Gröbner deformations

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Fiber-full modules and flatness

In the following slides, A is a Noetherian flat $K[t]$-algebra and M a finitely generated A-module which is flat over $K[t]$, and both A and M are graded $K[t]$-modules (think at A and M like they were, with the previous notation, P and S).

Lemma

The following are equivalent:

1. $\mathrm{Ext}^i_A(M, A)$ is a flat over $K[t]$ for all $i \in \mathbb{N}$.
2. $\mathrm{Ext}^i_{A/t^mA}(M/t^mM, A/t^mA)$ is a flat over $K[t]/(t^m)$ $\forall\, i, m \in \mathbb{N}$.

Proof: (1) $\implies$ (2): Since A is flat over $K[t]$, there is a short exact sequence $0 \to A \xrightarrow{\cdot t^m} A \to A/t^mA \to 0$. Consider the induced long exact sequence of $\mathrm{Ext}_A(M, -)$:

$$\cdots \to \mathrm{Ext}^i_A(M, A) \xrightarrow{\cdot t^m} \mathrm{Ext}^i_A(M, A) \to \mathrm{Ext}^i_A(M, A/t^mA)$$
$$\to \mathrm{Ext}^{i+1}_A(M, A) \xrightarrow{\cdot t^m} \mathrm{Ext}^{i+1}_A(M, A) \to \dots$$

Fiber-full modules and flatness

By (1), $\mathrm{Ext}^k_A(M, A)$ does not have t-torsion for all $k \in \mathbb{N}$, so for all $i \in \mathbb{N}$ we have a short exact sequence

$$0 \to \mathrm{Ext}^i_A(M, A) \xrightarrow{\cdot t^m} \mathrm{Ext}^i_A(M, A) \to \mathrm{Ext}^i_A(M, A/t^m A) \to 0,$$

from which $\mathrm{Ext}^i_A(M, A/t^m A) \cong \dfrac{\mathrm{Ext}^i_A(M, A)}{t^m \,\mathrm{Ext}^i_A(M, A)}$. It is straightforward to check that the latter is flat over $K[t]/(t^m)$ because (1). Finally, a previous lemma implies that

$$\mathrm{Ext}^i_A(M, A/t^m A) \cong \mathrm{Ext}^i_{A/t^m A}(M/t^m M, A/t^m A).$$

(2) $\implies$ (1): By contradiction, suppose $\mathrm{Ext}^i_A(M, A)$ is not flat over $K[t]$. Because $K[t]$ is a PID, then $\mathrm{Ext}^i_A(M, A)$ has nontrivial torsion. So, by the graded structure of $\mathrm{Ext}^i_A(M, A)$, there exists a nontrivial class $[\phi] \in \mathrm{Ext}^i_A(M, A)$ and $k \in \mathbb{N}$ such that $t^k[\phi] = 0$.

Fiber-full modules and flatness

Let us take a A-free resolution $F_\bullet$ of M, and let $(G^\bullet, \partial^\bullet)$ be the complex $\mathrm{Hom}_A(F_\bullet, A)$, so that $\mathrm{Ext}^i_A(M, A)$ is the ith cohomology module of $G^\bullet$. Then $\phi \in \mathrm{Ker}(\partial^i) \setminus \mathrm{Im}(\partial^{i-1})$ and $t^k\phi \in \mathrm{Im}(\partial^{i-1})$.

Since M and A are flat over $k[t]$, $F_\bullet/t^mF_\bullet$ is a A/t^mA-free resolution of M/t^mM.

$$\mathrm{Ext}^i_{P/t^mP}(S/t^mS, P/t^mP) \cong H^i(\mathrm{Hom}_{P/t^mP}(F_\bullet/t^mF_\bullet, P/t^mP)).$$

Let $(\overline{G^\bullet}, \overline{\partial^\bullet})$ denote the complex $\mathrm{Hom}_{A/t^mA}(F_\bullet/t^mF_\bullet, A/t^mA)$, so that $\mathrm{Ext}^i_{A/t^mA}(M/t^mM, A/t^mA)$ is the ith cohomology module of $\overline{G^\bullet}$, and $\pi^\bullet$ the natural map of complexes from $G^\bullet$ to $\overline{G^\bullet}$. Of course $\pi^i(\phi) \in \mathrm{Ker}(\overline{\partial^i})$ and $t^k\pi^i(\phi) \in \mathrm{Im}(\overline{\partial^{i-1}})$. Now, it is enough to find a positive integer m such that $\pi^i(\phi)$ does neither belong to $\mathrm{Im}(\overline{\partial^{i-1}})$ nor to $t^{m-k}\,\mathrm{Ker}(\overline{\partial^i})$.

