# ADAMS OPERATIONS IN COMMUTATIVE ALGEBRA

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## Contents



## 1. Day 1: Classical Lambda and Adams operations on $K_0$

What follows is an expanded version of my talk on Monday, June 1, 2019. I've added details, additional examples, and some exercises.

A good reference for the material in this seciton (and much, much more) is the book "Riemann-Roch Algebra", by Fulton and Lang.

### 1.1. Definition of $K_0$.

For a (commutative noetherian) ring $R$, let $\mathcal{P}(R)$ denote the category of finitely generated projective $R$-modules.

**Definition 1.1.** $K_0(R)$ is the abelian group generated by classes $[P]$ of objects of $\mathcal{P}(R)$ with relations coming from short exact sequences of such. That is, for each short exact sequence $0 \to P' \to P \to P'' \to 0$ in $\mathcal{P}(R)$, we declare

$$[P] = [P'] + [P''].$$

*Remark* 1.2. Since each such short exact sequence is split exact, one may equivalently define $K_0(R)$ as the group completion of the additive monoid of isomorphisms classes in $\mathcal{P}(R)$ (with addition given by direct sum).

Since $[P_1] + [P_2] = [P_1 \oplus P_2]$ holds in $K_0(R)$, every element of $K_0(R)$ can be written as a formal difference $[P] - [P']$ with $P, P' \in \mathcal{P}(R)$.

More generally, when $X$ is a noetherian scheme, $K_0(X)$ is the abelian group generated by classes of locally free coherent sheaves on $X$, modulo relations coming from short exact sequences of such. Nearly everything we say in this section about $K_0$ of commutative rings generalizes to schemes, but we will sometimes leave such generalizations unspoken.

For any abelian group $A$ we have a bijection of sets

$$\operatorname{Hom}_{Ab}(K_0(R), A)) \cong \{\text{additive functions } \rho : \mathcal{P}(R) \to A\}$$

where "additive" means that $\rho$ assigns an element of $A$ to each object of $\mathcal{P}$ in such a whay that $\rho(P) = \rho(P') + \rho(P'')$ holds whenever there is a short exact sequence $0 \to P' \to P \to P'' \to 0$.

**Example 1.3.** If $\operatorname{Spec}(R)$ is connected, each projective $R$-module $P$ has a well-defined rank, written $\operatorname{rank}(P) \in \mathbb{Z}$, which may be defined as the rank of the free module obtained by localizing at any point of $\operatorname{Spec}(R)$. The rank function is additive on short exact seqeunce and hence indcues a homomorphism

$$\operatorname{rank} : K_0(R) \to \mathbb{Z}$$

that sends a typical element $[P_1] - [P_2]$ of $K_0(R)$ to $\operatorname{rank}(P_1) - \operatorname{rank}(P_2)$.

If $R$ is a local ring, then the rank function induces an isomorphism $K_0(R) \cong \mathbb{Z}$, since every projective module is free in this case.

If $R$ is a Dedekind domain (a regular domain of dimension 1), then the kernel of $\operatorname{rank}(R) \to \mathbb{Z}$ is isomorphic to the divisor class group of $R$ (and also to the Picard group of $R$). For other rings, the kernel is more difficult to describe.

The assignment $R \mapsto K_0(R)$ is functorial: Given a ring homomorhism $g : R \to S$, we define the map

$$g_* : K_0(R) \to K_0(S)$$

to be induced by extension of scalars: given $P \in \mathcal{P}(R)$, we have $g_*([P]) = [P \otimes_R S]$. The map $g_*$ is well-defined since the function $P \mapsto [P \otimes_R S]$ is additive on short exact sequences.

The abelian group $K_0(R)$ becomes a commuative ring under the operation induced by tensor poduct:

$$[P] \cdot [P'] := [P \otimes_R P'].$$

This is a well-defined operation since for a fixed $P$, the mapping $P' \mapsto [P \otimes_R P']$ is additive on short exact sequences, and similarly for $P \mapsto [P \otimes_R P']$ for each fixed $P'$. The idenity element is $[R]$. Moreover, this structure is natural, so that $K_0(-)$ is a covariant fucntor from commutative rings to commutative rings.

The same holds on the level of schemes, but the variance is opposite: $K_0(-)$ is a contravariant functor from noetheriean schemes to commutative rings.

1.2. **Lambda operations.** There is important additional sctructure on $K_0(R)$ that arises from the exterior powers. For $P \in \mathcal{P}(R)$, let $\Lambda_R^k(P)$ denote the $k$-th exterior power:

$$\Lambda_R^k(P) = \frac{\overbrace{P \otimes_R \cdots \otimes_R P}^{k}}{L}$$

where $L$ is the sub-module generated by

$$e_1 \otimes \cdots \otimes e_k - \operatorname{sign}(\sigma) e_{\sigma(1)} \otimes \cdots \otimes e_{\sigma(k)}$$

for each permuation $\sigma \in S_k$.

Since the $k$-th exterior power of a free module of rank $r$ is free of rank $\binom{r}{k}$, it follows that $\Lambda_R^k(P)$ is a projective bundle of rank equal to $\binom{\operatorname{rank}(P)}{k}$. In particular, it is the trivial bundle for $k > \operatorname{rank}(P)$.

It is important to notice that the exterior power functor is *not additive* on short exact sequences. However, there is a replacement for this lack of additivity: given a short exact sequence

$$0 \to P' \xrightarrow{\iota} P \xrightarrow{\pi} P'' \to 0 \tag{1.4}$$

there is a filtration

$$0 = F_{-1} \subseteq F_0 \subseteq \cdots \subseteq F_k = \Lambda_R^k(P)$$

such that $F_i/F_{i-1}$ is isomorphic to $\Lambda^{k-i}(P') \otimes_R \Lambda^i(P'')$, for each $0 \le i \le k$. Namely, define $F_i$ as the image of the map $(P')^{\otimes k-i} \otimes_R P^{\otimes i} \to \Lambda^k P$ defined by

$$x_1' \otimes \cdots \otimes x_{k-i}' \otimes y_1 \otimes \cdots \otimes y_i \mapsto \iota(x_1') \wedge \cdots \wedge \iota(x_{k-i}') \wedge y_1 \wedge \cdots \wedge y_i.$$

For example, when $k = 2$, we have the filtration

$$0 \subseteq \Lambda^2(P') \subseteq F_1 \subseteq \Lambda^2(P)$$

with

$$F_1 = \operatorname{im}(P' \otimes P \xrightarrow{x' \otimes y \mapsto \iota(x') \wedge y} \Lambda_R^2(P))$$

The isomrphism $F_1/\Lambda^2(P') \xrightarrow{\cong} P' \otimes P''$ is induced from the surjection $\mathrm{id} \otimes \pi : P' \otimes_R P \to P' \otimes_R P''$, and the isomorphism $\Lambda^2(P)/F_1 \cong \Lambda^2(P'')$ is induced by the surjection $\Lambda^2(\pi) : \Lambda^2(P) \twoheadrightarrow \Lambda^2(P'')$.

It follows that

$$[\Lambda^k(P)] = \sum_{i=0}^{k} [\Lambda^{k-i}(P')] \cdot [\Lambda^i(P'')] \tag{1.5}$$

holds in $K_0(R)$. For example

$$[\Lambda^2(P)] = [\Lambda^2(P')] + [P'][P''] + [\Lambda^2(P'')]$$

Using this we are able to prove:

**Lemma 1.6.** *For each commutative ring $R$, there are functions $\lambda^k : K_0(R) \to K_0(R)$, for $k \ge 0$, uniquely deteremined by the following properties:*

- *For each $P \in \mathcal{P}(R)$ and $k$, we have*
$$\lambda^k([P]) = [\Lambda_R^k(P)].$$
- *For all $a, \beta \in K_0(R)$ and all $k$, we have*
$$\lambda^k(\alpha + \beta) = \sum_{i=0}^{k} \lambda^i(\alpha) \lambda^{k-i}(\beta). \tag{1.7}$$

*Moreover, these operators are natural for ring maps.*

*Proof.* Let $t$ be a formal parameter, form the ring $K_0(R)[[t]]$ of power series with coefficients in the commutative ring $K_0(R)$, and for any $P \in \mathcal{P}(R)$, define

$$\lambda_t(P) := \sum_k [\Lambda^k(P)]t^k \in K_0(R)[[t]].$$

(The sum is actually finite since $\Lambda^k(P) = 0$ for $k > \text{rank}(P)$.) Since the constant term of $\lambda_t(P)$ is 1, $\lambda_r(P)$ belongs to the group of units of $K_0(R)[[t]]$. For each short exact sequence (1.4), it follows from (1.12) that

$$\lambda_t(P) = \lambda_t(P') \cdot \lambda_t(P'');$$

that is, $\lambda_t$ is additive on short exact sequences, provided we interpret it as taking valus in the multiplicative abelian group $K_0(R)[[t]]^\times$. Therefore, by the universal mapping property of the Grothendieck group, $\lambda_t$ induces a homomorphism

$$\lambda_t : K_0(R) \to K_0(R)[[t]]^\times$$

of abelian groups. Finally, we define

$$\lambda^k : K_0(R) \to K_0(R)$$

to be the compotition of $\lambda_t$ with the map $K_0(R)[[t]]^\times \to K_0(R)$ sending a power series to the coefficient of $t^k$. Equation (1.7) follows.

The uniqueness property is seen to hold by induction on $k$.

The naturality assertion follows from the fact that given a ring map $g : R \to S$, we have an isomorphism $\Lambda^k_R(P) \otimes_P S \cong \Lambda^k_S(P \otimes_R S)$. □

The operator $\lambda^0$ is the constant function with value $1 = [R] \in K_0(R)$, and $\lambda^1$ is the identity map. The operator $\lambda^k$ is *not* a homomorphism of abelian groups for $k \neq 1$.

**Example 1.8.** What is $\lambda^2(-[P])$? We have $0 = \lambda^2([P]+(-[P])) = \lambda^2([P])\lambda^0(-[P]) + \lambda^1([P])\lambda^1(-[P]) + \lambda^0([P])\lambda^2(-[P]) = [\Lambda^2[P]] - [P \otimes P] + \lambda^2(-[P])$ and hence

$$\lambda^2(-[P]) = [P \otimes P] - [\Lambda^2(P)).$$

**Example 1.9.** When $R$ is local, we have $K_0(R) \cong \mathbb{Z}$. Under this isomorphism the operator $\lambda^k$ corresponds to the operator on $\mathbb{Z}$, which we will also write at $\lambda^k$, that satisfies

$$\lambda^k(n) = \binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

at least when $n \geq 0$. This holds since $\Lambda^k(R^n)$ is free of rank $\binom{n}{k}$ for any $n \geq 0$. But what if $n < 0$? To figure this out, we use the notation and results of the proof of the lemma. We have

$$\lambda_t(-m) = 1/\lambda_t(m) \in \mathbb{Z}[[t]]$$

and so if $m > 0$, we get

$$\lambda_t(-m) = \frac{1}{1 + mt + \binom{m}{2}t^2 + \cdots} = \frac{1}{(1+t)^m} = (1 - t + t^2 - t^3 + \cdots)^m.$$

Thus $\lambda^k(-m)$ is the coefficient of $t^m$ in $(1 - t + t^2 - t^3 + \cdots)^m$. It turns out that this shows

$$\lambda^k(n) = \frac{n(n-1)\cdots(n-k+1)}{k!}$$

is also the correct formula, even when $n < 0$.

