Appendix

Symmetric vs divided powers. Let V denote a free module of finite rank over a ring $\mathbf{k}$, and for $d > 0$ consider the tensor power $T^dV := V^{\otimes d} = V \otimes V \otimes \cdots \otimes V$ with the natural action of the symmetric group $\mathfrak{S}_d$ by permuting the factors. The **divided power** $\mathrm{D}^d\, V$ is defined as the set of symmetric tensors in T^dV, that is,

$$\mathrm{D}^d\, V := \{\omega \in T^dV : \sigma(\omega) = \omega \text{ for all } \sigma \in \mathfrak{S}_d\}.$$

If we consider the subspace of T^dV defined by

$$\Sigma_d := \mathrm{Span}\{\sigma(\omega) - \omega : \sigma \in \mathfrak{S}_d \text{ and } \omega \in T^dV\},$$

then the **symmetric power** $\mathrm{Sym}^d\, V$ is defined as the quotient $\mathrm{Sym}^d\, V := T^dV/\Sigma_d$. If V has a basis $(x_1, \ldots, x_n)$, then $\mathrm{Sym}^d\, V$ identifies with the space of homogeneous polynomials of degree d in $x_1, \ldots, x_n$, and as such it has a basis of monomials $x_1^{a_1} \cdots x_n^{a_n}$, where $a_1 + \cdots + a_n = d$. To get a basis for D^dV we first consider the orbits

$$O_{a_1,\ldots,a_n} := \mathfrak{S}_d \cdot x_1^{\otimes a_1} \otimes x_2^{\otimes a_2} \otimes \cdots \otimes x_n^{\otimes a_n}$$

and for $a_1 + \cdots + a_n = d$ consider the **divided power monomials**

$$x_1^{(a_1)} \cdots x_n^{(a_n)} := \sum_{\omega \in O_{a_1,\ldots,a_n}} \omega.$$

They form a basis for $D^d(V)$, and in particular we have that $\dim(\mathrm{Sym}^d\, V) = \dim(\mathrm{D}^d\, V)$. By composing the inclusion of $\mathrm{D}^d\, V$ into T^dV with the projection onto $\mathrm{Sym}^d\, V$ we obtain a natural map

$$\mathrm{D}^d\, V \longrightarrow \mathrm{Sym}^d\, V, \quad x_1^{(a_1)} \cdots x_n^{(a_n)} \mapsto \binom{d}{a_1, \ldots, a_n} x_1^{a_1} \cdots x_n^{a_n}, \text{ where } \binom{d}{a_1, \ldots, a_n} = \frac{d!}{a_1! \cdots a_n!}. \quad (23)$$

This map is an isomorphism when the multinomial coefficients are invertible (for instance, when $\mathbf{k}$ is a field with $\mathrm{char}(\mathbf{k}) = 0$ or $\mathrm{char}(\mathbf{k}) > d$). In general it is neither injective, nor surjective. There is a natural $\mathfrak{S}_n$-invariant perfect pairing

$$T^d(V) \times T^d(V^\vee) \longrightarrow \mathbf{k}, \quad (24)$$

defined on pure tensors via

$$\langle v_1 \otimes \cdots \otimes v_d, f_1 \otimes \cdots \otimes f_d \rangle = f_1(v_1) f_2(v_2) \cdots f_d(v_d), \text{ where } v_i \in V \text{ and } f_j \in V^\vee.$$

This induces a perfect pairing between $\mathrm{Sym}^d\, V$ and $D^d(V^\vee)$, giving a natural identification

$$(\mathrm{Sym}^d\, V)^\vee \cong \mathrm{D}^d(V^\vee).$$

The construction of tensor, symmetric and divided powers is functorial, so it can be applied to any locally free sheaf $\mathcal{E}$ on a variety X. It follows from the discussion above that when $\mathrm{char}(\mathbf{k}) > d$ we have $\mathrm{Sym}^d(\mathcal{E}) \simeq \mathrm{D}^d(\mathcal{E})$, and the inclusion $\mathrm{D}^d(\mathcal{E}) \hookrightarrow T^d(\mathcal{E})$ is split. In arbitrary characteristic, we have $(\mathrm{Sym}^d \mathcal{E})^\vee \simeq \mathrm{D}^d(\mathcal{E}^\vee)$, where $\mathcal{E}^\vee = \mathscr{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X)$.

