Exponential Functions - Population growth 6.1

Definition of Exponents

Definition

An exponent is a convenient way to write repeated multiplication. Given a natural number b the following notation represents a product of b many a's.

$$a^b = \overbrace{a \cdot a \cdot a \cdot .... \cdot a}^{b \text{ many } a\text{'s}}$$

Exponential Functions - Population growth 6.1

Question 6.1

Use exponents to represent the following:

a. $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

c. $x \cdot x \cdot x$

b. $3 \cdot 3 \cdot 3 \cdot 3 \cdot 3 \cdot 3$

d. $a \cdot a \cdot a \cdot a \cdot a \cdot a$

Exponential Functions - Population growth 6.1

Question 6.2

A colony of bacteria is being grown in a laboratory. It contains a single bacterium at $12:00$ noon (time 0), and the population is doubling every hour.

a. How long do you think it would take for the population to exceed 1 million? 2 million? Write down your guesses and compare with other students' guesses.

Exponential Functions - Population growth 6.1

Question 6.2

A colony of bacteria is being grown in a laboratory. It contains a single bacterium at $12:00$ noon (time 0), and the population is doubling every hour.

b. Make a table of values showing how this population of bacteria changes as a function of time. Find the population one hour from now, two hours from now, etc. Extend your table until you can answer the questions asked in Question 6.2 a. and graph your points. How close were your guesses?

Exponential Functions - Population growth 6.1

Question 6.2

t	Number of bacteria	
0	1	
1		
2		
3		
4		
5		
6		

Exponential Functions - Population growth 6.1

Question 6.2

t	Number of bacteria	
0	1	
1		
2		
3		
4		
5		
6		

c. In the third column in Question b. write the population each time as a power of 2 (for example, 4 is 2^2).

Exponential Functions - Population growth 6.1

Question 6.2

t	Number of bacteria	
0	1	
1	2	2^1
2	4	2^2
3	8	2^3
4	16	2^4
5	32	2^5
6	64	2^6

d. What would the population be after x hours? (Write this as a power of 2.)

Exponential Functions - Population growth 6.1

Question 6.2

e. Compare the population after 8 hours with the population after 5 hours.

f. How many times as much is it? (Compare by dividing.)

g. Which of your answers is a power of 2? What power of 2 is it?

Exponential Functions - Population growth 6.1

Question 6.2

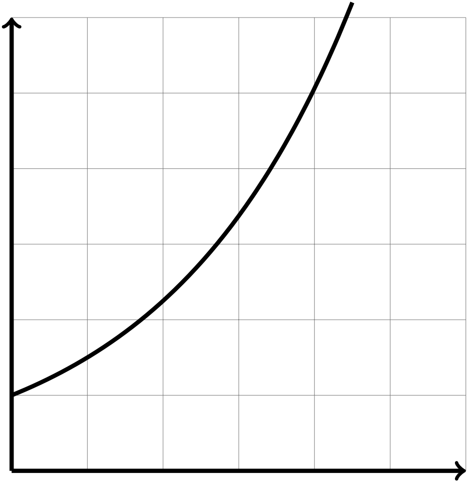
h. How many bacteria would there be after three and a half hours?

i. Why does Question f. demand that we depart from thinking of this as a sequence?

Exponential Functions - Population growth 6.1

Question 6.2

j. What does $2^{3.5} = 2^{\frac{7}{2}}$ mean? Can you use the graph to estimate this number?



Exponential Functions - Population growth 6.1

Question 6.2

k. How long exactly do we have to wait to see at least 1000000 bacteria?

