Gröbner deformations

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Squarefree monomial ideals and fiber-full modules

Our next goal is to show that, for an ideal $I \subset R$ such that $\mathrm{in}_w(I)$ is a squarefree monomial ideal, then $S = P/\hom_w(I)$ is a fiber-full P-module. To do so, we need to recall some notion:

Let $J \subset R$ a monomial ideal minimally generated by monomials $\mu_1, \ldots, \mu_r$. For all subset $\sigma \subset \{1, \ldots, r\}$ we define the monomial $\mu(J, \sigma) := \mathrm{lcm}(\mu_i | i \in \sigma) \in R$. If v is the qth element of σ we set $\mathrm{sign}(v, \sigma) := (-1)^{q-1} \in K$. Let us consider the graded complex of free R-modules $F_\bullet(J) = (F_i, \partial_i)_{i=0,\ldots,r}$ with

$$F_i := \bigoplus_{\substack{\sigma \subset \{1,\ldots,r\} \\ |\sigma| = i}} R(-\deg \mu(J, \sigma)),$$

and differentials defined by $1_\sigma \mapsto \sum_{v \in \sigma} \mathrm{sign}(v, \sigma) \frac{\mu(J,\sigma)}{\mu(J,\sigma \setminus \{v\})} \cdot 1_{\sigma \setminus \{v\}}$. It is well known and not difficult to see that $F_\bullet(J)$ is a graded free R-resolution of R/J, called the *Taylor resolution*.

For any positive integer k we introduce the monomial ideal

$$J^{[k]} = (\mu_1^k, \ldots, \mu_r^k).$$

Notice that $\mu_1^k, \ldots, \mu_r^k$ are the minimal system of monomial generators of $J^{[k]}$, so $\mu(J^{[k]}, \sigma) = \mu(J, \sigma)^k$ for any $\sigma \subset \{1, \ldots, r\}$.

Theorem

If $J \subset R$ is a squarefree monomial ideal, for all $i \in \mathbb{Z}$ and $k \in \mathbb{N}_{>0}$ the map $\mathrm{Ext}^i_R(R/J^{[k]}, R) \to \mathrm{Ext}^i_R(R/J^{[k+1]}, R)$, induced by the projection $R/J^{[k+1]} \to R/J^{[k]}$, is injective.

Corollary

Let $I \subset R$ be an ideal such that $\mathrm{in}_w(I)$ is a squarefree monomial ideal. Then $S = P/\hom_w(I)$ is a fiber-full P-module.

Proof of the corollary: Notice that $\hom_w(I) + tP = \mathrm{in}_w(I) + tP$ is a squarefree monomial ideal of P. So, by the previous theorem, the maps $\mathrm{Ext}^i_P(S/tS, P) \to \mathrm{Ext}^i_P(P/(\hom_w(I) + tP)^{[m]}, P)$ are injective for all $m \in \mathbb{N}_{>0}$. Since $(\hom_w(I) + tP)^{[m]} \subset \hom_w(I) + t^m P$, these maps factor through $\mathrm{Ext}^i_P(S/tS, P) \to \mathrm{Ext}^i_P(S/t^m S, P)$, hence the latter are injective as well. □

Proof of the theorem: Let $u_1, \dots, u_r$ be the minimal monomial generators of J. For all $k \in \mathbb{N}_{>0}, \sigma \subset \{1, \dots, r\}$ set $\mu_\sigma[k] := \mu(J^{[k]}, \sigma)$ and $\mu_\sigma := \mu_\sigma[1]$. Of course μ_σ is a squarefree monomial and, for what we said above, $\mu_\sigma[k] = \mu_\sigma^k$.