The commutative ring $K_0(R)$ equipped with the operators $\lambda^k, k \geq 0$, form what's called a *(special) lambda ring.* Roughly, this means that they are natural operators satisfying

$$\lambda^k(a+b) = \sum_{i=0}^{k} \lambda^{k-i}(a)\lambda^i(b),$$

and such that the rules describing how the operators interact with multipliation and composition are given by certain universal polynomials. In detail, we have

$$\lambda^k(a \cdot b) = P_k(\lambda^1(a), \dots, \lambda^k(a), \lambda^1(b), \dots, \lambda^k(b))$$

for some polynomial $P_k$ in $2k$ variables, and similarly for $\lambda^k(\lambda^j(a))$. We will not need the details.

More generaly, starting with the exterior power functors for locally free coherent sheaves, one defines lambda operators on $K_0(X)$, making it into a a special lambda ring. Even if one is only interested in Grothendieck groups of commutative rings, passing to schemes is valuable due to the following important fact:

> **The Splitting Princple**: For each noetherian ring $R$ and $P \in \mathcal{P}(R)$, there exists a morphism of noetherian schemes $p : X \to \operatorname{Spec}(R)$ such that
> (1) the induced map $p^* : K_0(R) \to K_0(X)$ is injective and
> (2) $p^*[P] = \sum_{i=1}^{\operatorname{rank}(P)} [L_i]$ where the $L_i$'s are line bundles on $X$ (i.e., coherent sheaves that are locally free of rank 1).
>
> More generally, the analogous result holds for schemes: starting with any noetherian scheme $Y$ and locally free coherent sheaf $P$ on $Y$, there is a morphism $p : X \to Y$ such that the above two properties hold.

In fact, $X$ may be taken to be the flag variety over $R$ associated to $P$.

With the notation of the Splitting Principle, since $\lambda^k(L_i) = 0$ for all $k \geq 2$, we have

$$p^*\lambda^k([P]) = \lambda^k([p^*P]) = \sum_{i_1 < \cdots < i_k} [L_{i_1}] \cdots [L_{i_k}]$$

for eny $k$. The right side of this equation a priori belongs to $K_0(X)$, but since the left hand side is in the image of $p^*$, we may intepret this equatin as occuring in $K_0(X)$ (but not so for the individual terms of the right-hand side). In other words, $\lambda^k([P])$ is the $k$-th elementary symmetric polynomial evaluated on $[L_1], \dots, [L_k]$.

1.3. **Adams operations.** The Adams operations are classically defined by combining the lambda operations together in a specific way so as to produce operators that are linear. They are based on work of Frank Adams, working in the context of topological vector bundles.

The $k$-th Adams operation $\psi^k$ on $K_0(R)$ is defined as follows. Recall that homomorphism of abelian groups $\lambda_t : K_0(R) \to K_0(R)[[t]]^\times$ defined in the proof of Lemma 1.6. For any commutative ring $A$ and variable $t$, write $\operatorname{dlog} : A[[t]]^\times \to A[[t]]$ for the function sending $p(t) = \sum_i a_i t^i$ to

$$\operatorname{dlog}(p(t)) := \frac{p'(t)}{p(t)} = \frac{\sum_i i a_i t^{i-1}}{\sum_i a_i t^i}$$

("the derivative of the natural log of $p(t)$"). Then dlog converts multiplication to addition; i.e. it is a homomorphiam of abelian groups, where the operation on the source is power series multiplication and the operation on the the target is addition of power series.

For $\alpha \in K_0(R)$, set

$$\psi_t(\alpha) = \operatorname{rank}(\alpha) - t \operatorname{dlog}(\lambda_{-t}(\alpha))$$

Since dlog converts multiplicaiton to addition, $\psi_t : K_0(R) \to K_0(R)[[t]]$ is a homomorphism of additive groups. We define

$$\psi^k : K_0(R) \to K_0(R)$$

as the composition of $\psi_t$ with the map that sends a power series to its $t^k$ coefficient. By construction, $\psi^k$ is a homomorphism of abelian groups for each $k \geq 0$.

**Example 1.10.** Let's compute $\psi^k$ for $k \leq 3$. We have

$$\begin{aligned}
\operatorname{dlog}(\lambda_{-t}(\alpha)) &= \frac{\lambda_{-t}(\alpha)'}{\lambda_{-t}(\alpha)} \\
&= \frac{-\lambda^1(\alpha) + 2\lambda^2(\alpha)t - 3\lambda^3(\alpha)t^2 + \cdots}{1 - \lambda^1(\alpha)t + \lambda^2(\alpha)t^2 - \cdots} \\
&= (-\lambda^1(\alpha) + 2\lambda^2(\alpha)t - 3\lambda^3(\alpha)t^2 + \cdots)(1 + \lambda^1(\alpha)t + (\lambda^1(\alpha)^2 - \lambda^2(\alpha))t^2 + \cdots) \\
&= -\lambda^1(\alpha) + (-\lambda^1(\alpha)^2 + 2\lambda^2(\alpha))t + (-\lambda^1(\alpha)^3 + \lambda^1(\alpha)\lambda^2(\alpha) + 2\lambda^2(\alpha)\lambda^1(\alpha) - 3\lambda^3(\alpha)^2)t^2 + \cdots
\end{aligned}$$

Since $\lambda^1(\alpha) = \alpha$, it follows that

$$\begin{aligned}
\psi^0(\alpha) &= \operatorname{rank}(\alpha) \\
\psi^1(\alpha) &= \alpha \\
\psi^2(\alpha) &= \alpha^2 - 2\lambda^2(\alpha) \\
\psi^3(\alpha) &= \alpha^3 - 3\lambda^2(\alpha)\alpha + 3\lambda^3(\alpha)
\end{aligned}$$

*Remark* 1.11. Let's double check that $\psi^2$ is additive on short exact sequences directly: Given $0 \to P' \to P \to P'' \to 0$, we have

$$\begin{aligned}
\psi^2(P) &= [P]^2 - 2[\Lambda^2(P)] \\
&= ([P'] + [P''])^2 - 2([\Lambda^2(P')] + [P' \otimes P''] + [\Lambda^2(P'')]) \\
&= [P']^2 - 2[\Lambda(P')] + [P'']^2 - 2[\Lambda(P'')] \\
&= \psi^2(P') + \psi^2(P'').
\end{aligned}$$

In general, the functional equation defining the Adams operations leads to the recursive formula

(1.12)
$$\psi^k(\alpha) - \psi^{k-1}(\alpha)\lambda^1(\alpha) + \psi^{k-2}(\alpha)\lambda^2(\alpha) - \cdots + (-1)^{k-1}\psi^1(\alpha)\lambda^{k-1}(\alpha) + (-1)^k k\lambda^k(\alpha) = 0.$$

The key properties of the Adams operations are summarized in:

**Proposition 1.13.** *For each $k \geq 0$,*

1. *$\psi^k : K_0(R) \to K_0(R)$ is a ring endomorphism for each $R$ (and likewise for $K_0(X)$ for each $X$),*
2. *$\psi^k$ is natural for ring maps (and more generally for morphisms of schemes),*
3. *if $L$ is a rank one projective $R$-module (or, more generally, a coherent sheaf locally free of rank 1 on a scheme), then $\psi^k([L]) = [L^{\otimes k}]$.*

*Moreover, these properties uniquely characterize the operator $\psi^k$.*

*Proof.* We've already oberved that $\psi^k$ preserves addition.

Property (2) follows by induction on $k$ using (1.12) and the fact that the lambda operations are natural.

Property (3) also follows from (1.12) by induction on $k$, using also that $\lambda^j([L]) = 0$ for all $j \geq 2$.

The multiplicativity follows from the additivity and the splitting principle. In detail, it suffices to check $\psi^k([P][P']) = \psi^k([P])\psi^k([P'])$. In the case when $[P]$ and $[P']$ are sums of classes of line bundles, this holds by (3) and the fact that $\psi^k$ is additive. The general case follows by naturalty and the Splitting Principle.

The last assertion also follows from the Splitting Principle. □

*Remark* 1.14. Given $P \in \mathcal{P}(R)$, let $\pi : X \to \operatorname{Spec}(R)$ be as in the Splitting Principle, so that $\pi^*([P]) = \sum_{i=1}^r [L_i]$. Then

$$\pi^*\psi^k([P)] = \sum_{i=1}^r [L_i^{\otimes k}].$$

In other words, $\psi^k$ takes $[P]$ to the $k$-th Newton polynomial evaluated at $[L_1], \dots, [L_r]$.

The $k$-th Newton polynomial is symmetric, and every symmetric polynomial can be written in terms of the elementary symmetric polynomials. This gives another method of expressing $\psi^k$ in terms of $\lambda^1, \dots, \lambda^k$.

**Corollary 1.15.** *If $R$ has characteristic $p$ for some prime $p$, then $\psi^p : K_0(R) \to K_0(R)$ coincides with the map induced by extension of scalars along the Frobenius map $\phi : R \to R$.*

*Proof.* The map $\phi_* : K_0(R) \to K_0(R)$ is a natural ring homomoriphism and one can check that $\phi_*(L) \cong L^{\otimes p}$ for any line bundle $L$ on a scheme of characteristic $p$. The result follows from the Proposition. □

**Exercise 1.16.** Prove $\phi_*(L) \cong L^{\otimes p}$ for any rank one projective $R$-module $L$.

**Exercise 1.17.** Use the proposition to prove

$$\psi^k \circ \psi^j = \psi^{kj} \tag{1.18}$$

for all $k, j \geq 0$. (In particular, this shows $\psi^k$ and $\psi^j$ commute.)

### 1.4. **A Theorem of Grothendieck.**

Finally, we mention a famous theorem of Grothendieck. (Technically, Grothendieck did not phrase this Theorem in terms of Adams operations.)

**Theorem 1.19** (Grothendieck, 1950's)**.** *If $X$ is a smooth variety over a field, then for any integer $k \geq 2$, the action of $\psi^k$ on $K_0(X)_{\mathbb{Q}}$ is diagonalizable with eigenvalues in the set $k^0, \dots, k^{\dim(X)}$. In other words we have an internal direct sum decomposition*

$$K_0(X)_{\mathbb{Q}} = \bigoplus_{j=0}^{\dim(X)} K_0(X)_{\mathbb{Q}}^{(j)}$$

*where we define $K_0(X)_{\mathbb{Q}}^{(j)} := \ker((\psi^k - k^j) : K_0(X)_{\mathbb{Q}} \to K_0(X)_{\mathbb{Q}})$, the eigenspace of the operator $\psi^k_{\mathbb{Q}}$ of eigenvalue $k^j$.*

*Moreover, for each $j$, we have isomorphisms*

$$K_0(X)_{\mathbb{Q}}^{(j)} \cong CH^j(X)_{\mathbb{Q}}$$

*where $CH^j(X)$ is the Chow group of codimension $j$ cycles modulo rational equiavalence.*

**Exercise 1.20.** Prove the subspace $K_0(X)^{(j)}_{\mathbb{Q}}$ is independent of the choice of $k \geq 2$. Tip: Use (1.18).

