4 Linear Functions

Essential questions

1. If a function $f(x)$ has a constant rate of change, what does the graph of $f(x)$ look like?

2. What does the slope of a line describe?

3. What can be said about the intersection of two lines?

4.1 Interpolating a Discrete Set of Data

Zion Bank on the corner of 4^{th} South and 7^{th} East has a sign that reports the time and temperature. The temperature is given in two ways, using both the Celsius and Fahrenheit temperature scales. Here is a log of the temperature at different times of the day for August 29, 2013:

Time	Temp (C)	Temp (F)
11:03	31	87
12:00	32	90
2:00	35	95
3:04	35	95
4:08	34	93
8:03	27	81

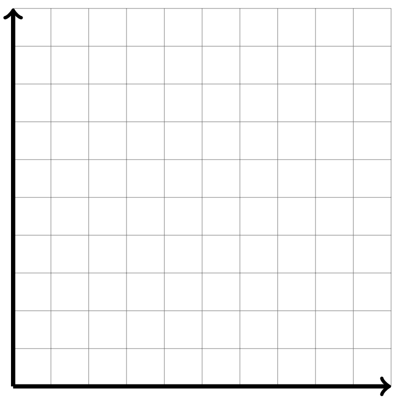
The weather report said that the low for the night had been 74° F at $4:30$ am and the high for the day was 97° F at $3:30$pm. Using the information in the table, estimate what you think the Celsius readings on the bank sign would have been at those two times. Explain how you got your answers.

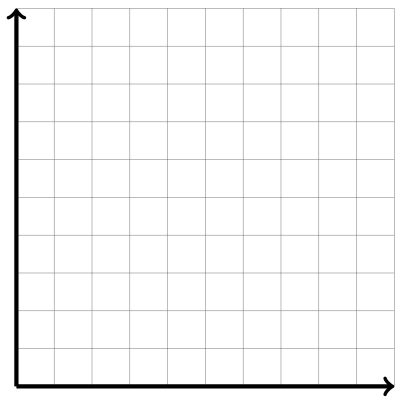
Question 4.1 Use the coordinate systems below to plot the data. There are few issues that you should be paying attention to:

a. Choose an appropriate scale and plot the points that show how the Celsius temperature changes with time. Your first point will be $(11:03, 31)$.

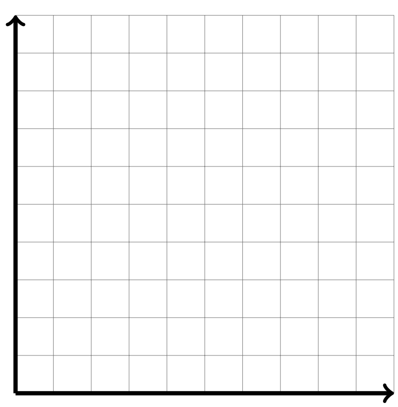
b. Plot the points that show how the Fahrenheit temperature changes with time. Your first point will be $(11:03, 87)$.

c. Write a short description of what your graphs show. Compare the two graphs.





Question 4.2 So far we have observed how the temperature reported in different scales depended on time. Now we will see how the Fahrenheit temperature changes with respect to the Celsius temperature. As before, choose an appropriate scale and plot the points from the table. Your first point will be (31, 87).



a. The points of your graph should fall approximately in a straight line. Draw a straight line that seems to go through most of the points.

b. What is the Fahrenheit temperature when the Celsius temperature is 25°?

c. What is the Celsius temperature when the Fahrenheit temperature is 50°?

d. Is there a temperature where a Fahrenheit and Celsius thermometer show the same number? If so, what is it?

Question 4.3 If you increase the Celsius temperature by one degree, by how much does the temperature increase on the Fahrenheit scale?

a. Explain how you know whether your answer to the previous question is accurate.

b. How would your answer be different if you knew that 25°C is 77°F, and that 50°C is 122°F?

Question 4.4 We want to come up with a general rule such that if we know the temperature in Celsius we can calculate the temperature in Fahrenheit.

a. From Question 4.3 we know the effect of increasing the Celsius temperature by one degree on the temperature in Fahrenheit.

b. Write down a rule that converts the temperature in Celsius to the temperature in Fahrenheit.

Question 4.5 Use the function you just obtained to find the rule that converts the temperature in Fahrenheit to the temperature in Celsius.

Question 4.6 There is another temperature scale called Kelvin. The scale is used because 0° Kelvin is the minimum temperature a system can have. If the temperature increases by one degree Kelvin, then the temperature also increases by one degree Celsius. Use the fact that $-273.15°C = 0°K$.

a. Find a function g that represents the conversion between Celsius and Kelvin.

b. Find a function h that represents the conversion between Fahrenheit to Kelvin.

c. If you haven't already, how can you use composition of functions to answer the previous question?

4.2 Slope

Question 4.7 Kingda Ka is a steel accelerator roller coaster located at Six Flags Great Adventure in Jackson, New Jersey, United States. It is the world's tallest roller coaster, the world's second fastest roller coaster, and was the second strata coaster ever built. The steepest portion of Kingda Ka is a 418 foot drop. During the 418 foot drop the train moves 25 feet horizontally.

