3 Functions

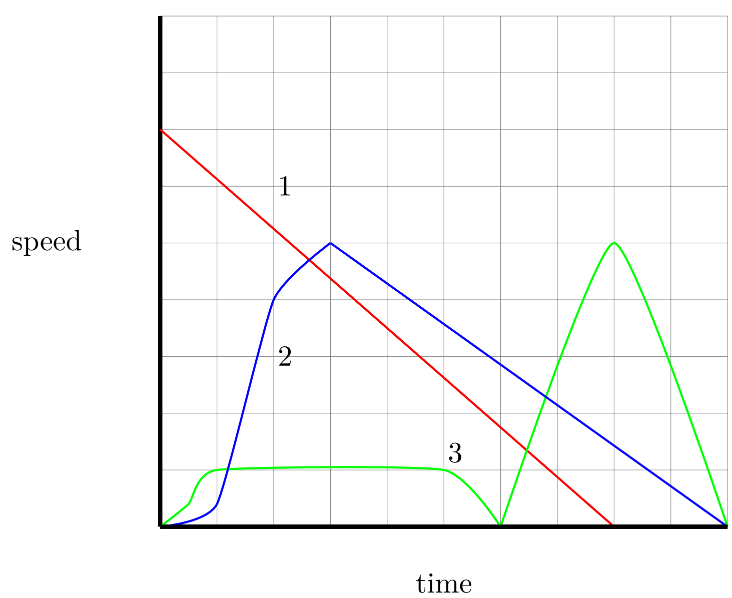
Essential Questions

1. How do we represent relationships?
2. How do you recognize if a relationship is a function?
3. How do we express a function?
4. How is a function related to its graph?
5. In what ways can functions be combined?
6. Given a discrete set of data how could we use this to describe a function?

3.1 Describing relationships

You've all taken photographs, and you know that those accurately represent some occurrence: a child sledding, a car driving by, a marble rolling on the floor. Those photographs depict a situation, but don't tell us a whole story: how fast was the child going, was the car stopped or moving, was the marble gaining speed or rolling to a stop. In order to describe certain characteristics of an object or a relationship between two quantities, the graphical representation is more telling. In the next few problems, we will try to decide what the graphs tell us about relationships.

Question 3.1 You remember those good ol' days when you sledded with your best buddies and pushed them off the sled? That was not a nice thing to do! Well, remember one of those times, and imagine yourself climbing that awesome hill, then sledding down. Which of the graphs depicted below best represents your trip up and down the hill?



a. Write a couple of sentences explaining why the graph you chose describes the situation above.

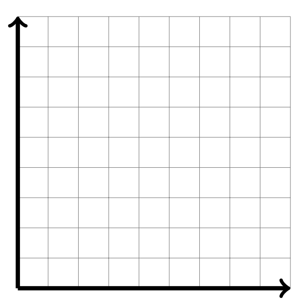
b. Write a couple of sentences explaining what the sledding experience would have been like if the other two graphs were the correct graphs.

c. Put a scale on your graph. On the horizontal axis place 0-10 minutes, and on the vertical axis place 0-5 miles per hour.

- How fast were you moving after 2 minutes? 5 minutes? After 7?

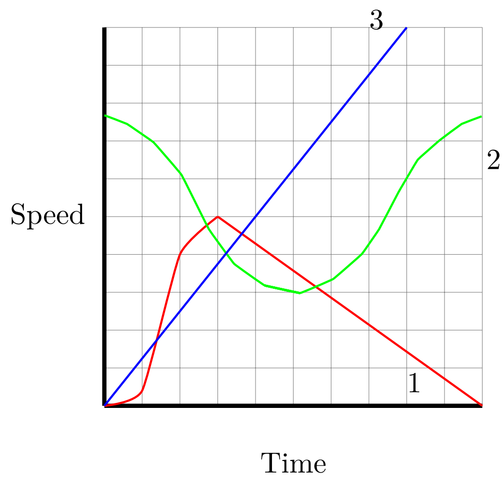
- What was the time when you moving 3 miles per hour? 5 miles per hour? 1 mile per hour?

- On the new set of coordinate axes and graph speed on the x-axis and time on the y-axis. Draw the same relationship on this new coordinate graph.



- How are the two graphs related? What kind of information does the new graph convey?

Question 3.2 Let's say you are sitting outside and a car passes by. It slows down as it passes you and then speeds up. Which of the graphs depicted here best represents the speed of the car?



a. Write couple of sentences explaining why the graph you chose describes the situation above.

b. Write couple of sentences explaining what situation you would observe if the car's speed had been described by the other graphs.