Exponential Functions - Rules of Exponents 6.2

Question 6.3

Rewrite the following expressions using just one exponent. To answer the question, think about how many twos would appear after you multiplied everything out.

a. $(2^2)^3$

b. $(2^4)^5$

c. $(2^5)2^7$

d. $(2^9)2^{10}$

Exponential Functions - Rules of Exponents 6.2

Question 6.4

Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $(5^5)^3$

b. $(5^4)^6$

c. $(5^4)5^6$

d. $(5^2)5^{10}$

Exponential Functions - Rules of Exponents 6.2

Question 6.5

Rewrite the following expressions using just one exponent. To answer the question, think about how many twos (or xs) would appear after you multiplied everything out. Think about a and b as positive integers.

a. $(2^a)^b$

b. $(2^a)2^b$

c. $(x^a)^b$

d. $x^a x^b$

Exponential Functions - Rules of Exponents 6.2

Question 6.6

Rewrite the following expressions using just one exponent. To answer the question, think about how many fives would appear after you multiplied everything out.

a. $\dfrac{5^5}{5^2}$

b. $\dfrac{6^4}{6^3}$

c. $\dfrac{x^a}{x^b}$

Exponential Functions - Rules of Exponents 6.2

Question 6.7

Evaluate this expression in two different ways: using the rule you just developed and by multiplying everything out:

$$\frac{5^7}{5^8}$$

Exponential Functions - Rules of Exponents 6.2

Question 6.8

a. What number is 2^{-1}?

b. What number is 3^{-1}?

c. What number is 2^{-2}?

d. What number is 3^{-2}?

Exponential Functions - Rules of Exponents 6.2

Question 6.8

f. The rule you developed for Question 6.5 Part d. is a rule we want to be true in general. Use that rule and the definition of $2^{-1}2 = 1$ to decide the value of 2^0.

Exponential Functions - Rules of Exponents 6.2

Question 6.8

e. This is the table you filled out recently. Use the patterns apparent in the table to decide why this definition makes sense:

x	2^x
5	32
4	16
3	8
2	4
1	2
0	1
-1	
-2	
-3	
-4	

Exponential Functions - Rules of Exponents 6.2

Question 6.8

What does $2^{3.5} = 2^{\frac{7}{2}}$ mean?

a. Calculate $(2^{\frac{7}{2}})^2$. Assume the rules for from 6.5 apply.

b. Explain what $\sqrt{2^7}$ means.

c. Combine Parts 1 and 2 to make sense of $2^{\frac{7}{2}}$

Exponential Functions - Rules of Exponents 6.2

Question 6.10

A colony of bacteria is being grown in a laboratory. It contains a single bacterium at $12:00$ noon (time 0), and the population is doubling every hour. How many bacteria are there after 3.5 hours.

Exponential Functions - Rules of Exponents 6.2

Question 6.11

Let us redo this for $a^{\frac{1}{2}}$:

a. Calculate $(a^{\frac{1}{2}})^2$

b. Explain what $\sqrt{a}$ means.

c. Combine Parts a. and b. to make sense of $a^{\frac{1}{2}}$:

Exponential Functions - Rules of Exponents 6.2

Question 6.12

Think about how $f(x) = x^{\frac{1}{2}}$ is the inverse function of $g : [0, \infty) \to \mathbb{R}$ defined by $g(x) = x^2$.

a. Why is the domain of g limited to $[0, \infty)$?

b. What would be the inverse function of $h : \mathbb{R} \to \mathbb{R}$ given by $h(x) = x^3$?

c. What would be the inverse function of $l : [0, \infty) \to \mathbb{R}$ given by $h(x) = x^4$?

d. What would be the inverse function of $p : [0, \infty) \to \mathbb{R}$ given by $p(x) = x^n$?

Exponential Functions - Rules of Exponents 6.2

Question 6.13

What does $2^{\frac{7}{5}}$ mean?

a. Calculate $(2^{\frac{7}{5}})^5$. Assume the rules for from 6.5 apply.

b. Explain what $\sqrt[5]{2^7}$ means.

c. Combine Parts 1 and 2 to make sense of $2^{\frac{7}{5}}$

Exponential Functions - Rules of Exponents 6.2

Question 6.15

In the following exercises, we will write the expression in a simplified version, which means that every power will be written using only positive exponents.