The module $\mathrm{Ext}^i_R(R/J^{[k]},R)$ is the ith cohomology of the complex $G^\bullet[k]=\mathrm{Hom}_R(F_\bullet[k],R)$ where $F_\bullet[k]=F_\bullet(J^{[k]})=(F_i,\partial_i[k])_{i=0,\dots,r}$ is the Taylor resolution of $R/J^{[k]}$. Let $F_i\xrightarrow{f_i}F_i$ be the map sending 1_σ to $\mu_\sigma\cdot 1_\sigma$. The collection $F_\bullet[k+1]\xrightarrow{f_\bullet=(f_i)_i}F_\bullet[k]$ is a morphism of complexes lifting $R/J^{[k+1]}\to R/J^{[k]}$ (since $\mu_\sigma[k]=\mu_\sigma^k$).

So the maps $\mathrm{Ext}^i_R(R/J^{[k]},R)\to\mathrm{Ext}^i_R(R/J^{[k+1]},R)$ we are interested in are the homomorphisms $H^i(G^\bullet[k])\xrightarrow{\overline{g^i}}H^i(G^\bullet[k+1])$ induced by $g^\bullet=\mathrm{Hom}(f_\bullet,R):G^\bullet[k]\to G^\bullet[k+1]$. Let us see how $\overline{g^i}$ acts: if $G^\bullet[k]=(G^i,\partial^i[k])$, then $G^i=\mathrm{Hom}_R(F_i,R)$ can be identified with F_i (ignoring the grading) and $\partial^i[k]:G^i\longrightarrow G^{i+1}$ sends 1_σ to $\sum_{v\in\{1,\dots,r\}\setminus\sigma}\mathrm{sign}(v,\sigma\cup\{v\})\left(\frac{\mu_{\sigma\cup\{v\}}}{\mu_\sigma}\right)^k\cdot 1_{\sigma\cup\{v\}}$ for all $\sigma\subset\{1,\dots,r\}$ and $|\sigma|=i$. The map $g^i:G^i\to G^i$, up to the identification $F_i\cong G_i$, is then the map sending 1_σ to $\mu_\sigma\cdot 1_\sigma$.

Squarefree monomial ideals and fiber-full modules

Want: $\overline{g^i}$ injective. Let $x \in \mathrm{Ker}(\partial^i[k])$ with $g^i(x) \in \mathrm{Im}(\partial^{i-1}[k+1])$. We need to show that $x \in \mathrm{Im}(\partial^{i-1}[k])$. Let $y = \sum_\sigma y_\sigma \cdot 1_\sigma \in G^{i-1}$ such that $\partial^{i-1}[k+1](y) = g^i(x)$. We can write y_σ uniquely as $y'_\sigma + \mu_\sigma y''_\sigma$ where no monomial in $\mathrm{supp}(y'_\sigma)$ is divided by μ_σ. If

$$y' = \sum_\sigma y'_\sigma \cdot 1_\sigma, \; y'' = \sum_\sigma y''_\sigma \cdot 1_\sigma,$$

$g^i(x) = \partial^{i-1}[k+1](y) = \partial^{i-1}[k+1](y') + \partial^{i-1}[k+1](g^{i-1}(y'')) = \partial^{i-1}[k+1](y') + g^i(\partial^{i-1}[k](y''))$. Writing $z = \sum_\sigma z_\sigma \cdot 1_\sigma$ for $\partial^{i-1}[k+1](y') \in G^i$, we have

$$z_\sigma = \sum_{v\in\sigma} \mathrm{sign}(v,\sigma) \left(\frac{\mu_\sigma}{\mu_{\sigma\setminus\{v\}}}\right)^{k+1} y'_{\sigma\setminus\{v\}}.$$

Since J is squarefree and $\mu_{\sigma\setminus\{v\}}$ does not divide $y'_{\sigma\setminus\{v\}}$ for any $v \in \sigma$, μ_σ cannot divide z_σ unless it is zero. On the other hand, μ_σ must divide z_σ by the green equality. Therefore $z_\sigma = 0$, and since σ was arbitrary $z = 0$, that is: $g^i(x) = g^i(\partial^{i-1}[k](y''))$. Being $g^i : G^i \to G^i$ obviously injective, we have found $x = \partial^{i-1}[k](y'')$. □

