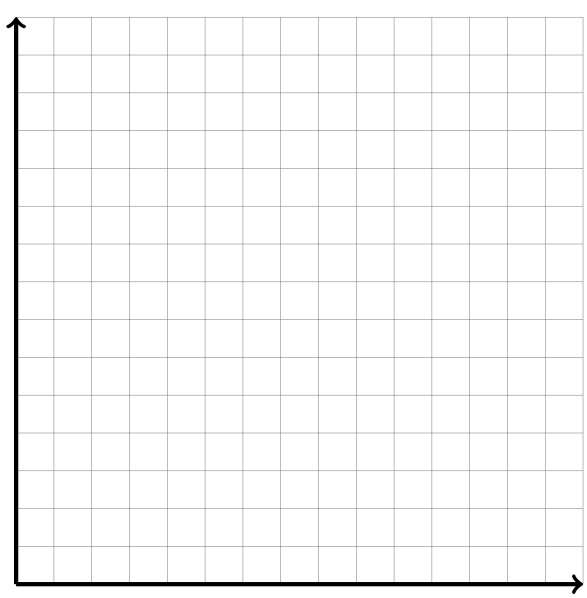
Question 2.13 If everyone in this room exchanged high-fives, guess how many high-fives there would be.

Question 2.14 Let n be the *combined* number of players on each of the two teams. Let H_n be the number of high-fives that are exchanged at the end of the game.
Complete the following table.

n players	H_n high-fives
1	
2	
3	
4	
5	
6	
7	

Question 2.15 Is the sequence $H_1, H_2, H_3, \ldots$ an arithmetic sequence? Is the sequence geometric?

Question 2.16 Sketch the graph of the high-five sequence, then describe what you see, comparing and contrasting it to the patio sequence from Question 2.3.



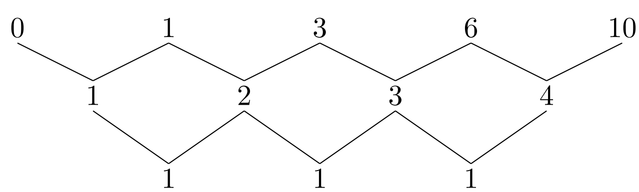
Question 2.17 What is the relationship between H_n and H_{n-1}.[2]

Question 2.18 Find an explicit formula for the sequence H_n.

Question 2.19 The defining quality of an arithmetic sequence is the constant difference between consecutive terms in the sequence.

a. Is there a constant difference between terms in the high fives sequence?

The following picture describes how to find the second difference of a sequence:



b. Add 3 terms to the top row and complete the picture.

c. What is the pattern?

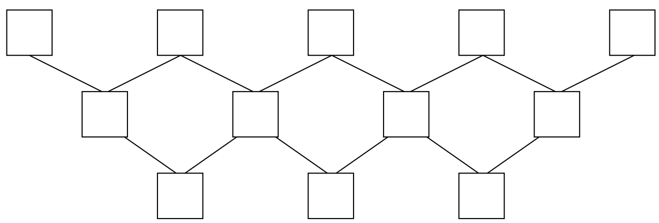
[2] Think about the context of the problem. If the 4^{th} player enters the scene, how many players are there with whom she has to exchange high fives? What about when the $5th$ player enters?

Question 2.20 Let $\{t_1, t_2, \ldots, t_n, \ldots\}$ be a sequence where the n-th term is given by $t_n = 3n^2 + n + 4$.

a. Fill in the following table:

n	t_n
1	
2	
3	
4	
5	
6	
7	

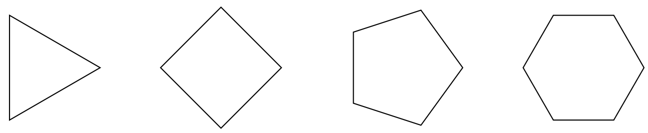
b. Calculate a few second differences of this sequence. To help you organize your thoughts, use the following diagram:



Question 2.21 Make up your own sequence which has a constant second difference.

Question 2.22 Make up your own sequence which has a constant third difference.

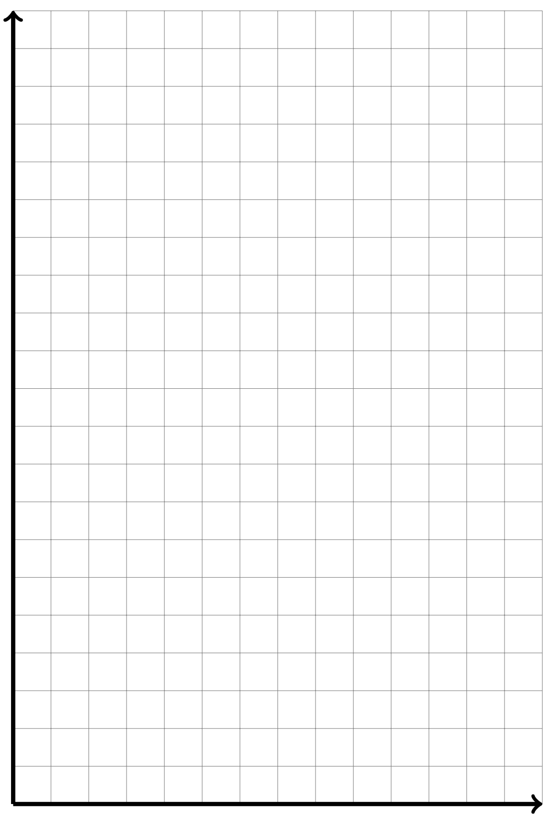
Question 2.23 A polygon is a closed shape consisting of line segments which pairwise share a common point. Below are drawn 3-sided, 4-sided, 5-sided and 6-sided polygons which you may know under different names.



