5 Quadratic Functions

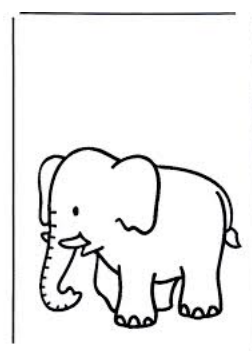
Essential Questions

1. What is the shape of the quadratic function and how can we use its features productively?
2. How can we find the zeros of a quadratic function?
3. How do we calculate the max or min of a quadratic function?

5.1 Rectangular fences

Question 5.1 You want to make a rectangular pen for Ellie, your pet elephant. What?! You don't have a pet elephant? That's rather unfortunate; they're quite cute. Well, imagine you have one. You want to make sure Ellie has as much space as possible. Unfortunately, you only have 28 feet of fencing available. If you use all of your fencing to make the pen, what is the biggest possible area you can achieve?

Outline here possible approaches to answering this question. What might you, or someone else, try to do to solve this problem?



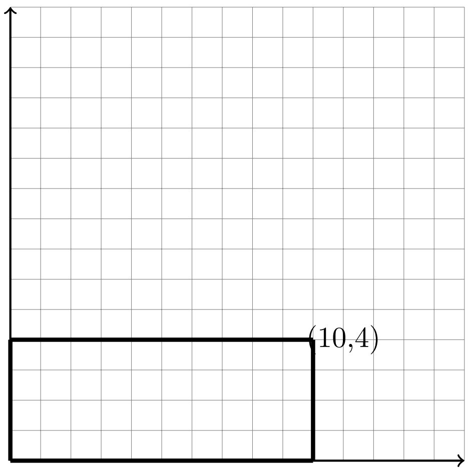
Hi! I'm Ellie, your pet elephant for a few days!

Question 5.2 We will investigate our main problem in several steps.

a. Draw 6 rectangular pens having a perimeter of 28 on the coordinate axis. Like below.

b. Label the coordinate in the upper right hand corner of each pen.

c. Make a table showing all the coordinates on your graph. Look for a pattern and make three more entries in the table.



length	height
10	4

d. Write an equation for the function described by your graph and table. This is a function that will relate the height of the rectangle as a function of the length.

Question 5.3 The point $(4, 10)$ is the upper right corner of a plausible pen.

a. What does the sum of these numbers represent in this problem?

b. What does the product of these two numbers represent in this problem?

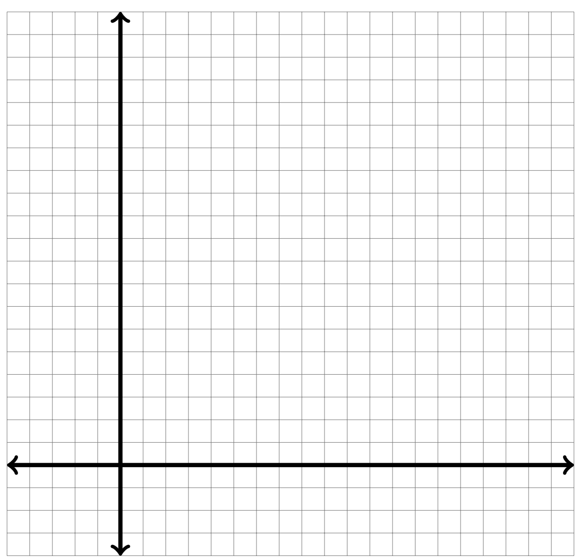
c. Of all the rectangular pens recorded on your chart, which rectangular pen enclosed the largest area?

d. How many rectangles are there whose perimeter is 28?

Question 5.4 For each rectangle from question c. compute the area.

length	height	area
10	4	

Question 5.5 Make a graph of the area as a function of length. Connect the points on your graph with a smooth curve[1]. What kind of curve is it?



Question 5.6 On the same coordinate system, make a graph of the area as a function of height. Connect the points on your graph with a smooth curve. What kind of curve is it? What else do you notice?

Question 5.7 Why did it make sense to connect the dots of both graphs?