The Eagon–Northcott and Koszul complexes. Suppose that F, G are free modules over a ring R, with $\mathrm{rank}(F) = n$, $\mathrm{rank}(G) = m$, $n \geq m$, and consider an R-linear map $\alpha : F \longrightarrow G$. The **Eagon–Northcott complex** of the map α is denoted $\mathbf{EN}_\bullet(\alpha)$, and its terms are given by

$$\mathbf{EN}_0(\alpha) = \bigwedge^m G,\ \ \mathbf{EN}_i(\alpha) = \bigwedge^{i+m-1} F \otimes \mathrm{D}^{i-1}(G^\vee) \text{ for } i = 1, \cdots, n-m+1.$$

The differential $d_i : \mathbf{EN}_i \longrightarrow \mathbf{EN}_{i-1}$ is constructed as follows. The first map is $d_1 = \bigwedge^m \alpha$, and for $i \geq 2$, d_i is given by

$$\bigwedge^{i+m-1} F \otimes \mathrm{D}^{i-1}(G^\vee) \longrightarrow \left(\bigwedge^{i+m-2} F \otimes F\right) \otimes (\mathrm{D}^{i-2}(G^\vee) \otimes G^\vee) \longrightarrow \bigwedge^{i+m-2} F \otimes \mathrm{D}^{i-2}(G^\vee),$$

where the first map is induced by the natural inclusions, and the second one is induced by

$$\alpha \in \mathrm{Hom}_R(F, G) = F^\vee \otimes G = \mathrm{Hom}_R(F \otimes G^\vee, R).$$

If α is surjective then $\mathbf{EN}_\bullet(\alpha)$ is an exact complex.

If we choose bases for F and G, then α is expressed by a $m \times n$ matrix, and $\bigwedge^m \alpha$ is given by a one-row matrix whose entries are the $m \times m$ minors of α. Writing $I_m(\alpha)$ for the ideal they generate, it follows that $\mathrm{coker}(d_1) \simeq S/I_m(\alpha)$. Suppose that R is a standard graded polynomial ring, that F, G are graded R-modules, and that α is a degree preserving map and it is minimal (that is, it has entries in the maximal homogeneous ideal). Under suitable genericity assumptions, such as $\mathrm{codim}(I_m(\alpha)) = n - m + 1$, $\mathbf{EN}_\bullet(\alpha)$ gives a minimal resolution of $S/I_m(\alpha)$.

The complex obtained in the special case $m = 1$ is called a **Koszul complex**, and is denoted $\mathbf{K}_\bullet(\alpha)$. The condition that $\mathrm{codim}(I_m(\alpha)) = n - m + 1$ is then equivalent to the requirement that the entries of α form a regular sequence, which is the familiar condition characterizing the exactness of the Koszul complex.

By functoriality, one can perform similar constructions in the relative setting, when F, G are replaced by locally free sheaves $\mathcal{F}, \mathcal{G}$ on a variety X. If α is surjective then $\mathbf{EN}_\bullet(\alpha)$, as remarked in the absolute setting.

The examples of interest for us are those coming from a rational normal curve, respectively from a rational normal scroll. If $R = S = \mathbf{k}[z_0, \cdots, z_g]$, $m = 2$, $n = g$ and α is given by the matrix Z in (2), then $n-m+1=g-1$ is the codimension of Γ_g in $\mathbb{P}^g$, and the corresponding Eagon–Northcott complex gives a minimal resolution of $S/I_2(\alpha)$.

Exercise 24. Choose bases and write down explicitly the Eagon–Northcott complex for the rational normal curves of degree $g=3$ and $g=4$.

For $a_1,\cdots,a_d\geq 1$, a **rational normal scroll** of type $(a_1,a_2,\cdots,a_d)$ is defined by the 2×2 minors of a matrix obtained by concatenating matrices of the form (2) in different sets of variables:

$$A(a_1,\cdots,a_d)=\left(\begin{array}{cccc|cc|cccc} x_{1,0} & x_{1,1} & \cdots & x_{1,a_1-1} & \cdots & \cdots & x_{d,0} & x_{d,1} & \cdots & x_{d,a_d-1} \\ x_{1,1} & x_{1,2} & \cdots & x_{1,a_1} & \cdots & \cdots & x_{d,1} & x_{d,2} & \cdots & x_{d,a_d} \end{array}\right)$$

The resulting scroll has dimension d and is a subvariety in $\mathbb{P}^{a_1+\cdots+a_d+d-1}$. Note that the matrix $A(a_1,\cdots,a_d)$ gives a map between free modules of ranks $n=a_1+\cdots+a_d$ and $m=2$, so $n-m+1$ is precisely the codimension of the scroll, and the Eagon–Northcott complex is again exact. The case of the rational normal curve of degree g can be recovered by taking $d=1$ and $a_1=g$.

