Gröbner deformations

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Initial ideals with respect to weights

Fix $w = (w_1, \ldots, w_n) \in \mathbb{N}^n$ a *weight vector*. If $\mu = X^u \in \mathrm{Mon}(R)$ with $u = (u_1, \ldots, u_n)$ then we set $w(\mu) := w_1 u_1 + \ldots + w_n u_n$. If $0 \neq f \in R$ we set $w(f) := \max\{w(\mu) : \mu \in \mathrm{supp}(f)\}$ and

$$\mathrm{init}_w(f) = \sum_{\substack{\mu \in \mathrm{supp}(f) \\ w(\mu) = w(f)}} a_\mu \mu,$$

where $f = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu$.

Example

If $w = (2, 1)$ and $f = X^3 + 2X^2Y^2 - Y^5 \in \mathbb{Q}[X, Y]$ then $\mathrm{init}_w(f) = X^3 + 2X^2Y^2$.

Given an ideal $I \subset R$ we set $\mathrm{in}_w(I) = (\mathrm{init}_w(f) : f \in I) \subset R$.

As we will see, the passage from an ideal I to $\mathrm{in}_w(I)$ can be seen as a "continuous" degenerative process. Before explaining it, we will show that, given a monomial order $<$ on R and an ideal $I \subset R$, we can always find a suitable $w \in (\mathbb{N}_{>0})^n$ such that $\mathrm{in}_w(I) = \mathrm{in}_<(I)$.

Example

Let us find a weight vector that picks the largest monomial in every subset of monomials of degree $\leq d$ in $K[X, Y, Z]$ for the lexicographic order determined by $X > Y > Z$. We give weight 1 to Z. Since $Y > Z^d$, we give weight $d+1$ to Y. Since $X > Y^d$ and $w(Y^d) = d(d+1)$, we must choose $w(X) = d(d+1)+1$. It is not hard to check that $w = (d(d+1)+1, d+1, 1)$ indeed solves our problem.

Initial ideals with respect to weights

Given $w \in \mathbb{N}^n$ and $<$ a monomial order, we define another monomial order on R as

$$\mu <_w \nu \iff \begin{cases} w(\mu) < w(\nu) \\ w(\mu) = w(\nu) \text{ and } \mu < \nu \end{cases} .$$

Lemma

For an ideal $I \subset R$, if $\mathrm{in}_w(I) \subset \mathrm{in}_<(I)$ or $\mathrm{in}_w(I) \supset \mathrm{in}_<(I)$, then $\mathrm{in}_w(I) = \mathrm{in}_<(I)$.

Proof: By applying $\mathrm{in}_<(-)$ on both sides we get, for example, $\mathrm{in}_{<_w}(I) = \mathrm{in}_<(\mathrm{in}_w(I)) \supset \mathrm{in}_<(\mathrm{in}_<(I)) = \mathrm{in}_<(I)$. So the equality $\mathrm{in}_<(\mathrm{in}_w(I)) = \mathrm{in}_<(\mathrm{in}_<(I))$ must hold, and because $\mathrm{in}_w(I) \supset \mathrm{in}_<(I)$ we must have $\mathrm{in}_w(I) = \mathrm{in}_<(I)$. $\square$

Initial ideals with respect to weights

Lemma

Let $P \subset \mathbb{R}^n$ be the convex hull of some vectors $u^1, \dots, u^m \in \mathbb{N}^n$. Then $X^u \leq \max\{X^{u^1}, \dots, X^{u^m}\}$ for any $u \in P \cap \mathbb{N}^n$.