Your friends Nancy and John are debating if Kingda Ka is steeper than Wicked, a roller coaster at Lagoon Amusement Park in Farmington, Utah. Lagoon does not advertise the specs of Wicked as well as Six Flags does. However, Nancy and John have a photograph of them on the ride. They measure the drop in the photograph 15 cm and after the drop the train has only been displaced 1 cm.

a. Is there enough information to determine which roller coaster is steeper?

b. If so calculate which coaster is steeper.

c. Is steepness all you look for in a roller coaster?

Question 4.8 Steep roads sometimes have a sign indicating how steep they are. For example, the sign may say 5% Grade. This means that you gain 5 units of altitude (the rise) for every 100 units you move in the horizontal direction (the run).

a. On a 5% grade, how many units of altitude do you gain for every 200 units you move in the horizontal direction.

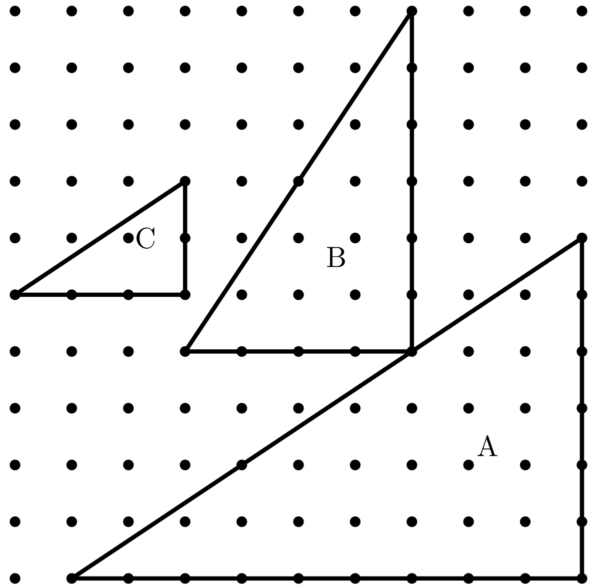
b. On a 5% grade, how many units in the horizontal direction would you have to move to increase your altitude by 100 units?

c. How would a mathematician report a 5% grade? What is the corresponding slope?

d. If the road up Little Cottonwood Canyon travels 8.26 miles horizontally and the elevation change is about 4000 feet, what is the average grade of canyon road? What is the average slope? (Use the fact that there are 5280 feet in a mile)

e. What is the grade when you are driving on the Salt Flats?

Question 4.9 Consider the following :



a. Find the slope of each hypotenuse in the above figure.

A :

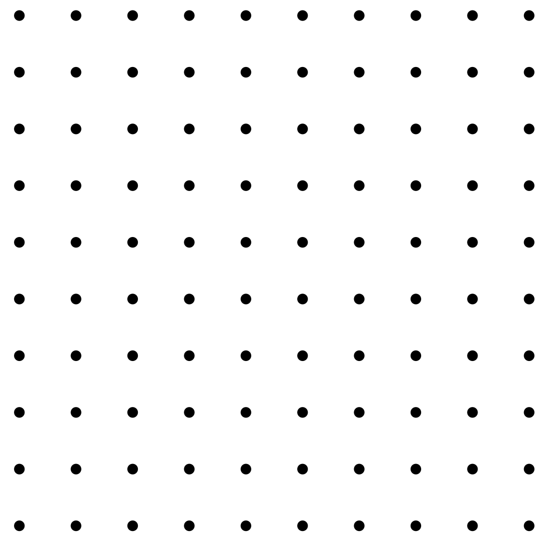
B :

C :

b. Which triangle has the steepest hypotenuse?

c. Two of the triangles' hypotenuse have the same slope. Why might someone make the mistake and report all three of the triangles have the same slope?

Question 4.10 Here is a geoboard:



a. Draw a triangle on the geoboard that would have a hypotenuse with the largest possible slope. Calculate the slope of the figure you drew. Explain how you know it is the requested triangle.

b. Draw a triangle on the geoboard that would have a hypotenuse with the smallest possible slope. Calculate the slope of the figure you drew. Explain how you know it is the requested triangle.

c. List all the possible slopes of the triangles you can draw on the geoboard. Report them as fractions.

Question 4.11 What can you say about the slope of a line if, when you follow the line from left to right

a. It goes up?

b. It goes down?

c. It doesn't go up or down?

Question 4.12 What can you say about the slope of a line that does not contain any points in the

a. First quadrant.

b. Second quadrant.

c. Third quadrant.

d. Fourth quadrant.

Question 4.13 The slope between two points is the quotient of the difference between their y-coordinates and the difference between their x-coordinates ($\frac{\Delta y}{\Delta x}$).

a. What does this mean for the slope of a vertical line?

b. What does this mean for the slope of a horizontal line?