## 2. Day 2: Algebraic $K$-theory with supports

A good reference for the material in this seciton is the paper "Intersection Theory using Adams Operations", *Inventiones Mathematicae*, by Gillet and Soulé.

Much of what we do in this section would work just as well for arbitrary schemes, but for the sake of concreteness we will stick to affine ones. Let us fix some notation:

- $R$ is a commutative Noetherian ring.
- $Z$ is a Zariski closed subset of $\operatorname{Spec}(R)$. So,
$$Z = V(I) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid I \subseteq \mathfrak{p}\}$$
for some ideal $I \subseteq R$.
- $\mathcal{P}^Z(R)$ is the category of bounded complexes of finitely generated and projective $R$-modules with homology supported on $Z$. That is, a typical object is a complex of the form
$$P := (\cdots \to 0 \to P_m \to \cdots \to P_n \to 0 \to \cdots)$$
with each $P_i$ a finitely generated and projective $R$-module, such that $P_{\mathfrak{p}}$ is an acyclic complex for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus Z$. Morphisms in $\mathcal{P}^Z(R)$ are chain maps. (Recall that before we wrote $\mathcal{P}(R)$ for the category of f.g. $R$-modules; to avoid confusion, we'll not use that notation from now on.)

### 2.1. Grothendieck group with supports.

**Definition 2.1.** $K_0^Z(R)$, the *Grothendieck group* of $\mathcal{P}^Z(R)$, is the abelian group generated by the set of classes $[P]$ for each object $P$ of $\mathcal{P}^Z(R)$, subject to two types of relations:

$[P] = 0$ if $P$ is exact, and

$[P] = [P'] + [P'']$ if there exists a short exact sequence of complexes of the form
$$0 \to P' \to P \to P'' \to 0.$$

**Exercise 2.2.** Establish the following facts:

(1) We have
$$[P] + [P'] = [P \oplus P'] \in K_0^Z(R), \tag{2.3}$$
for any pair of objects $P, P' \in \mathcal{P}^Z(R)$.

(2) For any $P \in \mathcal{P}^Z(R)$, we have $[\Sigma P] = -[P]$ in $K_0^Z$, where $\Sigma P$ is the suspension (shift) of $P$. (Tip: Use the short exact sequence $0 \to P \to \operatorname{cone}(\mathrm{id}_P) \to \Sigma P \to 0$.)

(3) Given $P, P' \in \mathcal{P}^Z(R)$, if there exists a quasi-isomorphism $\alpha : P \to P'$ in $\mathcal{P}^Z(R)$, then $[P] = [P']$ in $K_0^Z(R)$. (Consider $0 \to P' \to \operatorname{cone}(\alpha) \to P \to 0$.)

(4) Every element of $K_0^Z(R)$ is equal to one of the form $[P]$ for some $P \in \mathcal{P}^Z(R)$.

**Example 2.4.** Suppose $M$ is a finitely generated $R$-module of finite projective dimension, and choose a bounded resoltion $P \xrightarrow{\sim} M$ by finitely generted projective modules. Then $P$ is an object of $\mathcal{P}^{\mathrm{supp}(M)}(X)$ and hence determines a class $[P]$ in $K_0^{\mathrm{supp}(M)}(R)$. If $P'$ is another such resolution, then $P$ and $P'$ are homotopy equivalent and hence $[P] = [P']$. So $M$ determines a well-define element of $K_0^{\mathrm{supp}(M)}(R)$.

*Remark* 2.5. Associated to $\mathcal{P}^Z(X)$ we have its *homotopy category*, written $\mathrm{hot}(\mathcal{P}^Z(X))$. The objects of this category are the same as for $\mathcal{P}^Z(X)$, but morphisms are homotopy equivalence classes of chain maps. The category $\mathcal{P}^Z(X)$ has the structure of a triangulated category. The suspension functor $\Sigma$ is the usual shift functor for chain complexes. By definition, a triangle $P_1 \to P_2 \to P_3 \to \Sigma P_1$ is distinguished if it is isomorhic (in $\mathrm{hot}(\mathcal{P}^Z(X))$ to one of the form $P \xrightarrow{\alpha} P' \xrightarrow{\mathrm{can}} \mathrm{cone}(\alpha) \xrightarrow{\mathrm{can}} \Sigma P$.

$K_0^Z(R)$ may be equivalently defined as the Grothendieck group of this triangulated category. This means that

$$[\Sigma P] = -[P]$$

and

$$[P] = [P'] + [P'']$$

whenever there is a distinguished triangle $P' \to P \to P'' \to \Sigma(P')$ in $\mathrm{hot}(\mathcal{P}^Z(X))$.

**Example 2.6.** There is a canonical isomorphism $K_0^{\mathrm{Spec}\,R}(R) \xrightarrow{\cong} K_0(R)$, where $K_0(R)$ is the Grothendieck group of f.g. projective $R$-modules, that takes the class of an object $P \in \mathcal{P}(X)$ to the alternating sum of its components. The inverse takes the class of a f.g. projective $R$-module to the class of the complex obtained by regarding it as a complex concentrated in degree 0.

**Exercise 2.7.** Prove the assertions of the previous example.

Using the universal mapping property for presetations of abelian groups, we have the following: If $A$ is any abelian group and $\rho$ is a function assigning to each object of $\mathcal{P}^Z(X)$ an element of $A$ such that

$$\rho(P) = 0, \text{ if } P \text{ is acyclic}$$

and

$$\rho(P) = \rho(P') + \rho(P''), \text{ if there is a short exact sequence } 0 \to P' \to P \to P'' \to 0$$

then $\rho$ induces a unique homomorphism of abelian groups (also written as $\rho$)

$$\rho : K_0^Z(X) \to A,$$

defined by $\rho([P]) = \rho(P)$.

2.2. **Complexes with finite length homology.** Of particular interest for us will be the category $\mathcal{P}^{\mathfrak{m}}(R)$ and its associated Grothendieck group $K_0^{\mathfrak{m}}(R)$, when $(R, \mathfrak{m})$ is a local ring. (Technically the superscripts sould be "$\{\mathfrak{m}\}$" not "$\mathfrak{m}$", but we will use the latter.) We may equivalently describe an object of $\mathcal{P}^{\mathfrak{m}}(R)$ as a bounded complex of finitely generated projective $R$-modules having finite length homology. In particular, if $M$ is a finite length $R$-module of finite projective dimension, then $M$ determines a class in $K_0^{\mathfrak{m}}(R)$ by choosing a free resolution.

**Lemma 2.8.** *If $\mathfrak{m}$ is a maximal ideal of $R$, the map $\chi : \operatorname{ob} \mathcal{P}^{\mathfrak{m}}(R) \to \mathbb{Z}$ sending $P$ to $\sum_i (-1)^i \operatorname{length}_R H_i(P)$ induces a homomorpihsm*

$$\chi : K_0^{\{\mathfrak{m}\}}(R) \to \mathbb{Z}.$$

*If $(R, \mathfrak{m})$ is a regular local ring, this map is an isomorphism and $K_0^{\{\mathfrak{m}\}}(R)$ is the free abelian group of rank 1 generated by the class of the Koszul complex on a regular system of paramters.*

*Proof.* If $P$ is acyclic then $\chi(P) = 0$. Using the long exact sequence in homology associated to a short exact sequence of chain complexes, we see that $\chi$ is also additive on short exact seqeunces of complexes. It thus induces the indicated homomorphism $\chi$ on $K_0^{\mathfrak{m}}(R)$.

Assume $R$ is regular local and let $K \in \mathcal{P}^{\mathfrak{m}}(R)$ be the Koszul complex on a regular system of parameters. Then $H_0(K) = R/\mathfrak{m}$ and $H_i(K) = 0$ for all $i \neq 0$, and hence $\chi([K]) = 1$. In particular, $\chi$ is onto. Let $E$ be any object of $\mathcal{P}^{\mathfrak{m}}(R)$. It remains to show $[E]$ is in the subgroup generated by $[K]$. We proceed by induction on $h = h(E) = \sum_i \operatorname{length}_R H_i(E)$. If $h = 0$, then $E$ is acyclic and hence $[E] = [0] = 0$.

Assume $h > 0$. Pick $i$ as large as possible[1] so that $H_i(E) \neq 0$, and then pick $\alpha \in H_i(E)$ such that $\alpha \neq 0$ and $\mathfrak{m}\alpha = 0$. Since $H_j(E) = 0$ for $j > i$, it follows that there is a chain map $g : \Sigma^i K \to E$ such that the induced map on $H_i$ has the form $R/\mathfrak{m} \to H_i(E)$, sending 1 to $\alpha$. Let $C = \operatorname{cone}(g)$. Then we have an exact sequence $0 \to E \to C \to \Sigma^{i+1} K \to 0$ so that $[E] = [C] + (-1)^i [K]$. By construction, $h(C) = h(E) - 1$ and thus by induction $[C] \in \mathbb{Z} \cdot [K]$, and we are done. □

**Question 2.9.** *Does the analogue of the previous Lemma hold for $\mathbb{Z}/2$-graded complexes?*

*Remark* 2.10. When $R$ is not regular, $K_0^{\mathfrak{m}}(R)$ is in general much bigger than $\mathbb{Z}$. It is typically not at all easy to compute its value, and it is typically not finitely generated. Here is one case in which it can be computed: Assume $(R, \mathfrak{m})$ is a local domain of dimension 1, and let $F$ be its field of fractions. Then we have an isomorphism of abelian groups

$$\frac{F^\times}{R^\times} \cong K_0^{\mathfrak{m}}(R)$$

(with the left-hand side being mutliplicative.) The map sends $\frac{a}{b} \in F^\times$ to $[R \xrightarrow{a} R] - [R \xrightarrow{b} R] \in K_0^{\mathfrak{m}}(R)$ for any $a, b \in R \setminus \{0\}$. (Note that $R \xrightarrow{c} R$ has finite length homology for any $0 \neq c \in R$ since we assume $\dim(R) = 1$.)

Note that if $R$ is regular (i.e., a DVR) then the valuation mapping gives an isomorphism $F^\times / R^\times \cong \mathbb{Z}$, as the Lemma tells us.