Proof: If $u \in P \cap \mathbb{N}^n$, then $u = \sum_{i=1}^m \lambda_i u^i$ with $\lambda_i \in \mathbb{Q}_{\geq 0}$ and $\sum_{i=1}^m \lambda_i = 1$. If $\lambda_i = a_i/b_i$ with $a_i \in \mathbb{N}$, $b_i \in \mathbb{N} \setminus \{0\}$, then we have

$$bu = \sum_{i=1}^m a_i' u^i,$$

where $b = b_1 \cdots b_m$ and $a_i' = a_i(b/b_i)$. If, by contradiction, $X^u > X^{u^i}$ for all $i = 1, \dots, m$, then

$$(X^u)^b > (X^{u^1})^{a_1'} \cdots (X^{u^m})^{a_m'}$$

(because $b = \sum_{i=1}^m a_i'$) but this contradicts the fact that these two monomials are the same. $\square$

Initial ideals with respect to weights

Proposition

Given a monomial order $>$ on R and $\mu_i, \nu_i \in \mathrm{Mon}(R)$ such that $\mu_i > \nu_i$ for $i = 1, \dots, k$, there exists $w \in (\mathbb{N}_{>0})^n$ such that $w(\mu_i) > w(\nu_i)\ \forall\ i = 1, \dots, k$. Consequently, given an ideal $I \subset R$ there exists $w \in (\mathbb{N}_{>0})^n$ such that $\mathrm{in}_<(I) = \mathrm{in}_w(I)$.

Proof: Notice that $\mu_i > \nu_i \iff \prod_j \mu_j > \nu_i \prod_{j \neq i} \mu_j$ and $w(\mu_i) > w(\nu_i) \iff w(\prod_j \mu_j) > w(\nu_i \prod_{j \neq i} \mu_j)$, so we can assume that μ_i is the same monomial μ for all $i = 1, \dots, k$. If $\mu = X^u$ and $\nu_i = X^{v^i}$, consider $C = u + (\mathbb{R}_{\geq 0})^n \subset \mathbb{R}^n$ and $P \subset \mathbb{R}^n$ the convex hull of u and $v^1, \dots, v^k$. We claim that $C \cap P = \{u\}$. Suppose that $v \in C \cap P$. We can assume that $v \in \mathbb{Q}^n$, so that there is $N \in \mathbb{N}$ big enough such that $Nv \in \mathbb{N}$. Let $\nu = X^{Nv}$. Since $v \in C$, ν is divided by $\mu^N = X^{Nu}$, so $\nu \geq \mu^N$. On the other hand, $v \in P \implies Nv \in NP$, so $\nu \leq \max\{Nu, Nv^i : i = 1, \dots, k\} = Nu$ by the previous lemma, so $\nu = \mu^N$, that is $v = u$.

Therefore there is a hyperplane passing through u separating C and P, that is there is $w \in (\mathbb{R}^n)^*$ such that

$$w(v) > w(u) > w(v^i)$$

for all $v \in C \setminus \{u\}$ and $i = 1, \ldots, k$. Of course we can pick $w = (w_1, \ldots, w_n) \in \mathbb{Q}^n$; furthermore the first inequalities yield $w_i > 0$ for all $i = 1, \ldots, n$. After taking a suitable multiple, so, we can assume $w \in (\mathbb{N}_{>0})^n$ is our desired weight vector.

For the last part of the statement, let $f_1, \ldots, f_m$ be a Gröbner basis of I. By the first part, there is $w \in (\mathbb{N}_{>0})^n$ such that $w(\mu) > w(\nu)$ where $\mu = \mathrm{in}(f_i)$ and $\nu \in \mathrm{supp}(f_i) \setminus \{\mu\}$ for all $i = 1, \ldots, m$. So $\mathrm{in}_<(I) \subset \mathrm{in}_w(I)$, hence $\mathrm{in}_<(I) = \mathrm{in}_w(I)$. $\square$

Initial ideals with respect to weights

Let us extend R to $P = R[t]$ by introducing a *homogenizing variable* t. The *w-homogenization* of $f = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu \in R$ is

$$\mathrm{hom}_w(f) = \sum_{\mu \in \mathrm{supp}(f)} a_\mu \mu t^{w(f) - w(\mu)} \in P.$$

Example

Let $f = X^2 - XY + Z^2 \in K[X, Y, Z]$. We have:

- $\mathrm{hom}_w(f) = X^2 - XY + Z^2 t^2$ if $w = (2, 2, 1)$.
- $\mathrm{hom}_w(f) = X^2 - XYt^2 + Z^2 t^6$ if $w = (4, 2, 1)$.