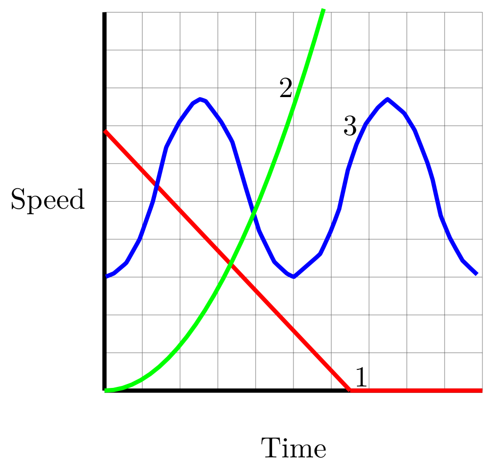
c. Put a scale on your graph. On the horizontal axis place 0-60 seconds, and on the vertical place 0-35 miles per hour.

- How fast was the car moving after 30 seconds? After 45 seconds ? After 60 seconds?

- At what time was the car going 20 miles per hour? At what time was the car going 0 miles per hour?

- Place any point at all on the graph and describe the moment that corresponds to that point.

Question 3.3 Imagine you roll a marble on the floor and you watch it roll to a stop (yes, you have an exciting life). Which of the graphs depicted below best represents the speed of the marble?

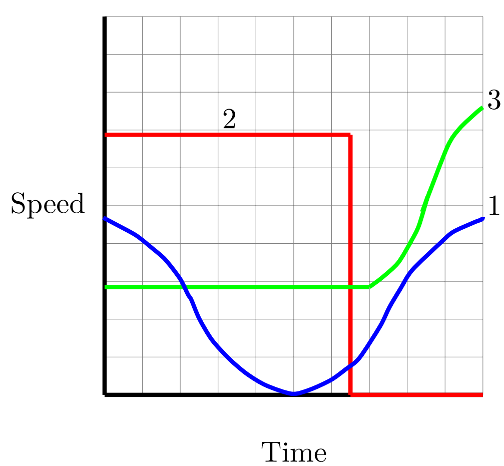


Put a scale on your graph. On the horizontal axis place 0-2 minutes, and on the vertical axis place 0-2 meters per second.

- How fast was the marble moving after 30 seconds? After 1 minute 30 seconds? After 1 minute 45 seconds?

- What was the time when the marble was moving 1 meter per second? At what time was the marble stopped?

Question 3.4 You're watching Formula 1 and the lead car slams into the wall at full speed. Which graph best depicts this sad story?



How fast was the car moving after 30 seconds? After 60 seconds? After 90 seconds? Put a scale on your graph. On the horizontal axis place 0-100 seconds, and on the vertical axis place 0- 120 meters per second.

- How fast was the car moving after 30 seconds? After 60 seconds? After 90 seconds?

- What was the time when the car was moving 90 meter per second? 60 m/s? 45 m/s? 30 m/s? 0 m/s?

3.2 Relations and Functions

The previous sections have described many relationships in different ways. We used words, graphs, tables, and expressions. What was common in all of those examples was that we knew which objects were related to each other and how. In other words, we knew how the objects were paired up. Let's look at some examples, old and new.

Example 3.1 In our pool problem we had the filled out the following table:

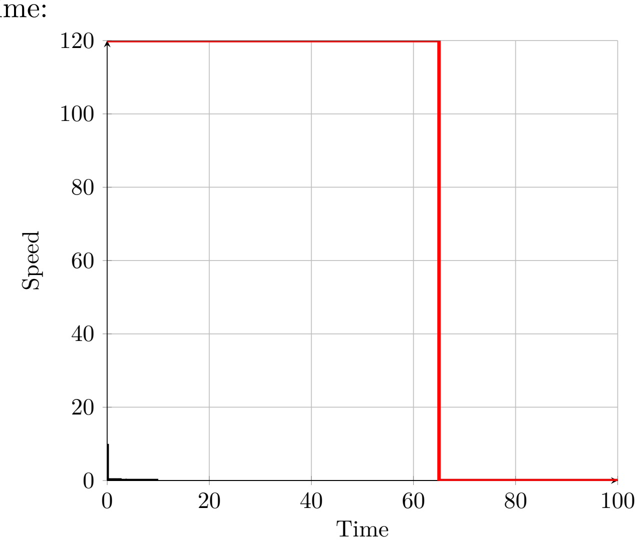
n	T_n
1	8
2	12
3	16
$\vdots$	$\vdots$
n	$4+4\cdot n$
$\vdots$	$\vdots$

The objects were counting numbers: $\{1,2,3,4,\cdots\}$, and they were paired up: to each counting number (which represented the side length of a pool) we associated another counting number: the number of tiles needed to tile the pool with that side length.
For example, for a pool of side length 3 feet we needed 16 tiles. So, we put the numbers 3 and 16 together in a "pair", $(3,16)$. The pair is "ordered" because we first thought about the length of the pool side length, and then proceeded to figure out the number of tiles needed; as a result, we chose to write the "3" as the first coordinate in our ordered pair, and the "16" as the second coordinate: $(3,16)$. In this way, our Pool Problem table is really the set of ordered pairs

$$\{(1,8),(2,12),(3,16),(4,20),(5,24),(6,28),\ldots,(n,4+4\cdot n),\ldots\}.$$

Secretly, thinking about this set of ordered pairs is probably how most of us created our graphs of this sequence: We typically think of the first coordinate as the horizontal axis coordinate, and the second coordinate as the vertical axis coordinate.