**Exercise 2.11.** Prove the assertions of the previous remark.

2.3. **Cup product.** Given a pair of closed subset $Z$ and $W$ of $\operatorname{Spec}(R)$, tensor product of complexes determines a bi-functor

$$\mathcal{P}^Z(R) \times \mathcal{P}^W(R) \to \mathcal{P}^{Z \cap W}(R)$$

given by $(P, P') \mapsto P \otimes_R P'$, that is bi-exact and preserves homotopy equivalences in each argument. It thus induces a bilinear pairing:

[1] In my talk, I neglected to assume $i$ is as large as possible, which is necessary for this argument to be valid.

**Definition 2.12.** The *cup product* pairing is the map on Grothendieck groups

$$- \cup - : K_0^Z(R) \times K_0^W(R) \to K_0^{Z\cap W}(R)$$

given by $[P] \cup [P'] = [P \otimes_R P']$. (Here $P \otimes_R P'$ refers to the tensor product complex.)

This is a well-defined pairing, since $P \otimes_R -$ maps ayclic complexes to acyclic complexes, and short exact sequences to short exact sequences, and similarly for $- \otimes_R P'$. Moreover, for $P \in \mathcal{P}^Z(R), P' \in \mathcal{P}^W(R)$, if $\mathfrak{p} \in \operatorname{Spec}(R) \setminus (Z \cup W)$ then either $P_\mathfrak{p}$ or $P'_\mathfrak{p}$ is exact, and hence so is $(P \otimes_R P')_\mathfrak{p} \cong P_\mathfrak{p} \otimes_{R_\mathfrak{p}} P'_\mathfrak{p}$. This shows that the target of $- \cup -$ is indeed $K_0^{Z\cap W}(R)$.

The following is easy to prove by checking on generators:

**Lemma 2.13.** *The cup product operation is commutative, associative, and unital, in the appropriate senses.*

**Example 2.14.** Suppose $(R, \mathfrak{m})$ is a local ring of dimension $d$ and let $x_1, \dots, x_d$ is a system of paramters. Let $\operatorname{Kos}_R(x_i) = (\cdots 0 \to R \xrightarrow{x_i} R \to 0 \to \cdots)$ be the Koszul complex on $x_i$, for each $i$. Then $[\operatorname{Kos}_R(x_i)] \in K_0^{V(x_i)}(R)$ for each $i$ and we have an equation

$$[\operatorname{Kos}_R(x_1)] \cup \cdots \cup [\operatorname{Kos}_R(x_d)] = [\operatorname{Kos}_R(x_1, \dots, x_d)] \in K_0^{\mathfrak{m}}(R),$$

since $V(x_1, \cdots, x_d) = \{\mathfrak{m}\}$.

**Example 2.15.** Given a finitely generated $R$-module $M$ of finite projective dimension a chosen bounded resolution $P^M$ gives a class in $K_0^{\operatorname{supp}(M)}(R)$. For another such module $N$, we have $[P^N] \in K_0^{\operatorname{supp}(N)}(R)$ and

$$[P^M] \cup [P^N] = [M \otimes_R^{\mathbb{L}} N] \in K_0^{\operatorname{supp}(M)\cap\operatorname{supp}(N)}(R).$$

2.4. **Intersection multiplicity.** Building on the previous example, suppose in addition that $R$ is local and $M$ and $N$ are chosen such that $\operatorname{supp}(M) \cap \operatorname{supp}(N) \subseteq \{\mathfrak{m}\}$. (This is equivalent to $\operatorname{length}_R(M \otimes_R N) < \infty$.) Recall that there is a map

$$\chi : K_0^{\{\mathfrak{m}\}}(R) \to \mathbb{Z}$$

given by $\chi([P]) = \sum_j (-1)^j \operatorname{length} H_j(P)$. We get that

$$\chi([P_M] \cup [P_N]) = \sum_i (-1)^i \operatorname{length}_R(\operatorname{Tor}_i^R(M, N)) =: \chi(M, N).$$

This is *Serre's intersection multiplicity* formula.

The goemetric intution here is the following: Say $M = R/I$ and $N = R/J$ have finite projective dimension (e.g., say $R$ is regular) and $\sqrt{I + J} = \mathfrak{m}$. Geometrically, $V(I)$ and $V(J)$ meet only at the single point $\{\mathfrak{m}\}$ The integer $\chi(R/I, R/J)$ gives the multiplicity of the intersection.

**Example 2.16.** Let $k$ be a field and $R = k[[x, y]]$. Suppose $I = (f(x, y)$ and $J = (g(x, y))$ meet only at the maximal ideal. Then $f, g$ have no common factors, and hence form a regular seuqnece, so that $\operatorname{Tor}_i^R(R/f, R/g) = 0$ for $i \neq 0$. We get

$$\chi(R/f, R/g) = \operatorname{length}_R R/(f, g) = \dim_k R/(f, g).$$

2.5. **Functorality.** Let $\phi : R \to S$ is a ring map and write $\phi^\# : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ be the induced map on spectra (i.e., $\phi^\#(\mathfrak{p}) = \phi^{-1}(\mathfrak{p})$). Suppose $Z \subseteq \mathrm{Spec}(R)$ and $W \subseteq \mathrm{Spec}(S)$ closed subsets such that $(\phi^\#)^{-1}(Z) \subseteq W$. (That is, assume that if $\mathfrak{q} \in \mathrm{Spec}(S) \setminus W$ then $g^{-1}(\mathfrak{q}) \in \mathrm{Spec}(R) \setminus Z$.) Then $\phi$ induces a homomorpihsm

$$\phi_* = \phi_*^{Z,W} : K_0^Z(R) \to K_0^W(S).$$

To see that the target is correct, suppose $\mathfrak{q} \in \mathrm{Spec}(S) \setminus W$ and $P \in \mathcal{P}^Z(R)$. By assumption $\mathfrak{p} = \phi^{-1}(\mathfrak{q}) \notin Z$ and thus

$$(P \otimes_R S)_\mathfrak{q} \cong P_\mathfrak{p} \otimes_{R_\mathfrak{p}} S_\mathfrak{q}$$

is acyclic.

**Example 2.17.** If $\phi = \mathrm{id}_R$ and $Z \subseteq W \subseteq \mathrm{Spec}(R)$, $K_0^Z(R) \to K_0^W(R)$ is induced by the inclusion $\mathcal{P}^Z(R) \subseteq \mathcal{P}^W(R)$. Beware that this map on Grothendieck groups is often not injective, since upon enlarging the support, one not only enlarges the number of generators but also the numbers of relations.

For example, if $R$ is a local domain, $W = \mathrm{Spec}(R)$ and $Z$ is any proper closed subset, then $K_0^Z(R) \to K_0^{\mathrm{Spec}(R)}(R)$ is the zero map: For recall that $K_0^{\mathrm{Spec}(R)}(R) \cong K_0(R) \cong \mathbb{Z}$, with the composition sending $[P]$ to $\sum_i (-1)^i \mathrm{rank}_R(P_i)$. If $P \in K_0^Z(R)$ and $Z \subsetneq \mathrm{Spec}(R)$, then $P \otimes_R F$ is acyclic, where $F$ is the field of fractions of $R$. It follows that $\sum_i (-1)^i \mathrm{rank}_R(P_i) = 0$.

**Example 2.18.** For any $f \in$R, we have the localization map $\phi : R \to R[1/g]$, which induces the map

$$K_0^Z(R) \to K_0^{Z \setminus V(g)}(R[1/g])$$

sending $[P]$ to $[P[1/g]]$.

2.6. **A right exact sequence.**

**Theorem 2.19** (Gillet-Soulé)**.** *For a regular ring $R$, closed subset $Z$ of* Spec($R$)*, and element $g \in R$, the seqeunce*

$$K_0^{Z \cap V(g)}(R) \to K_0^Z(R) \to K_0^{Z \setminus V(g)}(R[1/g]) \to 0$$

*is exact.*

*Sketch of proof.* The proof relies on the following fact: For any regular ring $B$ and ideal $J$, we have an isomorphism $G_0(B/J) \xrightarrow{\cong} K_0^{V(J)}(B)$, where $G_0(B/J)$ denotes the Grothendieck group of all finitely generated $B/J$-modules and the map sends the class of such a module to the class of a projective resolution of it. The proof of this fact is not difficult, but we omit it.

Say $Z = V(I)$. Since $R$ is regular, using the fact above, the sequence in the statement is isomorphic to the sequence

$$G_0(A/f) \to G_0(A) \to G_0(A[1/f]) \to 0,$$

where $A = R/I$, the first map is induced by restriction of scalars and the second by localization. The latter sequence is a portion of the well-known localization long exact sequence in $G$-theory; we include a sektch of the proof, adapted from Weibel's book "An Intoduction to Algebraic $K$-theory".

The composition $G_0(A/f)) \to G_0(A) \to G_0(A[1/f])$ is the 0 map since $N[1/f] = 0$ for any $A/f$-module $N$. Given a finitely generated $A[1/f]$-module $M$, by choosing a presentation and clearing denominaors, we constrct a finitely generated $A$-module $N$ such that $N[1/f] \cong M$. It follows that $G_0(A) \to G_0(A[1/f])$ is onto.

Set $\Gamma = \mathrm{coker}(G_0(A/f) \to G_0(A))$. By what we've already shown, there is an induced surjection $\Gamma \to G_0(A[1/f]))$, and we need to show it is an isomorphism. We do so by constructing an inverse. For each finitely generated $A[1/f]$-module $M$, choose a "lift" to a finitely generated $A$-module $N$ such that there is an isomprhism $N[1/f] \cong M$ of $A[1/f]$-modules, and set $\alpha(M) \in \Gamma$ to be the image of the class $[N] \in G_0(A)$ under the canonical map $G_0(A) \twoheadrightarrow \Gamma$. If we can show that $\alpha$ induces a well-defined additive function that is independent of the choice of $N$, then the induced map $\alpha : G_0(A[1/f]) \to \Gamma$ will be a left inverse of $\Gamma \tau G_0(A[1/f])$ and the proof will be complete.

It remains to show $\alpha(M)$ does not depend on the choice of $N$ and that it is additive on ses's. If $N'$ is another such lift, then there exists an isomorhism $g : N[1/f] \xrightarrow{\cong} N'[1/f]$ of $A[1/f]$-modules. Multiplying though by a sufficiently high power of $f$, this map $g$ may be assumed to send $N$ to $N'$. That is, $g$ lifts to a homomorphim $\tilde{g} : N \to N'$, whose kernel and cokernel are annilated by a power of $f$. So, each admits a filtration by objects annilhated by $f$ itself, and thus $[\ker(\tilde{g})]$ and $[\mathrm{coker}(\tilde{g})]$ both lie in the image of $G_0(A/f) \to G_0(A)$. It follows that $\alpha(M)$ is a well-defined function, independent of the choice of $N$.