Given an ideal $I \subset R$, $\mathrm{hom}_w(I) \subset P$ denotes the ideal generated by $\mathrm{hom}_w(f)$ with $f \in I$. For its study, we extend the weight vector w to w' on P by $w'(t) = 1$, so that $\mathrm{hom}_w(I)$ is a w'-homogeneous ideal of P, where the grading is $\deg(X_i) = w_i$ and $\deg(t) = 1$.

Initial ideals with respect to weights

Because $P/\hom_w(I)$ is a w'-graded P-module, it is also a graded $K[t]$-module (w.r.t. the standard grading on $K[t]$). So $t-a$ is not a zero-divisor on $P/\hom_w(I)$ for any $a \in K \setminus \{0\}$. We want to show that also t is not a zero-divisor on $P/\hom_w(I)$ as well, and in order to do so it is useful to consider the *dehomogenization map*:

$$\pi : P \longrightarrow R$$
$$F(X_1, \ldots, X_n, t) \mapsto F(X_1, \ldots, X_n, 1).$$

Remark

1. $\pi(\hom_w(f)) = f \ \forall\ f \in R$. So, $\pi(\hom_w(I)) = I$.
2. If $F \in P \setminus tP$ is w'-homogeneous, then $\hom_w(\pi(F)) = F$; moreover, if $r \in \mathbb{N}$ and $G = t^r F$, $\hom_w(\pi(G))t^r = G$.

Summarizing, for $F \in P$ we have $F \in \hom_w(I) \iff \pi(F) \in I$.

Initial ideals with respect to weights

Proposition

Given an ideal I of R, the element $t - a \in K[t]$ is not a zero divisor on $P/\operatorname{hom}_w(I)$ for every $a \in K$. Furthermore:

- $P/\big(\operatorname{hom}_w(I) + (t)\big) \cong R/\operatorname{in}_w(I)$.
- $P/\big(\operatorname{hom}_w(I) + (t - a)\big) \cong R/I$ for all $a \in K \setminus \{0\}$.

Proof: For the first assertion, we need to show it just for $a = 0$: Let $F \in P$ such that $tF \in \operatorname{hom}_w(I)$. Then $\pi(tF) \in I$, so, since $\pi(F) = \pi(tF)$, $F \in \operatorname{hom}_w(I)$.

For $P/(\operatorname{hom}_w(I) + (t)) \cong R/\operatorname{in}_w(I)$ it is enough to check that $\operatorname{hom}_w(I) + (t) = \operatorname{in}_w(I) + (t)$. This is easily seen since for every $f \in R$ the difference $\operatorname{hom}_w(f) - \operatorname{init}_w(f)$ is divisible by t.

Initial ideals with respect to weights

To prove that $P/(\hom_w(I) + (t-a)) \cong R/I$ for every $a \in K \setminus \{0\}$, we consider the graded isomorphism $\psi : R \to R$ induced by $\psi(X_i) = a^{-w_i} X_i$. Of course $\psi(\mu) = a^{-w(\mu)}\mu \ \forall \ \mu \in \mathrm{Mon}(R)$ and $\hom_w(f) - a^{w(f)}\psi(f)$ is divisible by $t-a$ for all $f \in R$. So $\hom_w(I) + (t-a) = \psi(I) + (t-a)$, which implies the desired isomorphism. $\square$

Remark

Since a module over a PID is flat iff it has no torsion, the proposition above says that $P/\hom_w(I)$ is a flat $K[t]$-module, and that it defines a flat family over $K[t]$ with generic fiber R/I and special fiber $R/\mathrm{in}_w(I)$.