a. $6w^5(2w^{-2})$

b. $(3a^{-2}b^{-4})^2$

Exponential Functions - Rules of Exponents 6.2

Question 6.15

In the following exercises, we will write the expression in a simplified version, which means that every power will be written using only positive exponents.

c. $\dfrac{2^{-3}r^{-2}(r^{-1})^{-2}}{r(r^{3})^{-3}}$

d. $\left(\dfrac{3q}{4p^{2}}\right)^{2}\left(\dfrac{2p}{5q}\right)^{-2}$

Exponential Functions - Rules of Exponents 6.2

Question 6.16

A patient is administered 75 mg of DRUGX. It is known that 30% of the drug is expelled from the body each hour.

a. How many mg of DRUGX are present after 2 hours?

b. How many mg of DRUGX are present after 3 hours?

c. Develop an exponential function that models the amount of DRUGX in the body after t hours.

d. Use your model to calculate the amount of DRUGX in the body after 2.5 hours?

e. What does the fractional exponent you used in d. mean?

f. A patient needs to take another dose once the amount of DRUGX is less than 20 mg. How long should the patient wait before the first and second dose?

g. How long will it be when the model predicts that there will be exactly 20 mg of the drug in the body?

Exponential Functions - Rules of Exponents 6.2

Question 6.16

A patient is administered 75 mg of DRUGX. It is known that 30% of the drug is expelled from the body each hour.

a. How many mg of DRUGX are present after 2 hours?

b. How many mg of DRUGX are present after 3 hours?

c. Develop an exponential function that models the amount of DRUGX in the body after t hours.

Exponential Functions - Rules of Exponents 6.2

Question 6.16

A patient is administered 75 mg of DRUGX. It is known that 30% of the drug is expelled from the body each hour.

d. Use your model to calculate the amount of DRUGX in the body after 2.5 hours?

e. What does the fractional exponent you used in d. mean?

f. A patient needs to take another dose once the amount of DRUGX is less than 20 mg. How long should the patient wait before the first and second dose?

Exponential Functions - Rules of Exponents 6.2

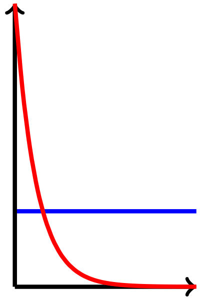
Question 6.16

a. How long will it be when the model predicts that there will be exactly 20 mg of the drug in the body?

Exponential Functions - Rules of Exponents 6.2

Question 6.16

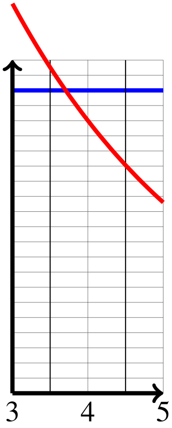
a. How long will it be when the model predicts that there will be exactly 20 mg of the drug in the body?



Exponential Functions - Rules of Exponents 6.2

Question 6.16

a. How long will it be when the model predicts that there will be exactly 20 mg of the drug in the body?



Graphs of Exponential Functions 6.3

Question 6.17

Let's make some predictions.

	Similarities	Differences
$f(x) = 2^x$ & $g(x) = 5^x$		
$f(x) = 2^x$ & $h(x) = (\frac{1}{2})^x$		
$h(x) = (\frac{1}{2})^x$ & $k(x) = (\frac{1}{5})^x$		

Graphs of Exponential Functions 6.3

Question 6.18

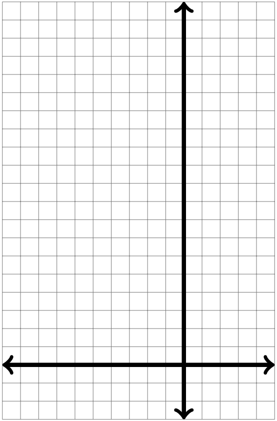
Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by the rule $f(x) = 2^x$.