Given a short exact sequence of finitely generated $A[1/f]$-modules $0 \to M' \to M \to M'' \to 0$, we may find finitely generated $A$-modules $N, N'$ and maps $N' \to N$, $N \to N''$ that lift these. The composition of $N' \to N \to N''$ is not a priori 0, but multiplciaty thorugh by a power $f$, this can be arranged. So, we have a complex $0 \to N' \to N \to N'' \to 0$, and its homology is annilated by a power of $f$. As before, it follows that

$$\overline{[N]} = \overline{[N']} + \overline{[N'']}$$

holds in $\Gamma$. □

2.7. **Adams operations.** We come to the central tool. In order to better understand the groups $K_0^Z(R)$, we need to decompose them into so-called "weight pieces" that have certain desireably property. These weight peices are obtained by taking eindepaces for certain Adams operators. All of this is analogous to what was done by Grothendieck for $K_0(R)$.

**Definition 2.20** (Gillet-Soulé)**.** Let $\mathcal{C}$ be a collection of commutative noetherian rings and let $k$ be a positive integer. An *Adams operation of degree $k$ defined on $\mathcal{C}$* is a collection of functions

$$\psi^k = \psi^k_{R,Z} : K_0^Z(R) \to K_0^Z(R)$$

for all $R \in \mathcal{C}$ and all closed subsets $Z \subseteq \mathrm{Spec}(R)$ such that four axioms hold:

(1) (Additivity) Each $\psi^k_{R,Z}$ is an endomorphism of abelian groups.
(2) (Multiplicativity) For any $R \in \mathcal{C}$ and closed subsets $Z, W$ of $\mathrm{Spec}(R)$, the diagram

$$\begin{tikzcd} K_0^Z(R) \times K_0^W(R) \arrow[r, "\cup"] \arrow[d, "\psi^k_{R,Z} \times \psi^k_{R,W}"] & K_0^{Z\cap W}(R) \arrow[d, "\psi^k_{R,Z\cap W}"] \\ K_0^Z(R) \times K_0^W(R) \arrow[r, "\cup"] & K_0^{Z\cap W}(R) \end{tikzcd}$$

commutes.

(3) (Naturality) Given a ring homomorphism $\phi : R \to S$, with $R, S \in \mathcal{C}$, and closed subsets $Z \subseteq \operatorname{Spec}(R)$, and $W \subseteq \operatorname{Spec}(S)$ such that $(\phi^{\#})^{-1}(Z) \subseteq W$, the diagram

$$\begin{tikzcd} K_0^Z(R) \arrow[r, "\phi_*^{Z,W}"] \arrow[d, "\psi^k_{R,Z}"] & K_0^W(S) \arrow[d, "\psi^k_{S,W}"] \\ K_0^Z(R) \arrow[r, "\phi_*^{Z,W}"] & K_0^W(S) \end{tikzcd}$$

commutes.

(4) (Normalization) For all $R \in \mathcal{C}$ and $a \in R$

$$\psi^k_{R,V(a)}([\operatorname{Kos}_R(a)]) = k \cdot [\operatorname{Kos}_G(a)] \in K_0^{V(a)}(R),$$

where $\operatorname{Kos}_R(a) = \left( \cdots \to 0 \to R \xrightarrow{a} R \to 0 \to \cdots \right)$, the Koszul complex on $a$.

*Remark* 2.21. These axioms do not uniquely specify the operator.

**Example 2.22.** Given an Adams operation on $\mathcal{C}$ of degree $k$, suppose $a_1, \dots, a_c \in R$ and $R \in \mathcal{C}$, and let

$$\operatorname{Kos}_R(a_1, \dots, a_c) = \bigotimes_i \operatorname{Kos}_R(a_i).$$

The multipilcative and normalization axioms give

$$\psi^k([\operatorname{Kos}_R(a_1, \dots, a_c)]) = k^c[\operatorname{Kos}_R(a_1, \dots, a_c)] \in K_0^{V(a_1,\dots,a_c)}(R).$$

2.8. **Frobenius.** One example of an Adams operation comes from the Frobenius, as we explain.

Let $p$ be a prime and $\mathcal{C}_p$ the collection of all commutative noetherian rings of characteristic $p$. For each $R \in \mathcal{C}_p$, let $F : R \to R$ denote the Frobenious endomorphism. Since $F^* : \operatorname{Spec}(R) \to \operatorname{Spec}(R)$ is the identity map of topological spaces, for each closed subset $Z$ we have an induced map

$$F_* : K_0^Z(R) \to K_0^Z(R)$$

that sends $[P]$ to $[P \otimes_R {}^F R]$.

**Proposition 2.23.** *$F_*$ is an Adams operation of degree $p$ defined on $\mathcal{C}_p$.*

*Proof.* Axioms 1 and 2 hold since $F_*$ is the homomorphism induced by extension of scalars. Axiom 3 holds by the naturality of Frobenius. For Axiom 4, we have

$$\operatorname{Kos}_R(a) \otimes_R {}^F R \cong \operatorname{Kos}_R(a^p)$$

and so it suffices to prove $[\operatorname{Kos}_R(a^p)] = p[\operatorname{Kos}_R(a)] \in K_0^{V(a)}(R)$ — this actually holds for any ring $R$, element $a \in R$ and integer $p$:

Let $\alpha : \operatorname{Kos}_R(a^{p-1}) \to \operatorname{Kos}_R(a^p)$ be the chain map given as the idenity in degree 1 and multiplication by $a$ in degree 0. Then $\operatorname{cone}(\alpha) \sim \operatorname{Kos}_R(a)$ and hence

$$[\operatorname{Kos}_R(a^p)] = [\operatorname{Kos}_R(a^{p-1})] + [\operatorname{Kos}_R(a)].$$

The result thus following by induction on $p$. $\square$

## 3. Day 3: Existence of Adams Operations, Weight Decompositions

### 3.1. **The Adams Operations of Gillet-Soulé.**

**Theorem 3.1** (Gillet-Soulé)**.** *For each $k \geq 1$, there exists an Adams operation $\psi^k_{GS}$ of degree $k$ defined on the collection of all commutative noetherian rings. Moreover,*

$$\psi^k_{GS} \circ \psi^j_{GS} = \psi^{jk}_{GS}.$$

*for all $j, k \geq 1$. In particular, any two such operations commute.*

Some comments:

- They first establish $\lambda$ operations on $K_0^Z(R)$ and define $\psi^k_{GS}$ via the same formula used for classical $K_0$.
- These $\lambda$ operations are defined by replacing complexes with simplicial modules (Dold-Kan correspondence) and taking exerior powers. This idea this goes back to Dold-Puppe.
- A difficult argument is needed to verify the axioms of a $\lambda$ ring.

We will not prove the Theorem of Gillet-Soulé.

### 3.2. **Cyclic Adams Operations.**

I present another way to create Adams operations due to [Brown-Miller-Thompson-W]. We call these "cyclic Adams operation".

An overview:

- The construction requires fixing a prime $p$ and we build the operators only for the cateogry of $\mathbb{Z}[1/p, \zeta_p]$-algebras, where $\zeta_p = e^{2\pi p}$, a primitive $p$-th root of unity. We build an Adams operation of degree $p$.
- Starting with an arbitrary local ring $R$, we could pick $p \notin \mathfrak{m}$ and pass to a finite étale extension $R \subseteq R'$ with $\zeta_p \in R'$. For many application, no important information is lost by this process and so these operations are usually good enough.
- $\psi^p_{cy}$ can be defined in other contexts too: e.g., on the Grothendieck group of matrix factorizations.

For a bounded complex $P$ of finitely generated, projective $R$-modules, and integer $n \geq 0$, define

$$T^n(P) := \overbrace{P \otimes_R \cdots \otimes_R P}^{n}$$

The symmetric group $\Sigma_n$ acts on $T^n(P)$ by permuting the tensor factors and adhering to the Koszul sign convention:

$$\tau \cdot (x_1 \otimes \cdots \otimes x_n) = \pm x_{\tau(1)} \otimes \cdots \otimes x_{\tau(n)}$$

where the $x_i$'s are homogenous elements of $P$. The sign is uniquely determined by the following rule: When $\tau$ is the adjacent transposition $\tau = (i\, i+1)$ for some $1 \leq i \leq n-1$, we have

$$\tau \cdot (x_1 \otimes \cdots \otimes x_n) = (-1)^{|x_i||x_{i+1}|} x_1 \otimes \cdots \otimes x_{i-1} \otimes x_{i+1} \otimes x_i \otimes x_{i+2} \otimes \cdots \otimes x_n.$$

(For a general $\tau$, the sign is $(-1)^e$ where $e = \sum_{i<j, \tau^{-1}(i) > \tau^{-1}(j)} |x_i||x_j|$.) Also, this sign rule gives that the action of $\Sigma_n$ commutes with the differential on $T^n(P)$; that is, we may regard $T^n(P)$ as a complex of left modules over the group ring $R[\Sigma_n]$.

Let $C_n = \langle \sigma \rangle$ be the cyclic supgroup of $\Sigma_n$ of order $n$ generated by the cyclic permutation $\sigma = (1\, 2\, \cdots\, n)$. Then $C_n$ acts on $T^n(P)$ by

$$\sigma \cdot (x_1 \otimes \cdots \otimes x_n) = (-1)^{|x_1|(|x_2| + \cdots + |x_n|)} x_2 \otimes \cdots \otimes x_n \otimes x_1$$

Now assume $n = p$, for a prime $p$, and that $R$ contain $\frac{1}{p}$ and a primitive $p$-th root of unity $\zeta_p$. These assumptions give that the group ring $R[C_p] \cong R[x]/(x^p - 1)$ decomposes as a product of copies of $R$, indexed by the primitive $p$-th roots of unity. We thus get an internal direct sum decomposition

$$T^p(P) = \bigoplus_{j=0}^{p-1} T^p(P)^{(\zeta_p^j)}$$

of complexes of $R$-modules, where we set

$$T^p(P)^{(\zeta_p^j)} = \ker(T^p(P) \xrightarrow{\sigma - \zeta_p^j} T^p(P)),$$

a subcomplex of $T^p(P)$.

In this document, we are primarily interested in the case $p = 2$. Note that $\zeta_2 = -1$ and for $P \in \mathcal{P}^Z(R)$ the group $\Sigma_2 = C_2 = \langle \sigma \rangle$ acts on $T^2(P)$ by $\sigma \cdot (x \otimes y) = (-1)^{|x||y|} y \otimes x$. So,

$$T^2(P)^{(1)} = S^2(P) := \{\alpha \in P \otimes_R P \mid \sigma \cdot \alpha = \alpha\}.$$

and

$$T^2(P)^{(-1)} = \Lambda^2(P) := \{\alpha \in P \otimes_R P \mid \sigma \cdot \alpha = -\alpha\}.$$

Since we assume $\frac{1}{2} \in R$, we have an internal direct sum decomposition

$$T^2(P) = S^2(P) \oplus \Lambda^2(P)$$

of chain complexes. In particular, we have

$$\operatorname{supp}(S^2(P)) \subseteq \operatorname{supp}(P) \text{ and } \operatorname{supp}(\Lambda^2(P)) \subseteq \operatorname{supp}(P). \tag{3.2}$$

For example, if $P$ is a projective module viewed as a complex with trivial differential concentrated in even degree $d$, then we may idenitfy $S^2(P)$ and $\Lambda^2(P)$ with the classical second symmetric and exterior powers of $P$, viewed as a complex in degree $2d$. In the same situation but with $d$ odd, the roles are flipped. In general, the graded module underlying $S^2(P)$ for an arbitrary $P$ is the tensor product of the classical second symmetric power of the even part of $P$ with the classical second exterior power of the odd part, and vice versa for $\Lambda^2(P)$.