Example 3.2 In the Formula 1 example the red graph (graph 2) showed how speed depended on time:



Here we associated numbers between 0 and 100, they represented the seconds during which the car was moving, with numbers between 0 and 120, which represented the speed in meters per second. The pairs we formed according to the graph above.

1. List at least three pairs from this relation:

2. Can you list all the pairs for this relation? Explain.

Definition 1 A **relation** is a collection of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target

We often call the first coordinates **inputs** and the second coordinates **outputs**.

Question 3.5 The following table gives several examples of relations, but they have been grouped into two categories. The ones in Group 1 share features that ones in Group 2 do not have. Study the two groups, and decide what the reason for such grouping may have been. [1]

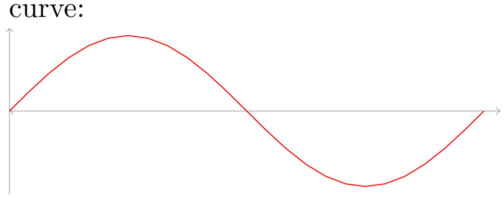
Group 1

a. $D = \{1,2,3,\ldots\}$, $T = \{1,2,3,\ldots\}$, $\{(1,2),(2,3),(3,4),(4,5),\ldots\}$

b. $D =$ all the people, $T =$ all the women, pairs (p,w) are formed whenever w is the biological mother of p.

c. $D = \{1,2,3\}$, $T = \{1,2,3\}$, $\{(1,2),(2,3),(3,3)\}$

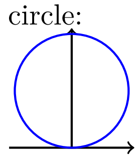
d.
$D =$ all the numbers between 0 and 2π,
$T =$ all the numbers between -1 and 1,
pairs (x,y) whenever they lie on the red curve:



Group 2

A. $D =$ all the parents, $T =$ all the people, pairs (p,c) are formed whenever p is a parent of c.

B. $D =$ all the numbers between -1 and 1, $T =$ all the numbers between 0 and 2, pairs (x,y) whenever they lie on the blue circle:



C. $D = \{a,b,c,d\}$, $T = \{1,2,3\}$

in	out
a	1
b	3
d	2

D. $D = \{1,2,3\}$, $T = \{1,2,3\}$, $\{(1,2),(1,3),(2,3),(3,2)\}$

[1] It may help to graph some of these relations, or to give examples of pairs that belong to them.

Question 3.6 Based on your reasons, into which group would you place these relations:

1. The Formula 1 example from above.
2. $D = \{1,2,3,4,\ldots,26645\}$, $T = \{0,1,2,3,\ldots,500\}$, an input is some number d from D which represents the d^{th} day in the life of Joe, and the corresponding output is the number of text messages Joe sent on day d.

Explain why you categorized them this way:

The relations in the first group are called functions. Informally, we can say: A function is a set of ordered pairs, such that each input is paired with exactly one output.

Question 3.7 Come up with two examples of relations one of which is a function and one which is not.

1. Function:
2. Non-function:

More formally, we have this definition:

Definition 2 A function is a set of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target
- every element in the domain is paired with exactly one element of the target.

We say that we have a function $f : D \to T$, and if an ordered pair (a,b) is in our function, then we say that $f(a) = b$.

You may be more familiar with the words "input" and "output". If the ordered pair (a,b) belongs to our function f, then, in addition to saying that $b = f(a)$ (reads: b is f of a), we say that a is the input, and b is the output for a. a comes from the set D, domain, and b comes from the set T, target. We also often write $a \mapsto b$.

3.3 Ways to represent a function

Question 3.8 For each description below make a table of at least five (x,y) pairs that fit the description. Then write down the algebraic equation that describes the relationship.

a. The y-coordinate is always equal to the x-coordinate.

x					
y					

b. The y-coordinate is always four less than the x-coordinate.

x					
y					

c. The y-coordinate is always opposite of the x-coordinate.

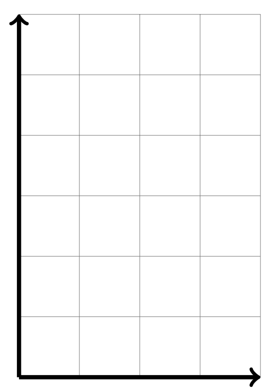
x					
y					

d. The y-coordinate is always the square of the x-coordinate.

x					
y					

Question 3.9 Let $f:\mathbb{R}\to\mathbb{R}$ be a function. Use the table below to guess which function f is.

x	$f(x)$
1	3
2	4
3	5
4	6



a. Graph the points listed in the table.

b. Draw in a possible graph for a function f such that the points in the table are included in the graph.

c. How many different choices for f are there in Question b.?