a. Fill out the table for f:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	

x	$f(x)$
1	
2	
3	
4	
5	

b. Sketch the graph



Graphs of Exponential Functions 6.3

Question 6.19

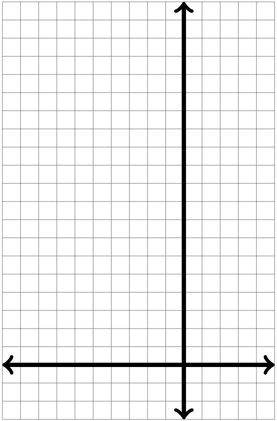
Let $f : \mathbb{R} \to \mathbb{R}$ be a function given by the rule $f(x) = \frac{1}{2}^x$.

a. Fill out the table: for f:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	

x	$f(x)$
1	
2	
3	
4	
5	

b. Sketch the graph



Graphs of Exponential Functions 6.3

Question 6.20

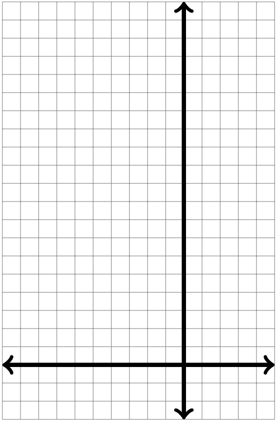
Let $f:\mathbb{R}\to\mathbb{R}$ be a function given by the rule $f(x)=5^x$.

a. Fill out the table: for f:

x	$f(x)$
-6	
-5	
-4	
-3	
-2	
-1	
0	

x	$f(x)$
1	
2	
3	
4	
5	

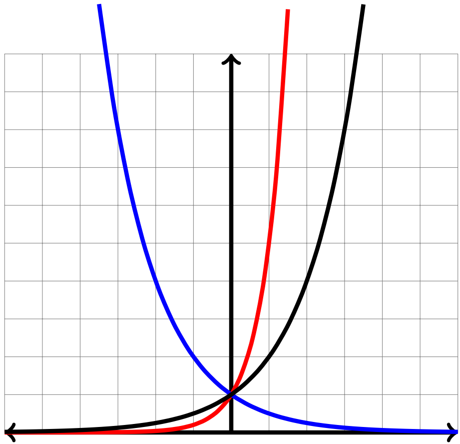
b. Sketch the graph



Question 6.21

Look at the graphs you drew in Questions 6.18, 6.19, and 6.20.

a. All three graphs share a common point. Which point is this?

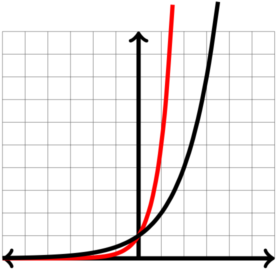


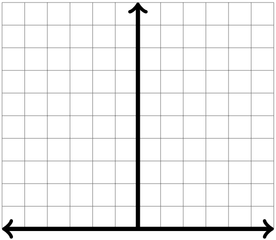
Graphs of Exponential Functions 6.3

Question 6.21

Look at the graphs you drew in Questions 6.18, 6.19, and 6.20.

c. Let $a > 1$. Use Questions 6.18 and 6.20 to help you sketch a graph of $f(x) = a^x$. Articulate why this is the general shape.



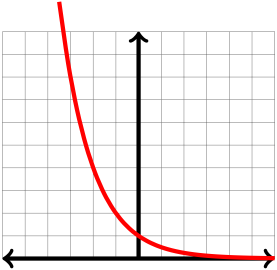


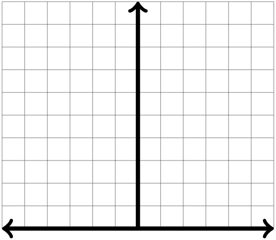
Graphs of Exponential Functions 6.3

Question 6.21

Look at the graphs you drew in Questions 6.18, 6.19, and 6.20.

d. Let $0 < b < 1$. Use Questions 6.19 to help you sketch a graph of $f(x) = b^x$. Articulate why this is the general shape.