Next we want to show that local cohomology cannot shrink passing to the initial ideal. We need the following first:

Lemma

Let A be a ring, M, N A-modules and $a \in \operatorname{ann}(N) \subset A$ a non-zero-divisor on M as well as on A. Then, for all $i \geq 0$,

$$\mathrm{Ext}^i_A(M, N) \cong \mathrm{Ext}^i_{A/aA}(M/aM, N).$$

Proof: Let $F_\bullet$ be a free resolution of M. The Ext modules on the left hand side are the cohomology modules of $\mathrm{Hom}_A(F_\bullet, N)$, which is a complex of A-modules isomorphic to $\mathrm{Hom}_{A/aA}(F_\bullet/aF_\bullet, N)$ because a annihilates N. However $F_\bullet/aF_\bullet$ is a free resolution of the A/aA-module M/aM since a is a non-zero-divisor on M as well as on A, so the cohomology modules of the latter complex are the Ext modules on the right hand side. $\square$

Initial ideals with respect to weights

Let us give a graded structure to $R = K[X_1, \ldots, X_n]$ by putting $\deg(X_i) = g_i$ where $g = (g_1, \ldots, g_n)$ is a vector of positive integers (so that $\mathfrak{m} = (X_1, \ldots, X_n)$ is the unique homogeneous maximal ideal of R). If $I \subset R$ is a g-homogeneous ideal, then $\hom_w(I) \subset P$ is homogeneous with respect to the *bi-graded* structure on P given by $\deg(X_i) = (g_i, w_i)$ and $\deg(t) = (0, 1)$. So $S = P/\hom_w(I)$ and $\mathrm{Ext}^i_P(S, P)$ are finetely generated bi-graded P-modules.

Notice that, given a finitely generated bi-graded P-module M, $M_{(j,*)} = \bigoplus_{k\in\mathbb{Z}} M_{(j,k)}$ is a finitely generated graded (w.r.t. the standard grading) $K[t]$-module for all $j \in \mathbb{Z}$. Finally, if N is a finitely generated $K[t]$-module, $N \cong K[t]^a \oplus T$ for $a \in \mathbb{N}$ and some finitely generated torsion $K[t]$-module T (since $K[t]$ is a PID). If N is also graded, then $T \cong \bigoplus_{k\in\mathbb{N}_{>0}}(K[t]/(t^k))^{b_k}$.

Initial ideals with respect to weights

From now, let us fix a g-homogeneous ideal $I \subset R$ and denote $P/\hom_w(I)$ by S. From the above discussion, for all $i, j \in \mathbb{Z}$:

$$\operatorname{Ext}^i_P(S, P)_{(j,*)} \cong K[t]^{a_{i,j}} \oplus \left(\bigoplus_{k \in \mathbb{N}_{>0}} (K[t]/(t^k))^{b_{i,j,k}} \right)$$

for some natural numbers $a_{i,j}$ and $b_{i,j,k}$. Let $b_{i,j} = \sum_{k \in \mathbb{N}_{>0}} b_{i,j,k}$.

Theorem

With the above notation, for any $i, j \in \mathbb{Z}$ we have:

- $\dim_K(\operatorname{Ext}^i_R(R/I, R)_j) = a_{i,j}$.
- $\dim_K(\operatorname{Ext}^i_R(R/\operatorname{in}_w(I), R)_j) = a_{i,j} + b_{i,j} + b_{i+1,j}$.

In particular, $\dim_K(\operatorname{Ext}^i_R(R/I, R)_j) \leq \dim_K(\operatorname{Ext}^i_R(R/\operatorname{in}_w(I), R)_j)$ and $\dim_K(H^i_{\mathfrak{m}}(R/I)_j) \leq \dim_K(H^i_{\mathfrak{m}}(R/\operatorname{in}_w(I))_j)$.