**Proposition 3.3** (Brown-Miller-Thompson-W)**.** *Assume $p$ is a prime and that $\frac{1}{p}, \zeta_p \in R$. For any $Z$, there is a well-defined endomorphism of abelian groups*

$$\psi^p_{cy} : K_0^Z(R) \to K_0^Z(R)$$

*given on generators by the formula*

$$\psi^p_{cy}([P]) = [T^p(P)^{(1)}] - [T^p(P)^{(\zeta_p)}].$$

*In particular, for $p = 2$, we have the operator*

$$\psi^2_{cy}([P]) = [S^2(P)] - [\Lambda^2(P)].$$

**Definition 3.4.** The function $\psi^p_{cy}$ described in the Proposition is called the *$p$-th cyclic Adams operation*

*Proof when $p = 2$.* For $P \in \mathcal{P}^Z(R)$ set $\psi^2_{cy}(P) = [S^2(P)] - [\Lambda^2(P)] \in K_0^Z(R)$. Note that $\psi^2_{cy}(P)$ does indeed belong to $K_0^Z(R)$ by (3.2). We must show this function respects the two defining relations of $K_0^Z(R)$.

Given a quasi-isomorphism $P \xrightarrow{\sim} P'$, we have a quasi-isomorhism $P \otimes_R P \xrightarrow{\sim} P' \otimes_R P'$. Since $\frac{1}{2} \in R$, this map decomposes as a direct sum of maps of the form $S^2(P) \to S^2(P')$ and $\Lambda^2(P) \to \Lambda^2(P')$, and each of these must also be quasi-isomorphisms. This proves $\psi_{cy}(P) = \psi_{cy}(P')$.

Suppose $0 \to P' \xrightarrow{i} P \to P'' \to 0$ is a short exact sequence in $\mathcal{P}^Z(R)$. We consider the filtration of complexes

$$0 = F_0 \subseteq F_1 \subseteq F_2 \subseteq F_3 = P \otimes_R P$$

where $F_1 = P' \otimes_R P'$ and $F_2 = P' \otimes_R P + P \otimes_R P'$. (Here, we interpret $i$ as an inclusion of a subconplex $P'$ into $P$.) This fitlration respects the action of $C_2$. We have $C_2$-equivariant isomorphisms of complexes

$$F_2/F_1 \cong P' \otimes_R P'' \oplus P'' \otimes_R P'$$

where the $C_2$ action on the right is determined by the action $\sigma(x' \otimes x'') = (-1)^{|x''||x'|} x'' \otimes x'$ on the first summand. Likewise, we have a $C_2$-equivariant isomorphism

$$F_3/F_2 \cong P'' \otimes_R P''.$$

It follows that we have isomorpihsm of complexes

$$\begin{aligned}
(F_1/F_0)^{(1)} &\cong S^2(P') \\
(F_1/F_0)^{(1)} &\cong \Lambda^2(P') \\
(F_2/F_1)^{(1)} &\cong P' \otimes_R P'' \\
(F_2/F_1)^{(1)} &\cong P' \otimes_R P'' \\
(F_3/F_2)^{(1)} &\cong S^2(P'') \\
(F_3/F_2)^{(1)} &\cong \Lambda^2(P'')
\end{aligned}$$

which gives

$$[S^2(P)] = [S^2(P')] + [P' \otimes_R P''] + [S^2(P'')]$$

and

$$[\Lambda^2(P)] = [\Lambda^2(P')] + [P' \otimes_R P''] + [\Lambda^2(P'')].$$

Taking the difference of the previous two equations yelds

$$\psi^2_{cy}(P) = \psi^2_{cy}(P') + \psi^2_{cy}(P'').$$

□

**Theorem 3.5** (Brown-Miller-Thompson-W). *$\psi^p_{cy}$ satsifies the four Gillet-Soulé axioms on the category $\mathcal{C}_p$.*

*Skethc of Proof when $p = 2$.* The first axiom, additivity, is given by Proposition 3.3.

I omit a proof of the second axiom, multiplicativity, although it is not too hard.

The third axiom, naturality, is a consequence of the naturality of the functors $S^2$ and $\Lambda^2$.

Let $a \in R$ and set $K = \mathrm{Kos}_R(a)$. Then $T^2(K) = \mathrm{Kos}_R(a, a) \cong \mathrm{Kos}_R(a, 0)$. Explicitly, if $K$ has $R$-basis $\alpha, \beta$ with $|\alpha| = 1$, $|\beta| = 0$ and $d(\alpha) = a\beta$, then $\mathrm{Kos}_R(a, 0)$ has basis $x = \alpha \otimes \alpha$, $y = \alpha \otimes \beta + \beta \otimes \alpha$, $z = \beta \otimes \alpha - \alpha \otimes \beta$, and $w = \beta \otimes \beta$, of degrees $2, 1, 1, 0$ respectively. Relative to this basis, we compute the action of the differntial and the transposition operators: $d(x) = az$, $d(y) = 2aw$, $d(z) = 0$, and $d(w) = 0$, and $\sigma \cdot x = -x$, $\sigma \cdot y = y$, $\sigma \cdot z = -z$, and $\sigma \cdot w = w$. It follows that

$$S^2(K) \cong \mathrm{Kos}_R(2a)$$

and

$$\Lambda^2(K) \cong \Sigma \operatorname{Kos}_R(-a).$$

Hence

$$\psi_{cy}(K) = [\operatorname{Kos}_R(2a)] + [\operatorname{Kos}_R(-a)] = [\operatorname{Kos}_R(a)] + [\operatorname{Kos}_R(a)] = 2[K].$$

This establishes the final axiom. □

It is not known (at least to me) if the cyclic Adams operations coincide with those defined by Gillet and Soulé in all situations in which the former are defined. But we do know this:

**Proposition 3.6** (BMTW)**.** *Let $R$ be a noetherian ring and $p$ a prime. If $p!$ is invertible in $R$, then $\psi^p_{cy}$ and $\psi^p_{GS}$ coincide as operators on $K_0^Z(R)$ for all $Z$. In particular, $\psi^2_{cy} = \psi^2_{GS}$ for local rings $(R, \mathfrak{m})$ with $\operatorname{char}(R/\mathfrak{m}) \neq 2$.*

3.3. **The weight decomposition.** As mentioned, the essential feature of Adams operations is that they decompose rationalized Grothendieck groups of regular rings into "weight pieces". The precise statement is:

**Theorem 3.7** (Gillet-Soulé)**.** *Assume $\psi^k$ is an Adams operation of degree $k$, for any $k \geq 2$, defined on some collection of commutative rings $\mathcal{C}$ that is closed under localization. If $R \in \mathcal{C}$ is regular and $Z = V(I) \subseteq R$ is any closed subset, then there exists an internal direct sum decomposition*

$$K_0^Z(R) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=\operatorname{height}(I)}^{\dim(R)} K_0^Z(R)^{(j)}$$

*where we define*

$$K_0^Z(R)^{(j)} := \ker(K_0^Z(R) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\psi^k - k^j} K_0^Z(R) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

In other words, the theorem says that the operator $\psi^k \otimes_{\mathbb{Z}} \mathbb{Q}$ is diagonalizable with eigenvalues conained in the set $\{k^j \mid \operatorname{height}(I) \leq j \leq \dim(R)\}$. In yet other words, $K_0^Z(R)_{\mathbb{Q}}$ is annihlated by $\prod_{\operatorname{height}(I) \leq j \leq \dim(R)}(\psi^k - k^j)$.

**Definition 3.8.** The subsapce $K_0^Z(R)^{(j)}_{\mathbb{Q}}$ the $j$-th *weight space* for the operator $\psi^k$. To indicate the (possible) dependence on the choice of operator, we sometimes write this $K_0^Z(R)^{(j)_{\psi^k}}_{\mathbb{Q}}$.

*Remark* 3.9. Before proving this Theorem, we make an observation that is simple but important in applications: the weight decomposition is multiplicative. That is, for all $R, Z, i, j$, given an Adams operations, the cup product pairing induces a pairing on the associate weight pieces of the form

$$K_0^Z(R)^{(i)}_{\mathbb{Q}} \times K_0^W(R)^{(j)}_{\mathbb{Q}} \to K_0^{Z \cap W}(R)^{(i+j)}_{\mathbb{Q}}.$$

This is immediate from Axiom 2 and the definitions.

*Sketch of Proof.* Let us regard $K_0^{V(I)}(R)_{\mathbb{Q}}$ as a $\mathbb{Q}[t]$-module with $t$ acting as $\psi^k$. We need to prove it is annihlated by the polynomial $\prod_{\operatorname{height}(I) \leq j \leq \dim(R)}(t - k^j)$.

If this is false for some regular noetherian ring $R$, then we may find an ideal $I$ that is maximal among those ideals for which it fails to hold. Let $\mathfrak{p} \supseteq I$ be a minimal prime containing $I$. By taking colimits over all $g \in R \setminus \mathfrak{p}$ in Theorem 2.19, we have the right exact sequence

$$\bigoplus_{g \in R \setminus \mathfrak{p}} K_0^{V(I,g)}(R)_{\mathbb{Q}} \to K_0^{V(I)}(R)_{\mathbb{Q}} \to K_0^{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})_{\mathbb{Q}} \to 0$$

of $\mathbb{Q}[t]$-modules (i.e., the the maps commute with the action of $\psi^k$ by Axiom 3.) By choice of $I$, the summand indexed by $g$ in the left-most term is annihilated by $\prod_{\text{height}(I,g) \leq j \leq \dim(R)}(t^k - k^j)$ and so, since $\text{height}(I,g) > \text{height}(I)$ for all such $g$, the left-most term is annihilated by $\prod_{\text{height}(I)+1 \leq j \leq \dim(R)}(t^k - k^j)$.

Since $R_{\mathfrak{p}}$ is regular local of dimension $\text{height}(I)$, by Lemma 2.8 the right-most term is generated by the class of a Koszul complex on $\text{height}(I)$ elements. Thus, by Example 2.22, this term isa nnihlated by $t^k - k^{\text{height}(I)}$.