Fiber-full modules and flatness

If $\pi^i(\phi) \in \operatorname{Im}(\overline{\partial^{i-1}})$, then

$$\phi \in \operatorname{Im}(\partial^{i-1}) + t^m G^i = \operatorname{Im}(\partial^{i-1}) + t^m \operatorname{Ker}(\partial^i).$$

Since ϕ does not belong to $\operatorname{Im}(\partial^{i-1})$, Krull's intersection theorem tells us that $\pi^i(\phi)$ cannot belong to $\operatorname{Im}(\overline{\partial^{i-1}})$ for all $m \gg 0$. Analogously, if $\pi^i(\phi) \in t^{m-k} \operatorname{Ker}(\overline{\partial^i})$, then

$$\phi \in t^{m-k} \operatorname{Ker}(\partial^i) + t^m G^i = t^{m-k} \operatorname{Ker}(\partial^i).$$

But $\phi \neq 0$, so, again using Krull's intersection theorem, $\pi^i(\phi) \notin t^{m-k}\overline{G^i}$ for all $m \gg 0$. $\square$

Fiber-full modules and flatness

In the above situation, we say that M is a *fiber-full* A-module if, for any $m \in \mathbb{N}_{>0}$, the natural projection $M/t^mM \to M/tM$ induces injective maps $\mathrm{Ext}^i_A(M/tM, A) \to \mathrm{Ext}^i_A(M/t^mM, A)$ for all $i \in \mathbb{Z}$.

Next we will see that, if M is a fiber-full A-module, then $\mathrm{Ext}^i_A(M, A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$. This circle of ideas are due (in slightly different contexts) to Ma-Quy and Kollar-Kovacs. After this, we will show that S is a fiber-full P-module provided that $\mathrm{in}_w(I)$ is a squarefree monomial ideal, and this will imply that

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) = \dim_K(H^i_{\mathfrak{m}}(R/\,\mathrm{in}(I))_j) \quad \forall\ i, j \in \mathbb{Z}$$

whenever $I \subset R$ is a homogeneous ideal such that $\mathrm{in}(I) \subset R$ is a squarefree monomial ideal, a result of Conca and myself.

Fiber-full modules and flatness

The previous lemma says that to show that $\mathrm{Ext}^i_A(M, A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$ it is enough to show that the $K[t]/(t^m)$-module $\mathrm{Ext}^i_{A/t^mA}(M/t^mM, A/t^mA)$ is flat for all $i \in \mathbb{Z}$ and $m \in \mathbb{N}_{>0}$. So we introduce the following helpful notation for all $m \in \mathbb{N}_{>0}$:

- $A_m = A/t^mA$.
- $M_m = M/t^mM$.
- $\iota_j : t^{j+1}M_m \to t^jM_m$ the natural inclusion $\forall\ j$.
- $\mu_j : t^jM_m \to t^{m-1}M_m$ the multiplication by t^{m-1-j} $\forall\ j$.
- $E^i_m(-)$ the contravariant functor $\mathrm{Ext}^i_{A_m}(-, A_m)$ $\forall\ i$.

Remark

A lemma of Rees implies that $E^i_m(M_k) \cong \mathrm{Ext}^{i+1}_A(M_k, A)$ whenever $k \leq m$. Hence we deduce that

$$E^i_m(M_k) \cong E^i_m(M_m) \ \ \forall \ \ k \leq m.$$

Fiber-full modules and flatness

Remark

Since t is a non-zero-divisor on M we have that:

$$M_j \cong t^{m-j} M_m \;\; \forall j.$$

Remark

The short exact sequences $0 \to t^{j+1} M_m \xrightarrow{\iota_j} t^j M_m \xrightarrow{\mu_j} t^{m-1} M_m \to 0$, if M is fiber-full, yield the following short exact sequences for all $i \in \mathbb{Z}$:

$$0 \to E_m^i(t^{m-1} M_m) \xrightarrow{E_m^i(\mu_j)} E_m^i(t^j M_m) \xrightarrow{E_m^i(\iota_j)} E_m^i(t^{j+1} M_m) \to 0.$$

Indeed, up to the above identifications, μ_j corresponds to the natural projection $M_{m-j} \to M_1$, therefore the map $E_m^i(\mu_j)$ is injective for all $i \in \mathbb{Z}$ by definition of fiber-full module.