[1] There should be no corners or sharp transitions¿

Question 5.8 In this question we will interpret the graph.

a. Label the highest point on your graph from 5.5 with its coordinates. Interpret these two numbers in terms of this problem. [2]

b. Where does the graph cross the x-axis? What do these numbers mean?

c. If you increase the length by one foot, does the area increase or decrease? Does it change the same amount each time? Explain.

Question 5.9 We will now articulate our findings algebraically.

a. Describe in words how you would find the area of the rectangular pen having perimeter 28, if you knew its length.

b. If the perimeter of the rectangular pen is 28 and its length is L, write an algebraic expression for its area in terms of L.

c. If you had 28 feet of fencing and wanted to make the largest possible rectangular pen, what would its length, width, and area be? Explain.

[2] This means: write a complete sentence explaining what your interpretation is.

Question 5.10 Now let us generalize our findings. You should make a sketch.

a. Describe in words how you would find the area of the rectangular pen having perimeter P if you knew its length.

b. If the perimeter of the rectangular pen is P and its length is L, write an algebraic expression for its area in terms of L and P.

c. If you had P feet of fencing and wanted to make the largest possible rectangular pen, what would its length, width, and area be? Explain.

5.2 Graphs of quadratic functions

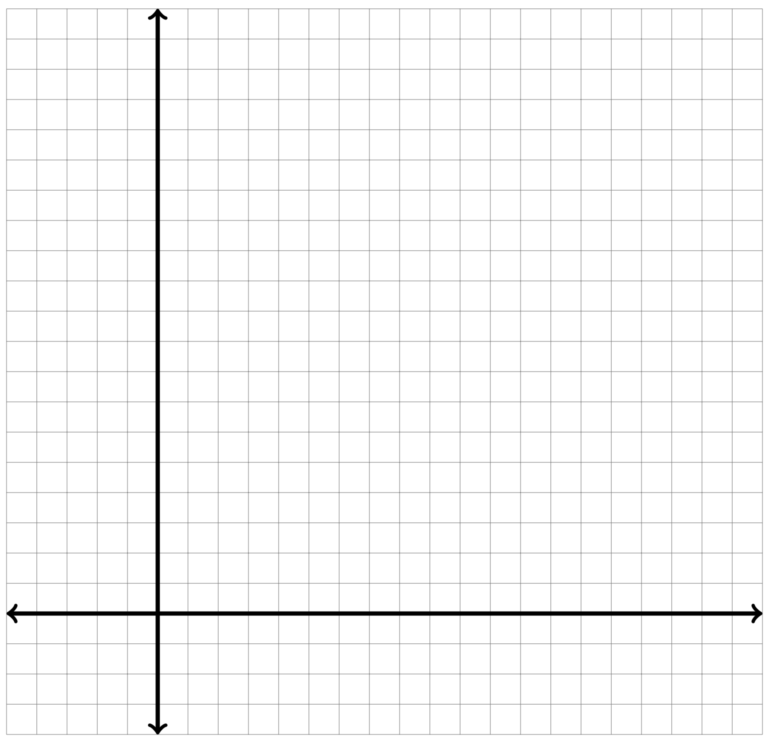
Question 5.11 Graph each of the following functions. Use a scale that will show values from -5 to 20 for the domain and from -20 to 100 for the target. To graph the functions, make a table and plot points.

a. $f(x) = x(8 - x)$

b. $g(x) = x(15 - x)$

c. $h(x) = x(12 - x)$

d. $k(x) = x(5 - x)$



Question 5.12 For each of the parabolas in question 5.11,

a. label the graph with its equation;

b. label the x-intercepts;

c. label the y-intercepts;

d. label the vertex;

e. draw the line of symmetry;

f. note if the graph opens up or opens down.

g. by looking at the graph note if any of the functions have an inverse function.

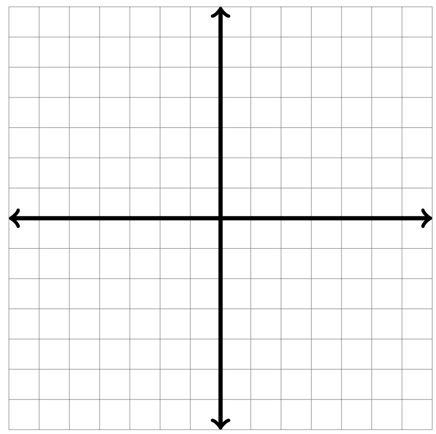
We will use the previous question to try to infer some general statements about the graphs of quadratic functions.