Since $\prod_{\text{height}(I)+1 \leq j \leq \dim(R)}(t^k - k^j)$ and $t^k - k^{\text{height}(I)}$ are relatively prime elements of $\mathbb{Q}[t]$, it follows that the middle term $K_0^{V(I)}(R)_{\mathbb{Q}}$ is annihlated by their product, namely $\prod_{\text{height}(I) \leq j \leq \dim(R)}(t - k^j)$. We have reached a contradiction. □

*Remark* 3.10. This proof does not show that $K_0^Z(R)_{\mathbb{Q}}$ admits such a weight decomposition for an arbitrary noetherian ring $R$. The most important things that fails in the proof is that $K_0^{\mathfrak{p}R_{\mathfrak{p}}}(R_{\mathfrak{p}})$ is not necessarily or "pure wieght" — i..e., $\psi^k$ need not act as multiplicatin by $k^h$ on it.

**Exercise 3.11.** Assume $\psi^k$, $\Phi^j$ are degree $k, j$ Adams operation with $k, j \geq 2$.

Prove that if $\psi^k$ and $\Phi^j$ commute, then

$$K_0^Z(R)_{\mathbb{Q}}^{(j)_{\psi^k}} = K_0^Z(R)_{\mathbb{Q}}^{(j)_{\Phi^j}}$$

In particular if $j = k$ and $\psi^k$ and $\Phi^k$ commute, then the operators themselves coincide: $\psi^k_{\mathbb{Q}} = \Phi^k_{\mathbb{Q}}$.

*Remark* 3.12. It is not known (at least to me) if the assumption that $\psi^k, \Phi^j$ commute is needed for the previous exercise. That is, it is unknown if the weight decomposition is independent of the choice of a Adams operation. Likewise, it is unknown if, for a fixed $k \geq 2$, the G-S axioms uniquely specify the rational degree $k$ Adams operator for a regular ring.

**Exercise 3.13.** Prove that if $p$ is a prime and $\mathcal{C}$ is the category of commutative Noetherian rings of charactistic $p$, then for any Adams operation $\psi^k$ of degree $k$ defined on $\mathcal{C}$, we have

$$K_0^Z(R)_{\mathbb{Q}}^{(j)_{\psi^k}} = K_0^Z(R)_{\mathbb{Q}}^{(j)_F}$$

where $F$ is the degree $p$ Adams operation induced by Frobenius. Conclude in particular that if $k = p$, then $\psi^k = F$.

## 4. Day 4: The Serre Vanishing Conjecture, the Total Rank Conjecture, and Speculations

We give some applications of the machinery developed in the first three lectures.

4.1. **Serre Vanishing Conjecture.** In this section we present the proof of the Serre Vanishing Conjecture (for regular rings) due to Gillet and Soulé. In fact, we present a slight modification of their original proof, one that uses the cyclic operations instead of the ones they used.

Let us recall the intersection pairing:

**Definition 4.1.** Let $(R, \mathfrak{m})$ be a local ring, and let $M$ and $N$ be finitely generated $R$-modules such that $\mathrm{supp}(M) \cap \mathrm{supp}(N) \subseteq \{\mathfrak{m}\}$ (or, equivalently, $\mathrm{length}_R(M \otimes_R N) < \infty$). Assume also that either $\mathrm{pd}_R(M) < \infty$ or $\mathrm{pd}_R(N) < \infty$. Then the *intersection multiplicity* of $M$ and $N$ is the integer

$$\chi(M, N) = \sum_j (-1)^j \,\mathrm{length}\, \mathrm{Tor}_j^R(M, N).$$

**Conjecture 4.2** (Serre)**.** *Suppose $R$ is a regular local ring and $M$ and $N$ are finitely generated $R$-modules such that* $\mathrm{supp}(M) \cap \mathrm{supp}(N) \subseteq \{\mathfrak{m}\}$*. If* $\dim(M) + \dim(N) <$ $\dim(R)$*, then* $\chi(M, N) = 0$*.*

Heuristically, the conjecture predicts that if the supports of $M$ and $N$ "ought not" meet, then their intersection multiplicity is 0.

**Example 4.3.** Two curves in three space meeting at the origin should have intersection multiplicity 0. That is, for the ring $R = k[[x, y, z]]$, given prime ideals $\mathfrak{p}$, $\mathfrak{q}$ such that $\dim(R/\mathfrak{p}) = 1$, $\dim(R/\mathfrak{q}) = 1$ and $\mathfrak{p} + \mathfrak{q}$ is $\mathfrak{m}$-primary, then we expect $\chi(R/\mathfrak{p}, R/\mathfrak{q}) = 0$. Let's check this holds for a pair of "axes": Let $\mathfrak{p} = (x, y), \mathfrak{q} = (y, z)$. We may resolve each by Koszul complexes and so their derived tensor product is $\mathrm{Kos}_R(x, y, y, z) \cong \mathrm{Kos}_R(x, y, z, 0)$, which is quasi-isomorphic to the complex $\cdots \to 0 \to k \xrightarrow{0} k \to 0 \to \cdots$, and hence $\chi = 0$.

*Remark* 4.4. The SVC was proven by Serre himself if $R$ is regular and contains a field.

The conjecture admits an evident generalization to the case when $R$ is local, but not necessarily regular, provided either $\mathrm{pd}_R(M) < \infty$ *or* $\mathrm{pd}_R(N) < \infty$. It is known to be false at that level of generality; see the famous example constructed by [Dutta-Hochster-McLaughlin].

However, it remains on open conjecture for an arbitrary local ring $R$ if it is assumed that *both* $M$ and $N$ have finite projective dimension. This has been proven to hold when $R$ is a complete intersection ring by Roberts and, independantly, by Gillet and Soulé. Roberts has also proven it for isolated singularities.

*The Gillet-Soulé Proof of SVC for all regular local rings.* Let $R$ be a regular local ring, $M, N$ finitely generated $R$-modules such that $\mathrm{length}(M \otimes_R N) < \infty$ and $\dim(M) + \dim(N) < d = \dim(R)$. We show $\chi(M, N) = 0$ by following Gillet and Soule's proof closely, except that we will use the cyclic Adams operations. This requires the addition of one preiminary step:

Pick a prime integer $p$ such that $p \neq \mathrm{char}(R/\mathfrak{m})$. Then $\frac{1}{p} \in R$. There exists a finite étale extension $R \subseteq R'$ of regular local rings such that $R'$ contains a primitive $p$-th root of unity $\zeta_p$. For such an extension we have

$$\chi_{R'}(M \otimes_R R', N \otimes_R R') = \dim_{R/\mathfrak{m}}(R'/\mathfrak{m}') \cdot \chi_R(M, N).$$

We may thus assume without loss of generality that $R$ contains $\zeta_p$, so that the cyclic Adams operator $\psi^p_{cy}$ is defined.

Choose projective resolutions $P_M$ and $P_N$ of $M$ and $N$. Then

$$[P_M] \in K_0^{\mathrm{supp}(M)}(R) \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=d-\dim(M)}^{d} K_0^{\mathrm{supp}(M)}(R)^{(j)}$$

by Theorem 4.6, and similarly for $[P_N]$. Since the cup product respects the weight decomposition (see Remark 3.9), we have

$$[P_M] \cup [P_N] = [P_M \otimes_R P_N] \in \bigoplus_{i \geq d-\dim(M), j \geq d-\dim(N)} K_0^{\mathfrak{m}}(R)^{(i+j)}.$$

The assumption $\dim(M) + \dim(N) < d$ implies $i + j > d$ for all such pairs, but we also know that $K_0^{\mathfrak{m}}(R) \otimes_{\mathbb{Z}} \mathbb{Q} = K_0^{\mathfrak{m}}(R)^{(d)}$. It follows that

$$[P_M] \cup [P_N] = 0 \in K_0^{\mathfrak{m}}(R)_{\mathbb{Q}}$$

and hence

$$\chi(M, N) = \chi([P_M] \cup [P_N]) = 0.$$

□

4.2. **The Total Rank Conjecture.** Let $(R, \mathfrak{m})$ be a local ring and assume $M$ is an $R$-module of finite length such that $M \neq 0$ and $\mathrm{pd}_R(M) < \infty$. Let $\beta_i^R(M)$ denote the $i$-th Betti number of $M$, defined as $\beta_i^R(M) = \dim_k \mathrm{Tor}_i^R(M, k)$ or, equivalently, as the rank of the $i$-th free module in the minimal free resolution of $M$ over $R$. A famous conjecture of Buchsbaum-Eisenbud and Horrochs predicts that $\beta_i^R(M) \geq \binom{d}{i}$ where $d = \dim(R)$. It remains open even for regular local rings, and even for polynomials rings in the graded setting.

Since $\sum_i \binom{d}{i} = 2^d$, an evident consequence of the BEH conjecture is:

**Conjecture 4.5** (Avramov's Total Rank Conjecture)**.** *For a local ring $R$ and a non-zero $R$-module $M$ of finite length and finite projective dimension, we have*

$$\sum_i \beta_i^R(M) \geq 2^{\dim(R)}.$$

**Theorem 4.6** (Walker)**.** *Let $(R, \mathfrak{m})$ be a regular local ring such that* $\mathrm{char}(k) \neq 2$. *If $P$ belongs to $\mathcal{P}^{\mathfrak{m}}(R)$ and $P$ is not exact, then*

$$\mathrm{rank}_R(P) \geq 2^{\dim(R)} \frac{|\chi(P)|}{h(P)},$$

*where* $\mathrm{rank}_R(P) = \sum_i \mathrm{rank}_R(P_i)$, $\chi(P) = \sum_i (-1)^i \mathrm{length}_R H_i(P)$, *and* $h(P) = \sum_i \mathrm{length}_R H_i(P)$.

*In particular, the Total Rank Conjecture holds for $R$.*

*Proof.* The last assertion follows from the first by taking $P$ to be the minimal free resolution of $M$, since $\mathrm{rank}(P) = \sum_i \beta_i^R(M)$ and $\chi(P) = h(P)$ in this case.

Since we assume $\mathrm{char}(k) \neq 2$, $R$ contains $\frac{1}{2}$ (and, obviously, $\zeta_2 = -1$), and thus the cyclic Adams operation $\psi_{cy}^2$ is defined for $R$. Since $R$ is regular, $K_0^{\mathfrak{m}}(R)$ is generated by the class of the Koszul complex. It follows that $\psi_{cy}^2$ acts as multiplication by $2^d$ on this group, and it follows that

$$\chi(\psi_{cy}^2(P)) = 2^{\dim(R)} \cdot \chi(P).$$

Recall that $\psi_{cy}(P) = [S^2(P)] - [\Lambda^2(P)]$ so that

$$\chi(\psi^2_{cy}(P)) = \chi(S^2(P)) - \chi(\Lambda^2(P)) \leq \sum_{i \text{ even}} \operatorname{length}_R H_i(S^2 P) + \sum_{i \text{ odd}} \operatorname{length}_R H_i(\Lambda^2 P).$$

Since $P \otimes_R P = S^2(P) \oplus \Lambda^2(P)$ it follows that

$$\chi(\psi^2_{cy}(P)) \leq h(P \otimes_R P)$$

I claim that

$$h(P \otimes_R P) \leq \operatorname{rank}(P) h(P). \tag{4.7}$$

To see this, consider the convergent spectral sequence

$$E^2_{i,j} = H_i(P \otimes_R H_j(P)) \Longrightarrow H_{i+j}(P \otimes_R P)$$

that arises from regarding $P \otimes_R P$ as a the toalization of bicomplex. Since $\operatorname{length}_R E^\infty_{i,j} \leq \operatorname{length}_R H_i(P \otimes_R H_j(P))$ for all $i, j$ and $\operatorname{length}_R H_n(P \otimes_R P) = \sum_{i+j=n} \operatorname{length}_R E^\infty_{i,j}$, we deduce

$$\operatorname{length}_R H_n(P \otimes_R P) \leq \sum_{i,j; i+j=n} \operatorname{length}_R H_i(P \otimes_R H_j(P))$$

for each $n$.