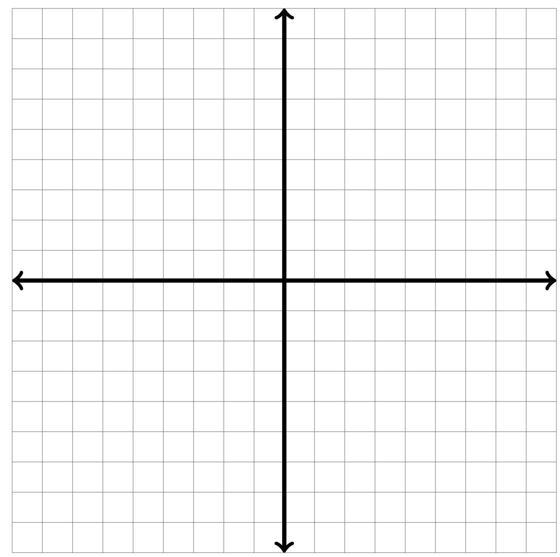
d. If this were data collected in a laboratory, which function would you choose for f?

Question 3.10 Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by $f(x)=2x+1$.

a. Construct a table for $f:\mathbb{R}\to\mathbb{R}$.

b. Plot the entries from your table on a graph.

x	$f(x)$



c. Evaluate $f(-5)$, and $f(a)$, where a represents any number.

d. Is there an input for which the output is 12? -13?

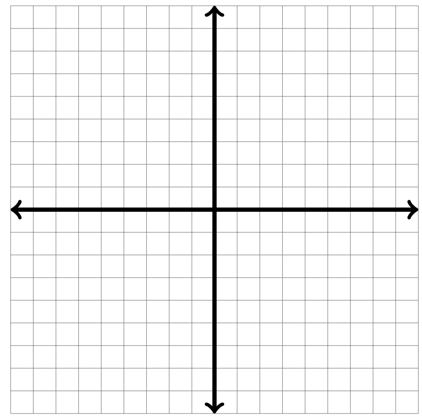
e. Use your graph to estimate $f(-3)$. Explain how this is done.

f. Use your graph to estimate the input whose output is $\frac{9}{2}$. Explain how this is done.

Question 3.11 Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by $f(x)=x^2+1$.

a. Construct a table and graph $f:\mathbb{R}\to\mathbb{R}$.

x	$f(x)$



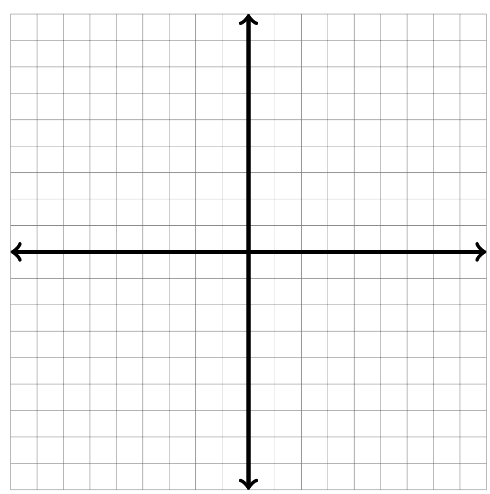
b. Evaluate $f(-2)$, and $f(a-1)$, where a represents any number.

c. Is there an input for which the output is 37? -10?

Question 3.12 Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by $f(x)=-3x+1$.

a. Construct a table and graph $f:\mathbb{R}\to\mathbb{R}$.

x	$f(x)$



b. Evaluate $f(-1)$, and $f(b+1)$, where b represents any number.

c. Is there an input for which the output is 25? -7?

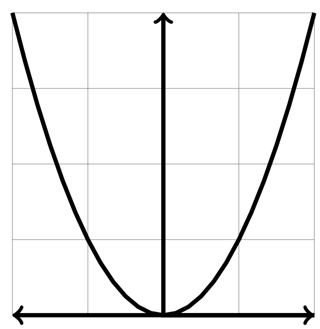
Question 3.13 Below is the graph of a function $g : \mathbb{R} \to \mathbb{R}$.

a What is $g(0)$?

b What is $g(1)$?

c For what values of x does $g(x) = 4$?

d Fill in the table.



x	$g(x)$

Question 3.14 Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = x^2 + 1$. Is the point $(3, 11)$ on the graph of f? How do you know?

Question 3.15 Let $h : \mathbb{R} \to \mathbb{R}$ be a function. How many times could a vertical line intersect the graph of h? Explain.