Question 5.13 You are given an arbitrary number b. Describe the graph of the quadratic equation $f(x) = x(b - x)$. Write an expression for the coordinates of its intercepts and maximum value in terms of b.

a. the x-intercepts:

b. the y-intercepts:

c. What is the line of symmetry for the graph?



Sketch the graph!

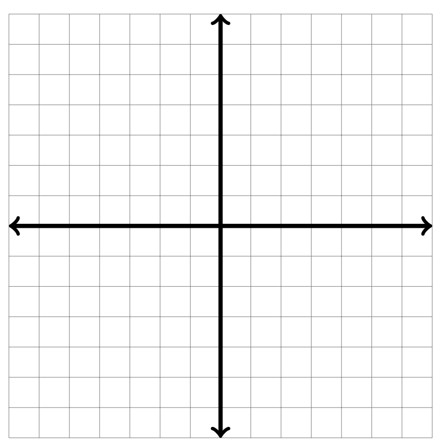
d. What is the maximum value of f?

Question 5.14 For an arbitrary number q, describe the graph of the quadratic equation $f(x) = x(x - q)$. Write an expression for:

a. the x-intercepts:

b. the y-intercept:

c. What is the line of symmetry for the graph?

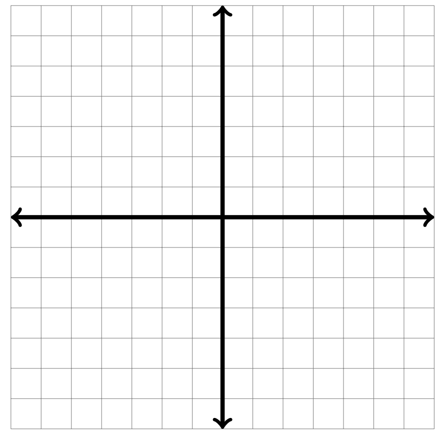


Sketch the graph!

d. What is the minimum value of f?

Question 5.15 For arbitrary numbers a, b, describe the graph of the quadratic equation $f(x) = (x-a)(x-b)$. Write an expression for:

a. the x-intercepts:



Sketch the graph!

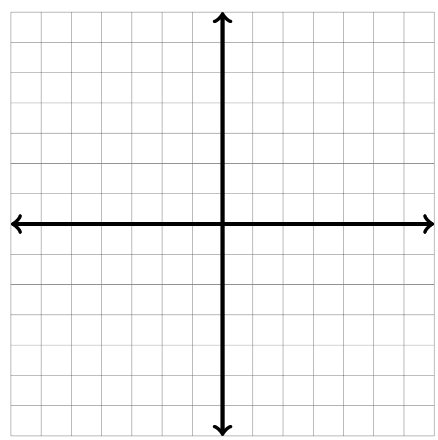
b. the y-intercept:

c. What is the line of symmetry for the graph?

d. What is the minimum value of f?

Question 5.16 Graph the function $f(x) = x^2 + x - 6$. Write an expression for

a. the x-intercepts;



Sketch the graph!

b. the y-intercept.

c. What is the line of symmetry for the graph?

d. What is the minimum value of f?

5.3 The Zero Product Property

Question 5.17 If $ab = 0$, which of the following is impossible? Explain.

a. $a \neq 0$ and $b \neq 0$

b. $a \neq 0$ and $b = 0$

c. $a = 0$ and $b \neq 0$

d. $a = 0$ and $b = 0$

> **Property 1** When the product of two quantities is zero, one of the quantities must be zero.