Graphs of Exponential Functions 6.3

Question 6.21

e. From looking at the graphs are the functions $f(x) = a^x$ and $g(x) = b^x$ invertible? Explain.

f. Why does it not make sense to talk about functions of the form $h(x) = c^x$ when $c < 0$?

Exponential Functions - Inverse function 6.4

Question 6.22

You have already discovered that exponential functions are invertible. Before we think about their inverse functions, let's solve a few problems as a warm up.

a. $f(x) = 4x + 5$

b. $g(x) = (x+5)(x-4)$

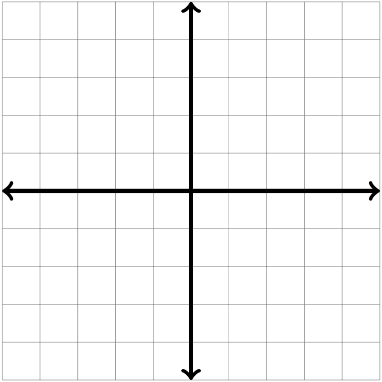
c. $h(x) = x^3 + 4$

d. $f_2(x) = 2^x$

Exponential Functions - Inverse function 6.4

Question 6.22

a. $f(x) = 4x + 5$



Click in!

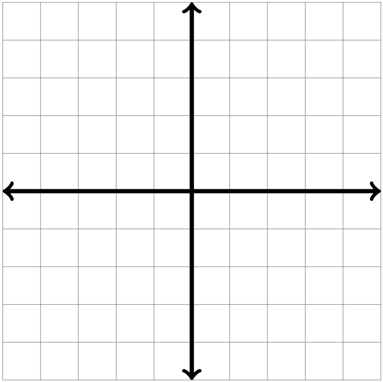
a. f is invertible

b. f is not invertible

Exponential Functions - Inverse function 6.4

Question 6.22

b. $g(x) = (x+5)(x-4)$



Click in!

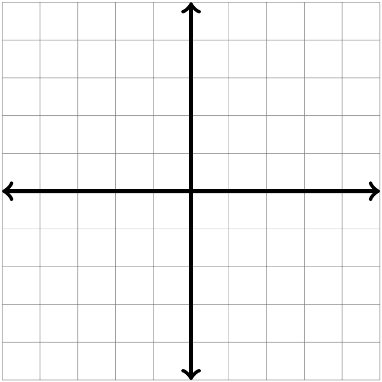
a. g is invertible

b. g is not invertible

Exponential Functions - Inverse function 6.4

Question 6.22

c. $h(x) = x^3 + 4$



Click in!

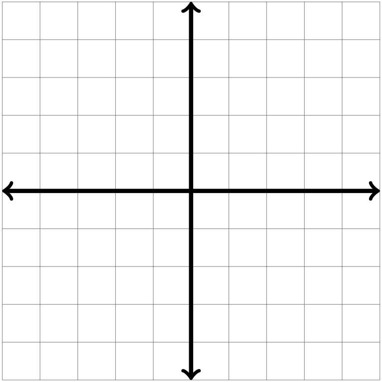
a. h is invertible

b. h is not invertible

Exponential Functions - Inverse function 6.4

Question 6.22

a. $f_2(x) = 2^x$



Click in!

a. f_2 is invertible

b. f_2 is not invertible

Exponential Functions - Inverse function 6.4

Question 6.23

Let $f_2 : \mathbb{R} \to \mathbb{R}$ be defined by $f_2(x) = 2^x$.

a. For what value of x does $f_2(x) = 4$?

b. For what value of x does $f_2(x) = 16$?

c. For what value of x does $f_2(x) = 128$?

d. For what value of x does $f_2(x) = \frac{1}{2}$?