3.4 Combining Functions

Question 3.16 You are in the market to buy a new TV. After carefully looking at consumer reports you choose the TV you want to buy:

Sony - 50" Class (49-1/2" Diag.) - LED - 1080p - 60Hz - HDTV

Luckily for you, you also just got a $100 coupon for this model from BestBuy. When you walk in the store you receive a special promotion for 10% off. We will consider four different scenarios:

1. You are allowed to take one discount only.
2. You are allowed to take both discounts off the original price.
3. You will use the $100 off coupon and then the 10% off coupon.
4. You will use the 10% off coupon and then the $100 off coupon.

Part 1: The current price tag on the TV is $599. For each of the scenarios above calculate the sale price of the TV as well as the amount you saved.

Scenario 1
Price Discount

Scenario 2
Price Discount

Scenario 3
Price Discount

Scenario 4
Price Discount

Part 2: How will the situation change as we vary the price of the TV? Let the cost of the TV be represented by the variable x. In each case articulate a reasonable domain and target for the function that represents the cost of the TV after the discount(s) have been applied as well as their real world meaning.

Scenario 1:

$f(x)$ is the cost of TV with \$100 off.

$g(x)$ is the cost of TV with 10% off.

Scenario 2:

$p(x)$ is the cost of TV with both discounts off the original price

Scenario 3:

$s(x)$ is the cost of TV with \$100 then 10% off

Scenario 4:

$r(x)$ is the cost of TV with 10% then \$100 off

We saw that we can combine functions using same operations we used for numbers. There is, however, a new way of combining functions that we have not had available to us until now: composition. Remember how Scenario 3 and 4 differed:

$$x \xrightarrow{\text{"}-100\text{"}} x-100 \xrightarrow{\text{"}\cdot 0.9\text{"}} 0.9(x-100)$$

$$x \xrightarrow{\text{"}\cdot 0.9\text{"}} 0.9x \xrightarrow{\text{"}-100\text{"}} 0.9x-100$$

Definition 3 Given two functions $f: D \to T$ and $g: T \to S$, we define a composition of functions f and g to be a new function which consists of ordered pairs (a,c) whenever a pair (a,b) belongs to f and pair (b,c) belongs to g, or

$$(g \circ f)(a) = g(f(a))$$

$$a \xrightarrow{f} f(a) \xrightarrow{g} g(f(a))$$

Question 3.17 We are given partial tables for two functions, $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$. Use these tables to fill out the partial table for $f \circ g: \mathbb{R} \to \mathbb{R}$ given below. In the event that information you need is not available explain what additional information you would need.

x	$f(x)$
-14.2	2
$-\sqrt{2}$	-4
-1	$-\frac{5}{8}$
0	-7
2	12
4	2
$\sqrt{5}$	-1

x	$g(x)$
-14.2	3
1	-14.2
-1	2
0	$-\sqrt{2}$
2	4
4	4
$\sqrt{5}$	0

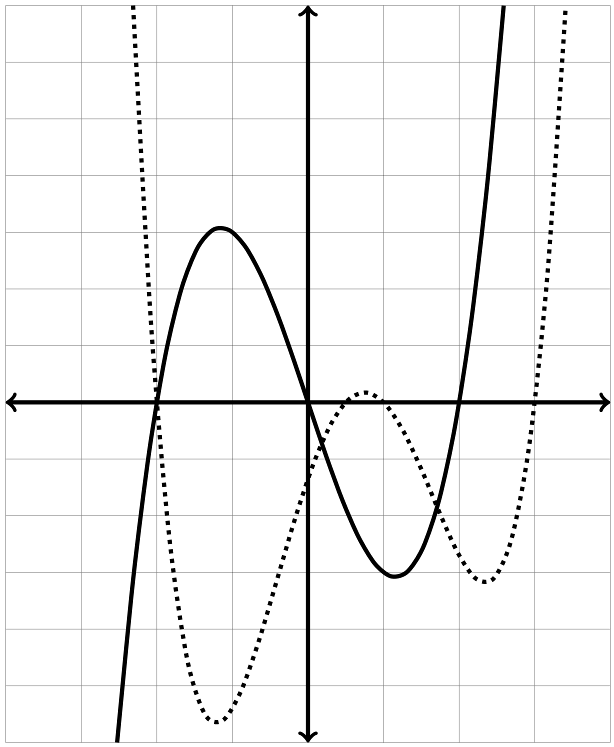
x	$(f \circ g)(x)$
-14.2	
1	
-1	
0	
2	
4	
$\sqrt{5}$	

Question 3.18 For the two functions $h: \mathbb{R} \to \mathbb{R}$ and $l: \mathbb{R} \to \mathbb{R}$ given by their algebraic rules $h(x) = 2x+1$ and $l(x) = x^2 - 1$ find the algebraic expressions for:

a. $h \circ l: \mathbb{R} \to \mathbb{R}$

b. $l \circ h: \mathbb{R} \to \mathbb{R}$

Question 3.19 Two functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$ are given by their graphs. The graph of f is dotted.