A diagonal of a polygon is a line segment which connects non-adjacent vertices of the polygon. Draw all diagonals for each polygon pictured above. Let's consider the sequence $\{d_n\}$ where d_n is the number of diagonals of a polygon with n sides.

a. Fill in the following table and sketch a graph.

n sides	d_n diagonals
1	
2	
3	
4	
5	
6	
7	



b. Can you come up with a recursive formula for the sequence?

c. Can you come up with an explicit formula for the sequence? [3]

[3]Once again, it's useful to think about the context of the problem.

2.5 Summary

Definition 3 *An infinite list of numbers is called a* ***sequence****. Sequences are written in the form*

$$a_1, a_2, a_3, \ldots$$

a_n *is called the* $\mathbf{n^{th}}$ ***term of the sequence****.*

Generally people think of sequences as having a nice pattern that one can describe either using words or mathematical expressions. For example, $a_n =$ "the number of rainy days in each month starting with January of 3017" describes a sequence (this is of course assuming that people or a natural disaster do not swipe the Earth away). $b_n = 3n - 4$, where n is a whole number, also describes a sequence. The first sequence, however, does not have a nice symbolic description. If you wrote down the sequence by listing its terms, you wouldn't notice any patterns that you can describe by a neat little formula. Many sequences are just lists of random numbers. Other sequences can neatly be described in mathematical, symbolic, language.

Definition 4 *A sequence* $a_1, a_2, a_3, \ldots$ *is an* ***arithmetic sequence*** *if there is a number* d *such that*

$$a_n = a_{n-1} + d.$$

We can say that this is the sequence where each term is obtained from the previous one by adding a constant number, d. We call this a *recursive* definition of a sequence. The n-th term of an arithmetic sequence can be described *explicitly* as well:

$$a_n = a_1 + (n-1)d.$$

Here you see that, in order to describe the sequence completely it is necessary to give its first term a_1 as well as the common difference d.
Some people like to start their sequences with 0^{th} term of the sequence: $a_0, a_1, a_2, \ldots$. In that case the explicit formula for the arithmetic sequence seems less complicated:

$$a_n = a_0 + nd.$$

Another example of a special type of a sequence is the one where each element is obtained from the previous one by multiplying it by a constant number, r.

Definition 5 *A sequence* $a_1, a_2, a_3, \ldots$ *is a* ***geometric sequence*** *if there is a number* r *such that*

$$a_n = r \cdot a_{n-1}$$

The explicit formula for the n-th term of a geometric sequence is given by

$$a_n = a_1 \cdot r^{n-1}.$$

Here, too, we might get somewhat simpler expression if we start with 0^{th} term instead of the 1^{st}:

$$a_n = a_0 \cdot r^n.$$

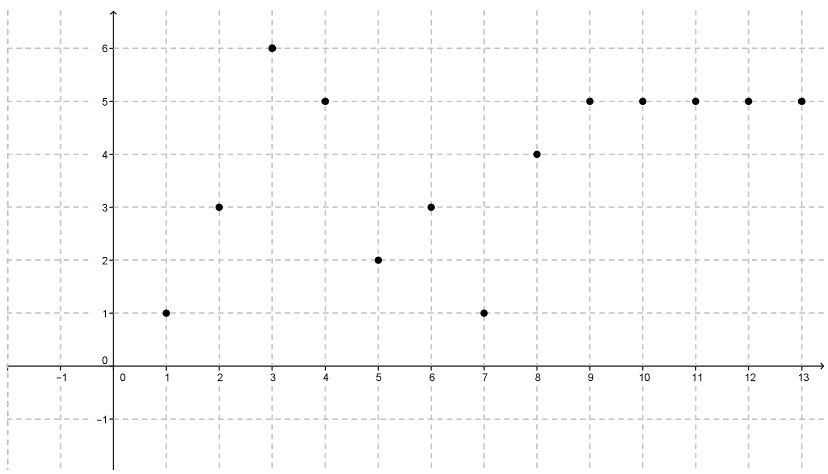
Apart from describing a sequence by giving either recursive or explicit formula that tells us what a general term looks like, we note that we can also simply use a list $\{a_1, a_2, a_3, \ldots\}$. For example: $\{1, 3, 6, 5, 2, 3, 1, 4, 5, 5, \ldots\}$.

Sequences can be organized in tables:

n	a_n
1	a_1
2	a_2
3	a_3
4	a_4

n	1	2	3	4	5	6	7	8	9	10	...
a_n	1	3	6	5	2	3	1	4	5	5	...

We can graphically represent a sequence by graphing ordered pairs (n, a_n) where the position in the sequence is represented on the x-axis, and the corresponding term of the sequence is represented on the y-axis. For example:



2.6 Student learning outcomes

1. Students will be able to identify arithmetic and geometric sequences.
2. Students will be able to use algebraic expressions, graphs, tables and verbal ques to identify and work with sequences.
3. Students will be able to compute the n-th term in a geometric/arithmetic sequence.
4. Students will be willing to engage and work with a pattern that they are unfamiliar with.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.