Exponential Functions - Inverse function 6.4

Question 6.23

With this knowledge fill out the following table:

x	$f_2^{-1}(x)$
4	
16	
128	
$\frac{1}{2}$	
$\frac{1}{4}$	
-5	
1	
2	
8	

Exponential Functions - Inverse function 6.4

Question 6.24

Evaluate the following:

a. $f_4^{-1}(16)$

b. $f_3^{-1}(81)$

c. $f_5^{-1}(125)$

d. $f_{\frac{1}{2}}^{-1}(4)$

Exponential Functions - Inverse function 6.4

Notation LOG

Definition

$$\log_b(x) \textit{ is the same as } f_b^{-1}(x)$$

Solving Exponential and Logarithmic Equations 6.5

Notation LOG

Exponential equation is an equation of the form: $y = ab^x$. If you know a, b, and x, it is easy to calculate y, but sometimes you need to find the one of the other three variables is the unknown. Let's consider the three examples below.

Solving Exponential and Logarithmic Equations 6.5

Question 6.25

a. You want to know how much someone deposited in an account, seven years ago. The amount in the account today is \$287.17. The interest rate 2%, compounded annually. Write and solve the equation.

b. Solve $y = ab^x$ for a.

Solving Exponential and Logarithmic Equations 6.5

Question 6.26

a. You want to know the yearly decay rate of a chemical that is decaying exponentially. At time 0, there was 300 grams of the substance. 10 years later there was 221 grams left. Write and solve the equation.

b. Solve $y = ab^x$ for b.

Solving Exponential and Logarithmic Equations 6.5

Question 6.27

a. You want to know how long it will take for a bacteria population to triple, if the hourly growth rate is 160%.

b. Solve $y = ab^x$ for x.

Solving Exponential and Logarithmic Equations 6.5

Question 6.28

a. $2^x = 16$

b. $5^x = 125$

c. $3 \cdot 2^x = 24$

d. $2 \cdot 5^{x-2} + 1 = 51$

Solving Exponential and Logarithmic Equations 6.5

Question 6.29

a. $\log_3 27 = x$

b. $\log_4 x = -2$

c. $2\log_3 x = 4$

d. $3\log_4 x + 1 = 7$

Exponential Functions - Applications 6.6

Question 6.30

In 1975, the population of the world was about 4.01 billion and was growing at a rate of about 2% per year. People used these facts to project what the population would be in the future.

a. Complete the following table, giving projections of the world's population from 1976 to 1980, assuming that the growth rate remained at 2% per year.

Year	Calculation	Projection (billions)
1976	4.01 + (.02)4.01	4.09

Exponential Functions - Applications 6.6

Question 6.30

b. Find the ratio of the projected population from year to year. Does the ratio increase, decrease, or stay the same?

c. There is a number that can be used to multiply one year's projection to calculate the next. What is that number?

d. Use repeated multiplication to project the world's population in 1990 from the 1975 number, assuming the same growth rate.

Exponential Functions - Applications 6.6

Question 6.30

e. Compare your result to the previous problem with the actual estimate of the population made in 1990, which was about 5.33 billion.

Did your projection over-estimate or under-estimate the 1990 population?

Was the population growth rate between 1975 and 1990 more or less than 2%? Explain.

Exponential Functions - Applications 6.6

Question 6.30

f. Write an algebraic expression for $f(x)$ which predicts the population of the world x years after 1976.

g. At a growth rate of 2% a year, how long does it take for the world's population to double? We call this

Exponential Functions - Applications 6.6

Question 6.31

This is what Wikipedia tells us: Radiocarbon dating (or simply carbon dating) is a radiometric dating technique that uses the decay of carbon-14 to estimate the age of organic materials, such as wood and leather, up to about 58,000 to 62,000 years. Carbon dating was presented to the world by Willard Libby in 1949, for which he was awarded the Nobel Prize in Chemistry. Basically, the way it works is that we know how long carbon-14 takes to decompose to half the initial amount (this is called *half-life*), and by observing how much carbon is in a given sample, we can decide how old the sample is. It is known that carbon-14 has a half-life of 5730 years.

a. What kind of function do you expect will model the decay of carbon 14? Explain what evidence you have for your claim.

b. Write an algebraic expression (rule) for the function that models the decay of carbon-14.