Fill out the following tables as accurately as possible using the graphs of these two functions. In the event that information you need is not available, explain what additional information you would need.

x	$(f+g)(x)$
-2	
-1	
0	
1	
2	
3	

x	$(f-g)(x)$
-2	
-1	
0	
1	
2	
3	

x	$(f\circ g)(x)$
-2	
-1	
0	
1	
2	
3	

x	$(g\circ f)(x)$
-2	
-1	
0	
1	
2	
3	

Question 3.20 We are given two functions $f:\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\to\mathbb{R}$.

a. Does $(f\circ g)(x)=(g\circ f)(x)$ sometimes, always, never?

b. Find an example and/or counterexample.

Question 3.21 Below is the portion of the graph for the function $f : \mathbb{R} \to \mathbb{R}$.

a. What is $f(2)$?

b. What is $(f \circ f)(0)$

c. What is $(f \circ f \circ f)(3)$?

Question 3.22 We have two functions $f, g : \mathbb{R} \to \mathbb{R}$ given by their algebraic expressions: $f(x) = x^2$ and $g(x) = x^3$.

a. Write a formula for $f \circ g$.

b. Write a formula for $f \cdot g$.

c. How are they different?

3.5 Inverse Functions

Question 3.23 In Question 3.16 we found that we could calculate the sale price of any TV whose original price we knew using the following function rule: $r(x) = 0.9x - 100$.

a. Sam paid \$549 for her TV. What was the original price?

b. Sharmitsa paid \$269 for her TV. What was the original price?

c. Z paid \$$y$ for his TV. What was the original price?

Question 3.24 The following table gives information about function $f : \mathbb{R} \to \mathbb{R}$.

x	3	4	-3
$f(x)$	5	6	-1

a. Write a possible algebraic rule for f (The rule I have in mind is of the form $f(x) = ax + b$).

b. Suppose you want to find another function that will undo the effects of this one. That is, it will take 5 and turn it back into 3. Write a rule for this new function and call it g.

c. Evaluate $f(7)$.

d. Evaluate g at the answer you got for Question c.. Explain what happened.

Question 3.25 Let $h : \mathbb{R} \to \mathbb{R}$ be a function given by the rule $h(x) = 2x + 1$. Complete the table for the function h and the function that will **undo** h.

h

input	output
2	5
1	
0	
-1	

undo h

input	output
5	2
3	
1	
-1	

a. Call the "undo h" g. What can you say about the pairs in h and g?

We call g the *inverse relation* of h.

b. Find algebraic expression for g.

c. Calculate $(h \circ g)(x)$.

d. Calculate $(g \circ h)(x)$.

Question 3.26 Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by $f(x) = x^2$. Fill out the tables below:

f

input	output
-2	
-1	
0	
1	

inverse relation of f

input	output

a. Is the inverse relation of f also a function?

b. Can you find an algebraic expression for the inverse relation of f?

At this moment, we have two different ways of thinking about inverse functions.

Definition 4 ("Undoing") We say that a function $f : D \to T$ is invertible (has an inverse function) if there is a function $g : T \to D$ for which:

$$(g \circ f)(a) = a, \qquad \text{for all} \quad a \quad \text{in} \quad D$$

and

$$(f \circ g)(b) = b, \qquad \text{for all} \quad b \quad \text{in} \quad T$$

Definition 5 ("Switching") For a given function $f : D \to T$ we form an inverse relation $g : T \to D$ by exchanging the coordinate pairs belonging to f: if (a,b) is in f, then (b,a) is in g. If g is also a function, then we say that f is invertible, and that g is its inverse.

Note: If f is invertible, then its inverse function is denoted with f^{-1}.

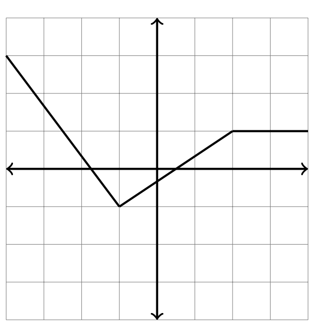
Question 3.27 Let $f : \mathbb{R} \to \mathbb{R}$ be an invertible function.

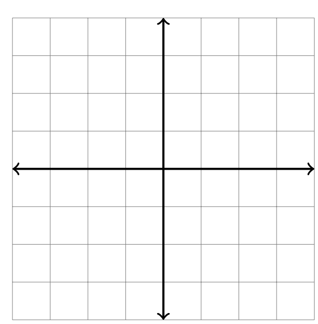
a. If you know that the point $(10,17)$ is on the graph of f, what will be a point on the graph of $f^{-1}(x)$.

b. What is $f(f^{-1}(4))$?

c. If $(x, f(x))$ is a point on the graph of f, what is the associated point on the graph for f^{-1}?

Question 3.28 Below is a graph for $f : [-4,4] \to \mathbb{R}$. Graph the inverse relation of f on the empty coordinate system. Is f invertible? Explain.