Question 5.18 If $(x-6)(-2x-1) = 0$, what are the possible values for x? [3]

Question 5.19 What would Property 1 say if the product of three quantities equaled 0?

$$a \cdot b \cdot c = 0$$

Question 5.20 Use Property 1 to solve the following equations:

a. $(3x+1)x = 0$

b. $(2x+3)(10-x) = 0$

c. $(3x-3)(4x+16) = 0$

d. $6x^2 = 12x$

[3] Hint: use Property 1

Definition 1 An integer q is a **factor** of the integer p if there is a third integer g such that

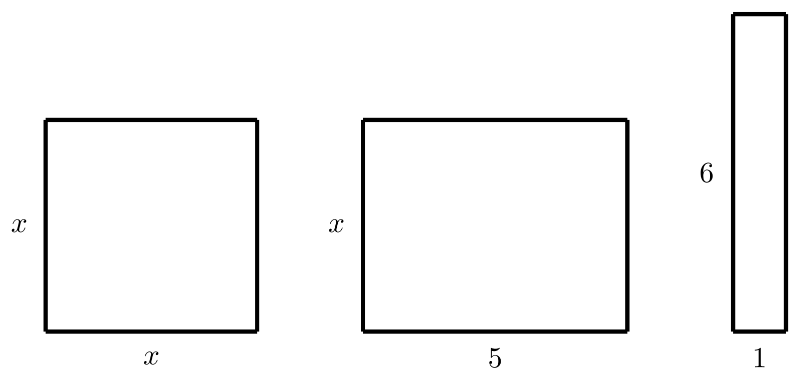
$$p = gq.$$

Definition 2 A polynomial $q(x)$ is a **factor** of the polynomial $p(x)$ if there is a third polynomial $g(x)$ such that

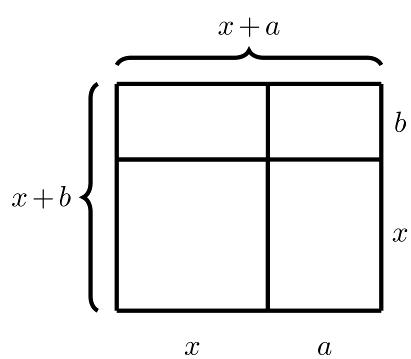
$$p(x) = q(x)g(x).$$

Question 5.21 Solve $x^2 + 5x + 6 = 0$. Our goal is to accomplish this by writing the left hand side as a product of two linear expressions, and then using the zero product property to find the solutions.

a. Each term on the left hand side of the equation has a geometric meaning. Explain how the figures from left to right represent $x^2, 5x$, and 6, respectively.



b. When we factor $x^2 + 5x + 6$ we are representing the above area as the area of a single rectangle.



c. Find a and b such that $x^2 + 5x + 6 = (x + a)(x + b)$. Use the picture.

d. Now that we have factored $x^2 + 5x + 6$, solve the equation $x^2 + 5x + 6 = 0$

Question 5.22 Solve the following equations by factoring. To help you factor draw the picture from Question 5.21.

a. $x^2 + 6x + 9 = 0$

b. $x^2 + 12x + 35 = 0$

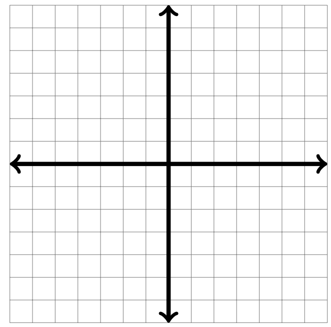
c. $x^2 + 9x + 20 = 0$

Question 5.23 Is $x = 4$ a solution to the equation $x^2 + 4x - 4 = 0$? Explain what it means to solve an equation.

Question 5.24 Not every quadratic polynomial can be factored. Which one of following polynomial functions can not be factored? You will want to graph each of them. Filling out a table should help.