Fiber-full modules and flatness

Theorem

With the above notation, if M is a fiber-full A-module, then $\operatorname{Ext}^i_A(M, A)$ is flat over $K[t]$ for all $i \in \mathbb{Z}$.

Proof: By the previous lemma, it is enough to show that $E^i_m(M_m)$ is flat over $K[t]/(t^m)$ for all $m \in \mathbb{N}_{>0}$. This is clear for $m = 1$ (because $K[t]/(t)$) is a field), so we will proceed by induction: thus let us fix $m \geq 2$ and assume that $E^i_{m-1}(M_{m-1})$ is flat over $K[t]/(t^{m-1})$. The local flatness criterion tells us that is enough to show the following two properties:

1. $E^i_m(M_m)/t^{m-1}E^i_m(M_m)$ is flat over $K[t]/(t^{m-1})$.
2. The map $\theta : (t^{m-1})/(t^m) \otimes_{K[t]/(t^m)} E^i_m(M_m) \to t^{m-1}E^i_m(M_m)$ sending $\overline{t^{m-1}} \otimes \phi$ to $t^{m-1}\phi$, is a bijection.

Fiber-full modules and flatness

By the previous Remark $E^i_m(\iota_k)$ is surjective for all k, so $E^i_m(\iota^j)$ is surjective where $\iota^j := \iota_j \circ \ldots \iota_{m-2} : t^{m-1}M_m \to t^j M_m$. Since $\iota^j \circ \mu_j$ is the multiplication by t^{m-1-j} on $t^j M_m$, we therefore have

$$\mathrm{Im}(E^i_m(\mu_j)) = \mathrm{Im}(E^i_m(\mu_j) \circ E^i_m(\iota^j)) = t^{m-1-j} E^i_m(t^j M_m).$$

Therefore $\mathrm{Ker}(E^i_m(\iota_j)) = t^{m-1-j} E^i_m(t^j M_m)$. Hence

$$E^i_m(t^{j+1} M_m) \cong \frac{E^i_m(t^j M_m)}{t^{m-1-j} E^i_m(t^j M_m)}.$$

Plugging in $j = 0$, we get that

$$\frac{E^i_m(M_m)}{t^{m-1} E^i_m(M_m)} \cong E^i_m(tM_m) \cong E^i_m(M_{m-1}) \cong E^i_{m-1}(M_{m-1})$$

is flat over $K[t]/(t^{m-1})$ by induction, and this shows (1).

Fiber-full modules and flatness

Concerning (2), from what said above it is not difficult to infer that the kernel of the surjective map $E_m^i(\iota^0) : E_m^i(M_m) \to E_m^i(t^{m-1}M_m)$ is equal to $tE_m^i(M_m)$. Since $E_m^i(\mu_0) \circ E_m^i(\iota^0)$ is the multiplication by t^{m-1} on $E_m^i(M_m)$ and $E_m^i(\mu_0)$ is injective, we get

$$0 :_{E_m^i(M_m)} t^{m-1} = \mathrm{Ker}(E_m^i(\iota^0)) = tE_m^i(M_m).$$

Since $\mathrm{Ker}(\theta) = \{\overline{t^{m-1}} \otimes \phi : \phi \in 0 :_{E_m^i(M_m)} t^{m-1}\}$, then θ is injective, and so bijective (it is always surjective). $\square$

Corollary

Let $I \subset R = K[X_1, \dots, X_n]$ be an ideal such that $S = P/\hom_w(I)$ is a fiber-full P-module ($P = R[t]$). Then $\mathrm{Ext}_P^i(S, P)$ is a flat $K[t]$-module. So, if furthermore I is homogeneous:

$$\dim_K(H_{\mathfrak{m}}^i(R/I)_j) = \dim_K(H_{\mathfrak{m}}^i(R/\operatorname{in}_w(I))_j) \ \ \forall\ i, j \in \mathbb{Z}.$$