4.3 Lines

Question 4.14 For each equation below find two pairs of numbers, (x,y), that satisfy the equation. Label the two points and calculate the slope of the line segment that connects the two points.[1]

a. $y = 1.5x + 3$

b. $y = -1.5x + 3$

c. $y = 2x + 3$

d. $y = -3x + 3$

e. How did your answer compare to people who chose different points?

[1] Do not simply copy down the slope from the equation!

Question 4.15 Find two (x,y) pairs that satisfy the equation $y = mx + b$ (your pairs should be in terms of m and b). Use the pair of points to calculate the slope of the line segment connecting the two points.

Question 4.16 Given an equation for a line $y = .5x + 3$, how do you calculate the y-intercept? Explore both a geometric technique and an algebraic technique.

Question 4.17 Given an equation for a line $y = mx + b$, calculate the y-intercept. Did you use an algebraic or geometric approach?

Question 4.18 For each of the following linear equations, fill out the following tables.

a. $y = x + 2$

x	y
0	
1	
2	
3	

x	y
0	
2	
4	
6	

x	y
1	
3	
6	
8	

When $x = 0$, what is y?

When x increases by 1, how much does y increase?[2]

b. $y = -4 - 3x$

x	y
0	
1	
2	
3	

x	y
0	
2	
4	
6	

x	y
1	
3	
6	
8	

When $x = 0$, what is y?

When x increases by 1, how much does y increase?

c. $y = 9$

x	y
0	
1	
2	
3	

x	y
0	
2	
4	
6	

x	y
1	
3	
6	
8	

When $x = 0$, what is y?

When x increases by 1, how much does y increase?

Where do you find these numbers in each of the tables for each equation?

[2] If y decreases, think of it as a negative increase.

Question 4.19 In the 2013-2014 academic year the tuition to attend the University of Utah is \$6400 a year (for 12 credits a semester). In the 2012-2013 academic year the cost of tuition was \$6000 a year (for 12 credits a semester).

a. Suppose that a linear function can model the tuition at the U. What will the tuition cost for the academic year $2014-2015$ (for 12 credits a semester)?

b. Write down a function f such that $f(t)$ represents the tuition in the academic year t (for 12 credits a semester). Discuss what a reasonable domain might be for your function by thinking about what $f(0)$ and $f(10,000,000)$ would represent.

c. For what values of t will $f(t)$ be most accurate?

d. In what year will tuition cost $\$10,000$ per semester? (according to our model)

4.4 Summary

One of the simplest, and very useful, functions are linear functions. You're used to seeing them as linear equations such as this one $y = 2x+1$, although those aren't the only linear equations. Other examples include $2x-3y = 4$ or $2(y-3), = 3(x+1)$. Each of these equations have one thing in common: if we choose any two pairs of solutions, and find the slope between them:

$$\frac{\text{change in} \quad y \quad \text{coordinates}}{\text{change in} \quad x \quad \text{coordinates}} = \frac{\Delta y}{\Delta x}$$

we will inevitably get the same number! Further, each of the equations can be placed into the following format: $y = ax+b$, for some real numbers a and b, which motivates us to give the following definition:

Definition 1 A **linear function** $f : \mathbb{R} \to \mathbb{R}$ is a function given by an algebraic rule of the form:

$$f(x) = ax+b$$

Here the number a represents the slope, the rate of change, of the function f. It tells us how much the output changes when the input changes by 1.

Line through two points We know that to completely determine a line it is enough to know two points that lie on it. Suppose then, that a line passes through two points: (x_1, y_1) and (x_2, y_2). We know that the slope between those two points is:

$$a = \frac{y_2 - y_1}{x_2 - x_1}.$$

Our function, f, then has the following rule:

$$f(x) = \frac{y_2 - y_1}{x_2 - x_1}x + b,$$

or, if you prefer, your equation of the line is:

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + b.$$

We still need to know b. Since both points, (x_1, y_1) and (x_2, y_2), lie on this line and belong to the function, we can use either of them to find b by substituting the values of its coordinates for x and y in the equation. For example, let's use the first point:

$$y_1 = \frac{y_2 - y_1}{x_2 - x_1}x_1 + b.$$

Now we have an equation in which only b is an unknown and we know how to solve those.

Line with a given slope through a given point We can similarly find an equation of a line if we know its slope and one point that lies on it. In essence, half of the work we did above has already been done for us. Say we know that the slope is a and a point that belongs to the line is (x_1, y_1). We have $y = ax+b$, we know a, so we just need to find b. Since we know our point satisfies the equation of the line we can easily find b from the following equation: $y_1 = ax_1 + b$.

4.5 Student learning outcomes.

1. Students will be able to use a discrete set of data and draw an interpolating graph of a function.

2. Students will be able to recognize linear functions from graphs, equations, tables and verbal descriptions.

3. Given two points in the plane a students will be able to write an equation for the line that passes through the two points.

4. Given a linear function $f(x)$, students will be able to draw the graph of $f(x)$.

5. Given two linear functions $f(x)$ and $g(x)$, students will be able to determine if the graphs of the functions intersect.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.