Question 3.29 Let $f : \mathbb{R} - \{\frac{3}{2}\} \to \mathbb{R}$ be a function given by $f(x) = \frac{-5-x}{3-2x}$. Find $f^{-1}(7)$, without finding $f^{-1}(x)$.

Question 3.30 Function $f : [2, \infty) \to \mathbb{R}$ is given by $f(x) = 2\sqrt{x-2}$.

a. Why does not f have all real numbers in its domain?

b. How would you determine whether f has an inverse function?

c. What would be the expression for f^{-1}?

Question 3.31 Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by the algebraic rule $f(x) = x^2$.

a. Is f invertible? How do you know?

b. Can you change the domain of f so that it would have an inverse? Explain.

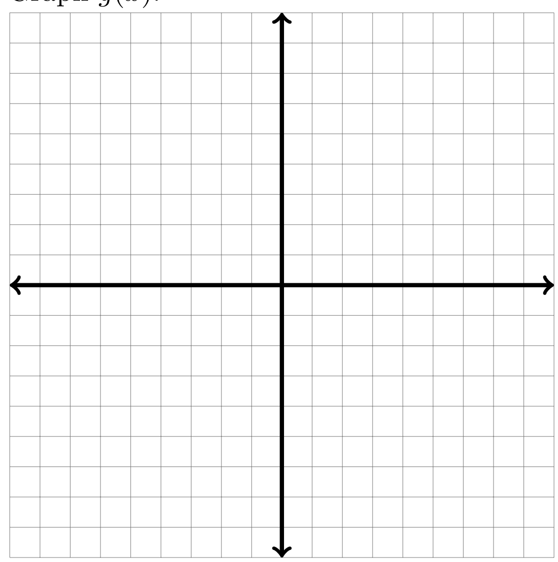
Question 3.32 Let $g : [0, \infty) \to \mathbb{R}$ be a function such that $g(x)$ is the side length of a square with area x.

a. Explain why the domain of g is $[0, \infty)$.

b. Fill in a table for $g(x)$.

x	$g(x)$
1	
4	
16	
$\frac{1}{4}$	
0	

c. Graph $g(x)$.



d. Is g invertible? Explain.

e. Find an algebraic formula for $g(x)$.

3.6 Summary

One of the most ubiquitous uses of mathematics in everyday life is describing relationships between quantities. You're used to seeing graphs in the newspapers that tell you how the stock prices change during a month, how temperature changes over time, the snowfall over the years, systolic blood pressure of people with various levels of glucose in their blood, etc. We are interested in seeing how quantities change, what inferences we can make about the relationships, and whether we can make predictions about future behavior.
Most ordinarily we consider relationships between two quantities, which means that we are interested in pairs of values: the value of Facebook stock on 12/31/13 was \$54.54, the minimum temperature in Salt Lake City on 12/13/13 was 13°F, the total snowfall at Alta during 2010-2011 season was 723.5", the recorded systolic pressure was 130mmHg and 145mmHg for people with 110mg/dL glucose in their blood. This inspires us to make the following definition:

Definition 6 A **relation** is a collection of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target

We often call the first coordinates **inputs** and the second coordinates **outputs**.

Our examples are then: $(12/31/13, \$54.54), (12/13/13, 13°)F$,
$(2010-11, 723.5”), (110mg/dL, 130mmHg)$, and $(110mg/dL, 145mmHg)$.
Notice that it is important for us to know what types of quantities we're interested in (domain and target) as well as in which order they appear (although we could consider either order, in which case we would have a different relation), but that there aren't any rules about how those pairs are made. Notice also that in all the examples but the last knowing the first value, determined the second one exactly. In the last problem, we couldn't claim that every person with the same blood sugars also has the same blood pressure. When we know exactly the output for a given input, we have a special type of relation which we call *function*:

Definition 7 A function is a set of ordered pairs, such that:

- the first entry comes from a set D called the domain
- the second entry comes from a set T called the target
- every element in the domain is paired with exactly one element of the target.

We say that we have a function $f : D \to T$, and if an ordered pair (a,b) is in our function, then we say that $f(a) = b$.

If the ordered pair (a,b) belongs to our function f, then, in addition to saying that $b = f(a)$ (reads: b is f of a), we say that a is the input, and b is the output for a. a comes from the set D, domain, and b comes from the set T, target. We also often write $a \mapsto b$.
Let us reiterate: in order to define a function we must know its domain, target and the pairings that belong to it.
We can represent functions in many different ways:

- We can use verbal descriptions, such as the ones at the beginning of this section.

- We can represent them by simply stating the ordered pairs that belong to the function, for example:

$$f = \{(a,1),(b,2),(c,3),(d,4),\ldots,(z,26)\}.$$

In this instance, even though it's not listed specifically, we can infer that the domain of f is the English alphabet and the target is the set of whole numbers $\{1,2,3,\ldots,26\}$.