Exercise 25. Verify that the columns of the matrices (6) (resp. (10)) can be rearranged in such a way that they agree, after a relabelling, with scrollar matrices $A(a_1,a_2)$ (resp. $A(a_1,\cdots,a_p)$).

The Buchsbaum–Rim complex. We continue with the notation from the previous section. The **Buchsbaum–Rim** complex associated with the map $\alpha : F\longrightarrow G$ is denoted $\mathbf{BR}_\bullet(\alpha)$, and its terms are given by

$$\mathbf{BR}_0(\alpha)=G,\ \mathbf{BR}_1(\alpha)=F,\ \mathbf{BR}_i(\alpha)=\bigwedge^{i+m-1}F\otimes\det(G^\vee)\otimes \mathrm{D}^{i-2}(G^\vee)\text{ for } i=2,\cdots,n-m+1.$$

The first differential is $d_1=\alpha$, while the second one is obtained by composing

$$\bigwedge^{m+1}F\otimes\bigwedge^{m}G^\vee\longrightarrow\left(\bigwedge^{m}F\otimes F\right)\otimes\bigwedge^{m}G^\vee\longrightarrow F$$

where the first map is the natural inclusion, while the second one is induced by

$$\bigwedge^{m}\alpha\in\mathrm{Hom}_R\left(\bigwedge^{m}F,\bigwedge^{m}G\right)=\mathrm{Hom}_R\left(\bigwedge^{m}F\otimes\bigwedge^{m}G^\vee,R\right).$$

The higher differentials are defined in analogy with the ones for the Eagon–Northcott complex. Just like in the case of the Eagon–Northcott complex, when α is surjective the complex $\mathbf{BR}_\bullet(\alpha)$ is exact. A similar statement holds if we replace F,G by locally free sheaves.

Cohomology of line bundles on projective space, and Castelnuovo–Mumford regularity. Let V be an n-dimensional vector space over ome field $\mathbf{k}$. Let $\mathbf{P} = \mathbb{P}(V^\vee)$ denote the projective space parametrizing one-dimensional subspaces of $V^\vee$, so that $H^0(\mathbf{P}, \mathcal{O}_\mathbf{P}(1)) = V$. We have

$$H^0(\mathbf{P}, \mathcal{O}_\mathbf{P}(d)) = \begin{cases} \mathrm{Sym}^d(V) & d \geq 0; \\ 0 & d < 0. \end{cases}$$

$$H^{n-1}(\mathbf{P}, \mathcal{O}_\mathbf{P}(d)) = \begin{cases} \mathrm{Sym}^{-d-n}(V)^\vee = \mathrm{D}^{-d-n}(V^\vee) & d \leq -n; \\ 0 & d > -n. \end{cases}$$

$$H^i(\mathbf{P}, \mathcal{O}_\mathbf{P}(d)) = 0 \text{ for all } d \in \mathbb{Z},\ i \neq 0 \text{ and } i \neq n-1.$$

A coherent sheaf $\mathcal{F}$ on $\mathbf{P}$ is d-**regular** if $H^i(\mathbf{P}, \mathcal{F}(d-i)) = 0$ for $i > 0$. It follows from the cohomological calculations above that for any $d \in \mathbb{Z}$, the sheaf $\mathcal{F} = \mathcal{O}_\mathbf{P}(-d)$ is d-regular.

In general, we have that $\mathcal{F}$ is d-regular if and only if it admits a resolution $\mathcal{F}_\bullet$ where $\mathcal{F}_i \simeq \bigoplus \mathcal{O}_\mathbf{P}(-d-i)$. More generally, $\mathcal{F}$ is d-regular if it admits a resolution $\mathcal{F}_\bullet$ where $\mathcal{F}_i$ is $(d+i)$-regular. If $\mathcal{F} = \mathcal{F}' \oplus \mathcal{F}''$ then $\mathcal{F}$ is d-regular if and only if both $\mathcal{F}', \mathcal{F}''$ are d-regular. If $\mathcal{E}$ is a locally free sheaf which is e-regular, and if $\mathcal{F}$ is d-regular, then the tensor product $\mathcal{F} \otimes \mathcal{E}$ is $(d+e)$-regular.