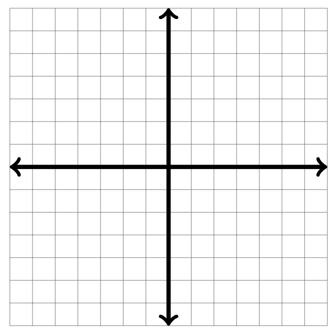
a. $f(x) = x^2 + 10x + 25$

x	$f(x)$



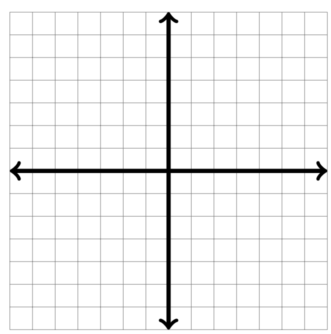
b. $g(x) = x^2 + 7x + 5$

x	$g(x)$



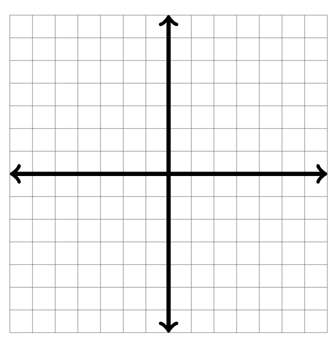
c. $h(x) = x^2 + 10x + 21$

x	$h(x)$



d. $k(x) = x^2 - 6x + 10$

x	$k(x)$



5.4 Completing the Square

Question 5.25 Solve the following equations[4]:

a. $x^2 = 4$

b. $x^2 = 25$

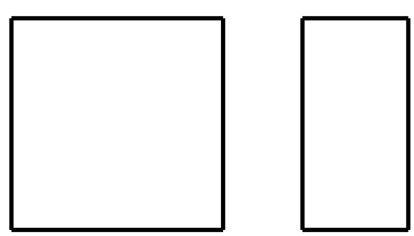
c. $x^2 = 7$

d. $(x+3)^2 = 16$

e. $(x+4)^2 = 5$

Question 5.26 In Question 5.25 we were able to solve the equation $x^2+8x+11 = 0$[5]. Try and factor $x^2+8x+11$. In this question we are going to investigate how to turn $x^2+8x+11 = 0$ into the more convenient form of $(x+4)^2 = 5$.

a. Label the sides of the square and rectangle below so that the total area is x^2+8x.

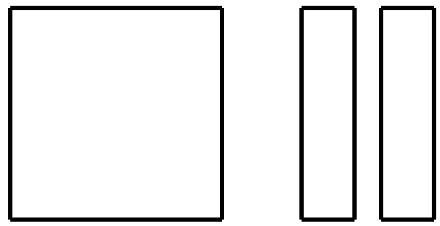


b. Our goal is to cut and rearrange the pieces we have so that the new shape resembles a square as much as possible. What would you do?

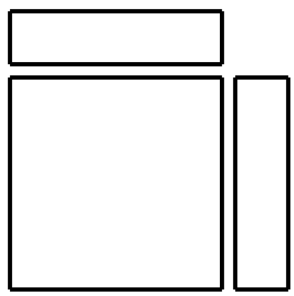
c. Why do we want a square?

[4]Hint: Each equation has two solutions
[5]Really?! I didn't see it there. Did you?

d. Here is how one student did this: she chopped the $8x$ rectangle in half. Label each side length. Has the area changed?



e. In the picture below one of the rectangles has been moved to the top. Label the side lengths. Has the area changed?



f. Notice that this arrangement almost makes a square. What would be the area of the entire square?

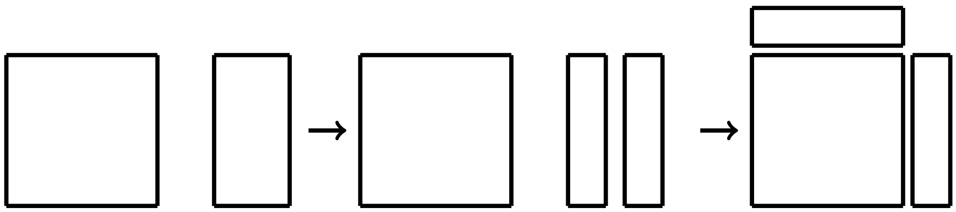
g. What is the area of the missing piece?

h. Write an algebraic equation that relates: the area of the entire square, the area of missing piece, and $x^2 + 8x$.

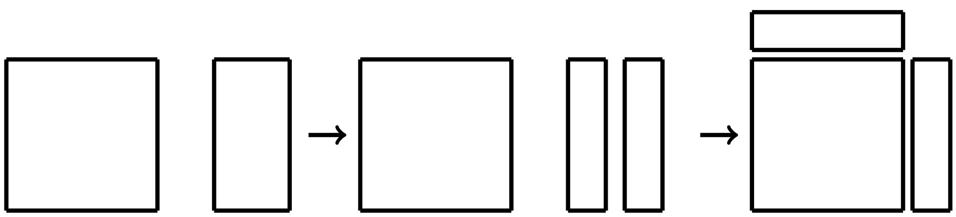
i. Use Part h. to substitute for $x^2 + 8x$ in the equation $x^2 + 8x + 11 = 0$.