Squarefree monomial ideals and fiber-full modules

Corollary

Let $I \subset R$ be an ideal such that $\mathrm{in}_w(I) \subset R$ is a squarefree monomial ideal. Then $\mathrm{Ext}^i_P(S, P)$ is a flat $K[t]$-module. So, if I is homogeneous and $\mathrm{in}(I)$ is a squarefree monomial ideal, then

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\,\mathrm{in}(I))_j) \ \ \forall\ i, j \in \mathbb{Z}.$$

This is the arrival point for these lectures, but it suggests also some open questions...

Open questions

During the lectures we proved the following:

Theorem

Let $I \subset R$ be an ideal such that $S = P/\operatorname{hom}_w(I)$ is a fiber-full P-module. Then $\operatorname{Ext}^i_P(S, P)$ is a flat $K[t]$-module. So, if furthermore I is homogeneous:

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\operatorname{in}_w(I))_j) \ \ \forall\ i, j \in \mathbb{Z}.$$

After this we proved that, if $\operatorname{in}_w(I) \subset R$ is a squarefree monomial ideal, then $S = P/\operatorname{hom}_w(I)$ is a fiber-full P-module. However, this is not the only instance: e.g., if $R/\operatorname{in}_w(I)$ is Cohen-Macaulay (equivalently if S is CM), then it is not difficult to see that S is fiber-full.

Open questions

More interestingly, we have:

- If K has positive characteristic, then S is fiber-full whenever $R/\operatorname{in}_w(I)$ is F-pure (Ma).
- If K has characteristic 0, then S is fiber-full whenever $R/\operatorname{in}_w(I)$ is Du Bois (Ma-Schwede-Shimomoto).
- S is fiber-full whenever $R/\operatorname{in}_w(I)$ is cohomologically full (a notion recently introduced by Dao-De Stefani-Ma).

Let us recall that, for a homogeneous ideal $J \subset R$, R/J is *cohomologically full* if, whenever $H \subset I$ such that $\sqrt{H} = \sqrt{J}$, the natural map $H^i_{\mathfrak{m}}(R/H) \to H^i_{\mathfrak{m}}(R/J)$ is surjective for all i.

Open questions

Often $\text{in}_w(I)$ is not a monomial ideal, rather a binomial ideal s.t.

$$R/\text{in}_w(I) \cong K[\mathcal{M}] := K[Y^u : u \in \mathcal{M}] \subset K[Y_1, \ldots, Y_m].$$

for some monoid $\mathcal{M} \subset \mathbb{N}^m$. This is the case when dealing with SAGBI (or Khovanskii) bases.

Problem

Find a big class of monoids $\mathcal{M} \subset \mathbb{N}^m$ such that $K[\mathcal{M}]$ is cohomologically full.

For example, if K has characteristic 0 and $\mathcal{M}$ is seminormal, then $K[\mathcal{M}]$ is Du Bois combining results of Bruns-Li-Römer and Schwede. So $K[\mathcal{M}]$ is cohomologically full for a seminormal monoid $\mathcal{M}$.

Open questions

We proved that, if A is a Noetherian graded flat $K[t]$-algebra, M is a f.g. A-module which is graded and flat over $K[t]$, then $\mathrm{Ext}^i_A(M, A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$ whenever M is fiber-full.

Problem

With the above notation, when is it true that $\mathrm{Ext}^i_A(M, A)$ is fiber-full whenever M is fiber-full?

For example, together with D'Alì we proved that, if M/tM is a squarefree R-module then M is fiber-full, and this implies a positive answer to the above problem when M/tM is a squarefree R-module. A consequence of this, is that the homological degrees (a notion introduced by Vasconcelos) of R/I and $R\,\mathrm{in}(I)$ are the same provided that $\mathrm{in}(I)$ is a squarefree monomial ideal.