- We can use tables:

x	$f(x)$
a	1
b	2
c	3
d	4

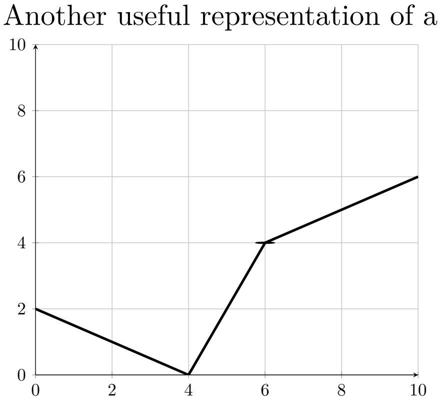
The domain here is less clear, and would need to be specified. It is possible that we only showed a partial table, or that the whole one is shown. This representation also clearly shows the ordered pairs.

- It is sometimes possible to represent a function with an algebraic expression. For instance, $f : \mathbb{R} \to \mathbb{R}_+$ is given by

$$f(x) = \frac{x^2+3}{2}.$$

The ordered pairs are less obvious here, but can be found. Remember that the pairs are made in such a way that the first coordinate is the input, an element from the domain, for example $x = -3$, and the second coordinate is its output, the corresponding element from the target: $f(-3) = \frac{(-3)^2+3}{2} = 6$, so one pair is $(-3,6)$.

- Another useful representation of a function is the graph:



the graph for f

Here, as well, the pairs aren't immediately obvious, but can be found by looking at the points on the graph and reading their x and y coordinates. For instance, we can tell that $f(6) = 4$ because a point $(6,4)$ lies on the graph of f.

Functions are not unlike numbers which you're used to, in the sense that we can combine them in various ways. Suppose for a minute that f and g are functions with equal domain and target $f,g : D \to T$. We can define new functions:

$$f+g : D \to T \quad \text{is defined by} \quad (f+g)(x) = f(x)+g(x)$$
$$f-g : D \to T \quad \text{is defined by} \quad (f-g)(x) = f(x)-g(x)$$
$$f \cdot g : D \to T \quad \text{is defined by} \quad (f \cdot g)(x) = f(x) \cdot g(x)$$
$$f \div g : D' \to T \quad \text{is defined by} \quad (f \div g)(x) = f(x) \div g(x).$$

In the last function the domain, D', is either D or D with all the elements where the value of g is 0 excluded.
A combination of functions that does not have an analogous combination of numbers is *composition*:

Definition 8 Given two functions $f : D \to T$ and $g : T \to S$, we define a composition of functions f and g to be a new function which consists of ordered pairs (a,c) whenever a pair (a,b) belongs to f and pair (b,c) belongs to g, or

$$(g \circ f)(a) = g(f(a))$$
$$a \xrightarrow{f} f(a) \xrightarrow{g} g(f(a))$$

Regardless of which combination you need, you can find the values of the new function given different representations of the original functions. For example, if two functions are given by their graphs, you can find the graph or table of their sum or composition. Likewise, if you have algebraic expression, you could find the algebraic expression or a table of their product or the difference.

A closely related concept to that of composition is an inverse relation, and then inverse function. We can define an inverse relation of any relation: it simply consists of ordered pairs which when the order is reversed belong to the original relation. In other words, if (a,b) belongs to a given relation, then (b,a) belongs to the inverse relation. When our original relation is a function, it is possible that the inverse relation is also a function, but it is not always the case. When it is, we talk about invertible function.

Definition 9 ("Undoing") We say that a function $f : D \to T$ is invertible (has an inverse function) if there is a function $g : T \to D$ for which:

$$(g \circ f)(a) = a, \qquad \text{for all} \qquad a \quad \text{in} \quad D$$

and

$$(f \circ g)(b) = b, \qquad \text{for all} \qquad b \quad \text{in} \quad T$$

This really tells us that the composition of the function and its inverse is the identity function, a function that returns the output which is the same as the input.

Definition 10 ("Switching") For a given function $f : D \to T$ we form an inverse relation $g : T \to D$ by exchanging the coordinate pairs belonging to f: if (a,b) is in f, then (b,a) is in g. If g is also a function, then we say that f is invertible, and that g is its inverse.

Note: If f is invertible, then its inverse function is denoted with f^{-1}. Once again, different representations of the original function can be used to find the inverse function.

3.7 Student learning outcomes

1. Students will be able to determine if a relationship is a function.
2. Students will be able to seamlessly move between different representations of functions.
3. Given two functions $f(x)$ and $g(x)$ students will be able to calculate the composition of $f \circ g(x)$ using a table, graph and algebraic expressions.
4. Student understands a meaning of inverse function, can determine when it exists based on different representations of the given function and find some representation of the inverse.

Have you accomplished these outcomes? Use the homework exercises to ensure that you have. Are there any questions that remain? Make sure to clarify those in class or while collaborating with your peers.