Question 5.27 The process that we carried out in Question 5.26 is called completing the square. It is **wonderfully useful**. Complete the square for the following expression. Use the pictures provided to organize your thoughts.

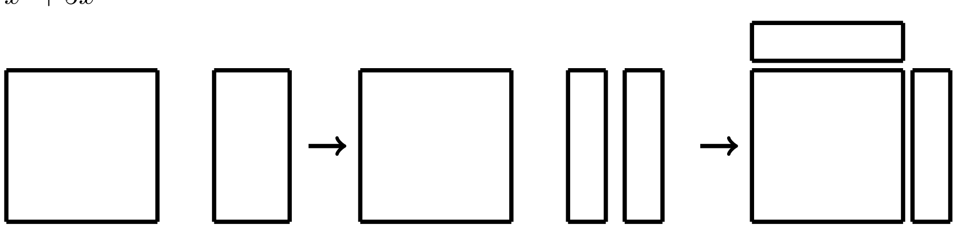
a. $x^2 + 10x$



b. $x^2 + 12x$



c. $x^2 + 5x$



Question 5.28 In Question 5.27 we completed the square for several expressions. Use that information to solve the following equations:

a. $x^2 + 10x = 10$

b. $x^2 + 12x = 14$

c. $x^2 + 5x = 7$

Question 5.29 Let's take this up a notch. Solve the following equations:

a. $2x^2 - 4x - 16 = 0$

b. $2x^2 + x - 6 = 0$

5.5 Calculating Maximum and Minimum Values of Quadratic Functions

Question 5.30 David Ortiz of the Boston Red Sox has an average off the bat speed of 102.2 miles per hour in the 2013 play off season. The average vertical speed off the bat is 67.5 miles per hour. This means that the height of the ball is given by $h(t) = -16t^2 + 99t$.

a. How long is the ball in the air?

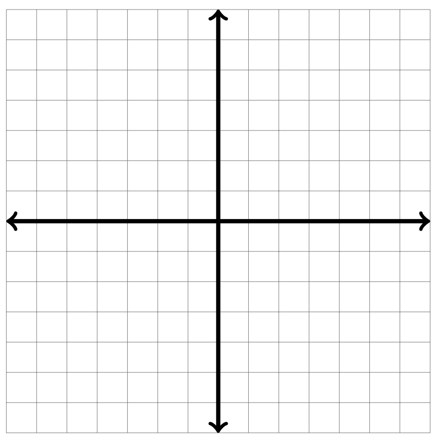
b. What is the maximum height of the ball?

Question 5.31 Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2 + 3x - 4$. Does f have a maximum or a minimum? How were you able to tell?

Question 5.32 The following questions lead us to discover what the minimum value of $f(x) = x^2 + 3x - 4$ is.

a. Given that $f(x) = x^2 + 3x - 4$ what are the values of x such that $f(x) = 0$?

b. Use the symmetry of the graph of f to calculate the value of x where f achieves its minimum or maximum.



c. Use Part b. to find the minimum or maximum value of f.

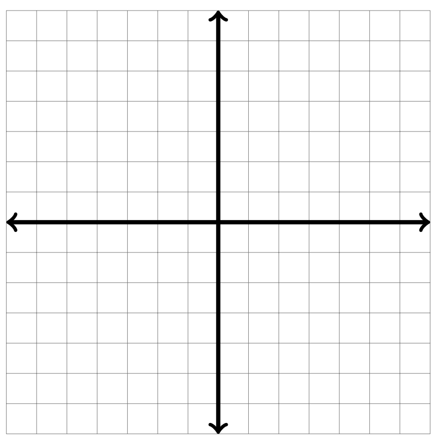
d. Write $f(x)$ in a completed square form.