Now, $H_i(P \otimes_R H_j(P))$ is a subquotient of the finite lenght module $P_i \otimes_R H_j(P)$, and hence

$$\operatorname{length}_R H_i(P \otimes_R H_j(P)) \leq \operatorname{rank}_R(P_i) \operatorname{length}(H_j). \tag{4.8}$$

We conclude

$$h(P \otimes_R P) \leq \sum_n \sum_{i+j=n} \operatorname{rank}_R(P_i) \operatorname{length}(H_j) = \sum_{i,j} \operatorname{rank}_R(P_i) \operatorname{length}_R(H_j) = \operatorname{rank}(P) h(P).$$

To conlude the proof, we just put the inequalites above together to get

$$\operatorname{rank}(P) h(P) \geq 2^{\dim(R)} \chi(P),$$

and since $h(P) > 0$ we conclude

$$\operatorname{rank}_R(P) \geq 2^{\dim(R)} \frac{\chi(P)}{h(P)}.$$

Appling this inequality to $\Sigma P$ in place $P$, and using that $\operatorname{rank}_R(\Sigma P) = \operatorname{rank}_R(P)$, $h(\Sigma P) = h(P)$, and $\chi(\Sigma P) = -\chi(P)$, we also get

$$\operatorname{rank}_R(P) \geq -2^{\dim(R)} \frac{\chi(P)}{h(P)}. \quad \square$$

*Remark* 4.9. The inequality (4.7) can also be proven without using spectral sequences. More generally, if $M$ is any bounded complex of $R$-modules having finite length homology, I claim $h(P \otimes_R M) \leq \operatorname{rank}(P) h(M)$. Both sides are unaffected by replacing $M$ with any complex that is quasi-isomorphic to it. In particular, it holds when $M$ is acylic. If $M$ is not acyclic, proceed by induction on $w = w(M) := \max\{n \mid H_n(M) \neq 0\} - \min\{n \mid H_n(M) \neq 0\}$. When $w = 0$, $M$ is quasi-isomorphic to a module viewed as a complex concentrated in one degree. In this case the inequality holds for the same reason (4.8) holds. For $w > 0$, by using a "soft truncation", we may find a short exact sequence

$$0 \to M' \to M \to M'' \to 0 \tag{4.10}$$

of non-acyclic, bounded complexes with finite length homology such that the map induced map on homology $H(M) \to H(M'')$ is surjective, $w(M') < w(M)$, and $w(M'') < w(M)$. These conditions imply that $h(M) = h(M') + h(M'')$. Moreover, the long exact seqeuence in homology associated to the short exact sequence obtained by applying $P \otimes_R -$ to (4.10) yields the inequality

$$h(P \otimes_R M) \leq h(P \otimes_R M') + h(P \otimes_R M'').$$

By induction and the equation $h(M) = h(M') + h(M'')$ we get

$$h(P \otimes_R M) \leq \beta(P)h(M') + \beta(P)h(M'') = \beta(P)h(M).$$

*Remark* 4.11. I also have a proof of the Total Rank Conjecture in the following cases:

(1) $R$ is a "Roberts ring" (for example, a complete intersection ring) and $\mathrm{char}(R/\mathfrak{m}) \neq 2$ or
(2) $R$ is any local ring such that $\mathrm{char}(R) = p$ for an odd prime $p$.

4.3. **Marix Factorizations.** I provide no details, but would mention that [Brown-Miller-Thompson-W] also use cyclic Adams operations to prove a conjecture due to Hailong Dao concerning the vanishing of the theta invariant for isolated hypersurface singularities. Our proof requires developing a good notion of Adams operations for matrix factorizations. Since this involves working with $\mathbb{Z}/2$-graded complexes, the Adams operations of Gillet-Soulé do not exist in this context.

4.4. **Speculations.** I close with some slightly speculative statements.

Fix a local ring $(R, \mathfrak{m})$ and assume there exists a surjection $\pi : Q \twoheadrightarrow R$ with $Q$ regular local, and let $I = \ker(\pi)$. Such a surjection exists if, for example, $R$ is complete. Then for each finitely generated $R$-module $M$, we may choose a projective resolution $P^M \xrightarrow{\sim} M$ of $M$ as a $Q$-module, obtaining a class

$$[P^M] \in K_0^{V(I)}(Q).$$

(In fact, the assignment $M \mapsto [P_M]$ induces isomorphism

$$\rho : G_0(R) \cong K_0^{V(I)}(Q),$$

where $G_0(R)$ is the Grothendieck group of finitely generated $R$-modules, but we will not need this fact here.)

Let $\psi^k$ be an Adams operation of degree $k$, $k \geq 2$, defined on $Q$ and its localizaitons.

**Definition 4.12** (Kurano)**.** With the notation above, an $R$-module $M$ is called a *test module* (relative to $\pi$ and $\psi^k$) if

(1) $M$ is a (f.g.) MCM $R$-module and
(2) the class $[P^M] \in K_0^{V(I)}(Q)$ has weight $c := \dim(Q) - \dim(R)$ for $\psi^k$; that is, $[P^M] \in K_0^{V(I)}(Q)^{(c)}_{\mathbb{Q}}$ or equivalently

$$\psi^k([P^M]) = k^c[P^M] \text{ modulo torsion.}$$

**Example 4.13.** Suppose $R = Q/(f_1, \dots, f_c)$ for some regular local ring $Q$ and regular seqeunce of elements $f_1, \dots, f_c$ in the maximal ideal of $Q$. Then $R$ is a test module over itself (relative to the canonical surjection $Q \twoheadrightarrow R$) for any Adams operation. Indeed, it is MCM since $R$ is Cohen-Macaulay, and the class of $R$ in

$K_0^{V(I)}(Q)$ is $[\mathrm{Kos}_Q(f_1, \dots, f_c)]$, which, as we've seen, has weight $c$. (Note that $c = \dim(Q) - \dim(R)$.)

The following is equivalent to a (very strong) conjecture of Kurano:

**Conjecture 4.14.** *Let $R$ be a complete, local domain. Then $R$ has a test module.*

I will call the following statement a "proto-theorem", since while I have a sketch of a proof, some details have not been checked carefully.

**ProtoTheorem 4.15.** *Suppose $(R, \mathfrak{m})$ is a local ring of dimension $d$ and $\mathrm{char}(R/\mathfrak{m}) \neq 2$. Assume $F \in \mathcal{P}^{\mathfrak{m}}(R)$ has the form*

$$F = (\cdots \to 0 \to F_d \to \cdots \to F_0 \to 0 \to \cdots)$$

*and that $H_0(F) \neq 0$.*

*If $R$ admits a test module relative to $\psi^2_{cy}$ (for some surjection $Q \twoheadrightarrow R$), then $\mathrm{rank}_R(F) \geq 2^d$. In particular, the Total Rank Conjecture holds for such a ring $R$.*

*Remark* 4.16. By the New Intersection Theorem [Roberts], the complex $F$ occuring in the statement is the "narrowest" possible complex of projective modules having non-zero, finite length homology. Also, the theorem would become false if $F$ were allowed to be any wider; see [Iyengar-Walker].

*Sketch of Proof.* The second assertion follows from first since by the Auslanger-Buchsbaum formula, the minimal free resolution of a non-zero $R$-module of finite length and finite projective dimension has the form of $F$.

Suppose $\pi : Q \twoheadrightarrow R$ is a surjection with kernel $I$ such that $Q$ is regular and that $M$ is a test module for $R$ relative to $\pi$ and $\psi^2_{cy}$. As usual, let $P^M \xrightarrow{\sim} M$ be the minimal $Q$-free resolution of $M$.

Modulo some careful checking of the details, I assert there is a pairing

$$- \cap - : K_0^{\mathfrak{m}}(R) \times K_0^{V(I)}(Q) \to \mathbb{Z},$$

called the "cap product" pairing, that has the following properties:

(1) For any $R$-module $N$, if $P^N \xrightarrow{\sim} N$ is a $Q$-free resolution of $N$, then

$$[E] \cap [P^N] = \chi(E \otimes_R N) = \sum_i (-1)^i \operatorname{length}_R H_i(E \otimes_R N),$$

for any $E \in \mathcal{P}^{\mathfrak{m}}(R)$.

(2) $\psi^k[E] \cap \psi^k[P] = k^{\dim(Q)}([E] \cap [P])$, for any Adams operation $\psi^k$.

We apply this using the Adams operation $\psi^2_{cy}$:

By assumption, $\psi^2_{cy}([P^M]) = 2^c[P^M]$ and thus by the second property of the cap product pairing we have

$$2^{\dim(Q)}[F] \cap [P^M] = \psi^2_{cy}[F] \cap \psi^k_{cy}([P^M]) = 2^c\, \psi^2_{cy}([F]) \cap [P^M]$$

which gives

$$\psi^2_{cy}([F]) \cap [P^M] = [F] \cap [P^M] = 2^{\dim(Q)-c}[F] \cap [P^M] = 2^{\dim(R)}\chi(F \otimes_R M).$$

(The last equality uses the first property of the cap product pairing.)

The rest of the proof is very similar to the proof of Theorem 4.6 given above. In detail, we have

$$\psi^2_{cy}([F]) \cap [P^M] = [S^2(F)] \cap [P^M] - [\Lambda^2(F)] \cap [P^M] = \chi(S^2(F) \otimes_R M) - \chi(\Lambda^2(F) \otimes_R M)$$

and the argument in that proof applies here verbatim to show

$$\psi^2_{cy}([F]) \cap [P^M] \leq \operatorname{rank}_R(F) h(F \otimes_R M).$$

Combining these facts gives

$$2^{\dim(R)} \chi(F \otimes_R M) \leq \operatorname{rank}_R(F) h(F \otimes_R M). \tag{4.17}$$

So far we have used neither that $M$ is an MCM $R$-module nor that $F$ is "narrow". These facts together imply that the homology of $F \otimes_R M$ is concentrated in degree 0:

$$H_i(F \otimes_R M) \cong \begin{cases} 0, & \text{if } i \neq 0 \text{ and} \\ H_0(F) \otimes_R M, & \text{if } i = 0. \end{cases}$$

In particular, $\chi(F \otimes_R M) = h(F \otimes_R M)$, and the Theorem follows from (4.17) by dividing by the positive integer $h(F \otimes_R M)$. □

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