Proof: Letting x be t or $t-1$ we have the short exact sequence

$$0 \to P \xrightarrow{\cdot x} P \to P/xP \to 0.$$

The long exact sequence of $\mathrm{Ext}_P(S,-)$ associated to it, gives us the following short exact sequences for all $i \in \mathbb{Z}$:

$$0 \to \operatorname{Coker} \alpha_{i,x} \to \mathrm{Ext}^i_P(S, P/xP) \to \operatorname{Ker} \alpha_{i+1,x} \to 0,$$

where $\alpha_{k,x}$ is the multiplication by x on $\mathrm{Ext}^k_P(S,P)$. We can restrict the above exact sequences to the degree $(j,*)$ for any $j \in \mathbb{Z}$ getting:

$$0 \to (\operatorname{Coker} \alpha_{i,x})_{(j,*)} \to (\mathrm{Ext}^i_P(S, P/xP))_{(j,*)} \to (\operatorname{Ker} \alpha_{i+1,x})_{(j,*)} \to 0.$$

Notice that we have:

- $(\operatorname{Coker} \alpha_{i,t})_{(j,*)} \cong K^{a_{i,j}+b_{i,j}}$ and $(\operatorname{Ker} \alpha_{i+1,t})_{(j,*)} \cong K^{b_{i+1,j}}$.
- $(\operatorname{Coker} \alpha_{i,t-1})_{(j,*)} \cong K^{a_{i,j}}$ and $(\operatorname{Ker} \alpha_{i+1,t-1})_{(j,*)} = 0$.

Therefore, for all $i, j \in \mathbb{Z}$, we got:

- $(\mathrm{Ext}^i_P(S, P/tP))_{(j,*)} \cong K^{a_{i,j}+b_{i,j}+b_{i+1,j}}$.
- $(\mathrm{Ext}^i_P(S, P/(t-1)P))_{(j,*)} \cong K^{a_{i,j}}$.

By a previous proposition both t and $t-1$ are non-zero-divisors on S as well on P, hence a previous lemma together with the same proposition imply:

- $(\mathrm{Ext}^i_P(S, P/tP))_{(j,*)} \cong (\mathrm{Ext}^i_{P/tP}(S/tS, P/tP))_{(j,*)}$, which is isomorphic to $(\mathrm{Ext}^i_R(R/\operatorname{in}_w(I), R))_j$.
- $(\mathrm{Ext}^i_P(S, P/(t-1)P))_{(j,*)} \cong (\mathrm{Ext}^i_{P/(t-1)P}(S/(t-1)S, P/(t-1)P))_{(j,*)}$, which is isomorphic to $(\mathrm{Ext}^i_R(R/I, R))_j$.

The thesis follows from this. For the local cohomology statement just observe that by Grothendieck graded duality $H^i_{\mathfrak{m}}(R/J)_j$ is dual as K-vector space to $\mathrm{Ext}^{n-i}_R(R/J, R)_{-|g|-j}$ for any g-homogeneous ideal $J \subset R$ and $i, j \in \mathbb{Z}$ (where $|g| = g_1 + \ldots + g_n$). $\square$

Corollary

If I is a homogeneous ideal of R, then for all $i, j \in \mathbb{Z}$

$$\dim_K(H^i_{\mathfrak{m}}(R/I)_j) \leq \dim_K(H^i_{\mathfrak{m}}(R/\operatorname{in}(I))_j).$$

Next we want to show that, if $\operatorname{in}_<(I)$ is squarefree, then we have equalities above. In order to do this, we will show that, if $\operatorname{in}_w(I)$ is a squarefree monomial ideal, then $\operatorname{Ext}^i(S, P)$ is a flat $K[t]$-module for all $i \in \mathbb{Z}$ (so that the numbers $b_{i,j}$ in the previous theorem would be 0 for all $i, j \in \mathbb{Z}$). Let us recall that a module is flat over a PID (such as $K[t]$) if and only if it has no torsion...