Question 5.33 Let $g(x) = -x^2 + 4x - 2$.

a. Does g have a maximum or a minimum? How are you able to tell?

b. What are the values of x such that $g(x) = 0$?

c. Use the symmetry of the graph of g to calculate the value of x where g achieves its maximum.



d. What is the maximum value of g?

e. Write $g(x)$ in a completed square form.

Question 5.34 So why might you bother with different forms of the function expression?

a. In what circumstances would the completed square form be more useful?

b. In what circumstances would the factored form be more useful?

c. In what circumstances would the standard form be more useful?

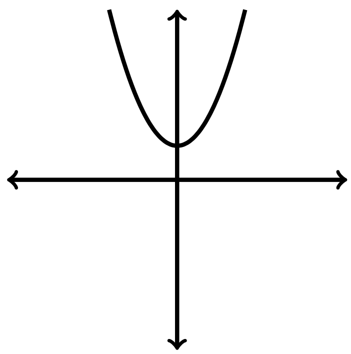
Question 5.35 Find a number between 0 and 1 such that the difference of the number and its square is maximum.

5.6 Summary

Definition 3 A **quadratic function** $f : \mathbb{R} \to \mathbb{R}$ is a function given by an algebraic rule of the form:

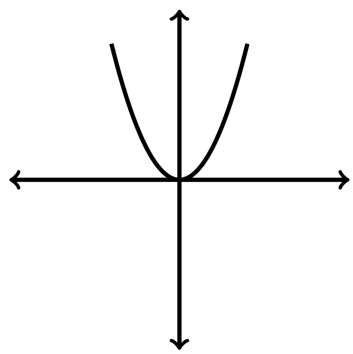
$$f(x) = ax^2 + bx + c \text{ with } a \neq 0.$$

The graph of a quadratic function f is a parabola. The y-intercept of the graph of a quadratic function f is the point $(0, f(0))$. This is a common feature with any function, since the y-intercept occurs when the x-coordinate is 0. A parabola can have $0, 1$, or 2 x-intercepts. This is demonstrated by the following three pictures:



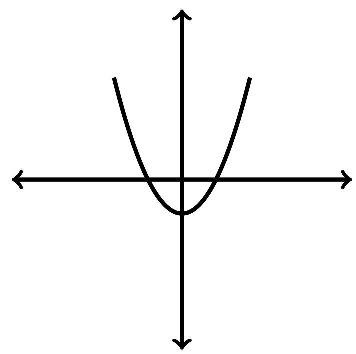
No x-intercepts:

$f(x) = x^2 + 1$



One x-intercept:

$f(x) = x^2$



Two x-intercepts:

$f(x) = x^2 - 1$

To find the x-intercepts of the graph of a quadratic function we solve the equation $f(x) = 0$. We have developed two techniques for solving a quadratic equation: factoring and completing the square, which we will outline below.
The parabola associated to a quadratic functions opens up if the leading coefficient is positive, and the parabola opens down if the leading coefficient is negative. A parabola has a vertical line of symmetry, and the location of the axis of symmetry can be calculated by looking at the average of the x-intercepts, if they exist. If the x-intercepts do not exist, it is still possible to locate the axis of symmetry by locating to inputs that yield the equal outputs. Again, averaging these two inputs will yield the x-coordinate of the points on the axis of symmetry.

Example 1 In this example we will work through how to graph a function $f(x) = x^2 + 2x - 3$.

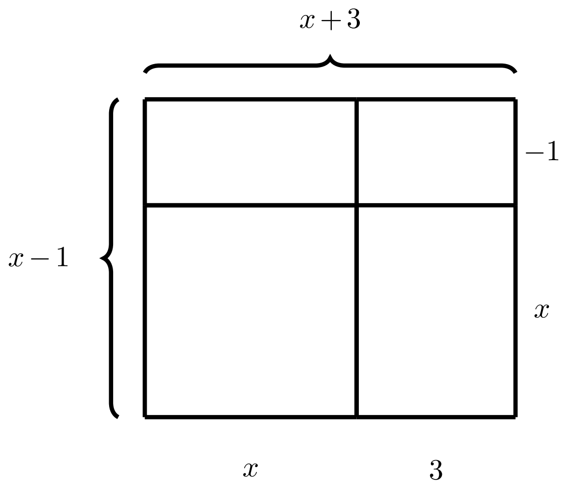
y-intercept: This is always the point $(0, f(0))$, so in this example we have the y-intercept:

$$(0, f(0)) = (0, 0^2 + 2 \cdot 0 - 3) = (0, -3)$$

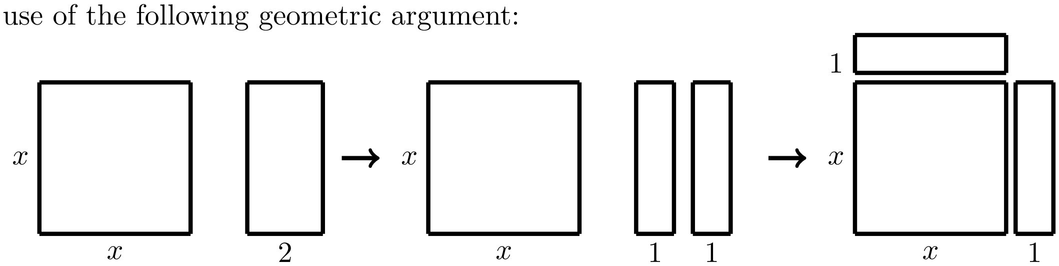
x-intercept: To find the x-intercepts we need to solve the equation

$$f(x) = x^2 + 2x - 3 = 0.$$

The following highlights how to use both the factoring method and the completing the square method. The quadratic expression $x^2 + 2x - 3$ factors as $(x+3)(x-1)$. This can be shown by drawing the following picture:



Now we use the zero product property to conclude that $x-1=0$ or $x+3=0$. Solving these two linear equations, gives the solutions to the original quadratic equation; $x=1$ or $x=-3$. This tells us that the x-intercepts for the graph are $(1,0)$ and $(-3,0)$.
To solve the equation $x^2+2x-3=0$ using the completing the square method we make use of the following geometric argument:



Which shows that $x^2+2x=(x+1)^2-1$. This is used to substitute in the original equation:

$$x^2+2x-3=(x+1)^2-1-3=(x+1)^2-4=0$$

Now we add 4 to both sides:

$$(x+1)^2=4$$

Take the square root of both sides:

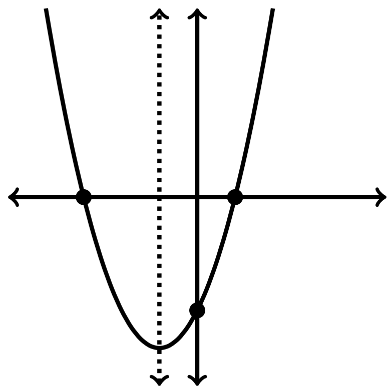
$$x+1=\pm 2$$

Subtract one from both sides:

$$x=1\pm 4=3,-1$$

Axis of symmetry: There are two ways to find the axis of symmetry. The first is to make use of the idea that it is a line of symmetry and therefore must lie half way between the two roots. So we take the average of the roots: $\frac{-3+1}{2}=-1$ and conclude that the axis of symmetry is the line given by $x=-1$. The second way to find the axis of symmetry is by looking at the completed square form: $(x+1)^2-4$. Here we can note that $(x+1)^2$ will achieve a minimum value at $x=-1$, since any number squared is always nonnegative. This means that the function f achieves its minimum at $x=-1$. The axis of symmetry always passes through the max/min so the line of symmetry is given by the equation $x=-1$.

max/min: We know that f has a local minimum because the parabola opens up. The minimum occurs on the axis of symmetry, and so the minimum is therefore the point $(-1,f(-1))=(-1,(-1)^2+2\cdot(-1)-3)=(-1,-4)$.



x-intercepts: $(-3,0)$ and $(1,0)$
y-intercepts: $(0,-3)$
axis of symmetry: $x=-1$
minimum at $(-1,-4)$

5.7 Student learning outcomes

1. Students will understand a geometric model for factoring/multiplying.
2. Students will be able to apply a geometric understanding to the process of completing the square.
3. Given a quadratic function in standard form students will be able to graph the function.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.