4. Hermite reciprocity, syzygies of curves

Let U denote a 2-dimensional $\mathbf{k}$-vector space with basis $(1, x)$ as before. The goal of this section is to describe an explicit isomorphism

$$\Theta_d^i : \mathrm{Sym}^d(\mathrm{D}^i U) \longrightarrow \bigwedge^i(\mathrm{Sym}^{d+i-1} U). \tag{22}$$

Since $\mathrm{D}^i U$ has a basis consisting of elements $x^{(t)}$, $0 \leq t \leq i$, we get a basis for $\mathrm{Sym}^d(\mathrm{D}^i U)$ consisting of elements

$$e_\mu(x) = x^{(\mu_1)} \cdot x^{(\mu_2)} \cdots x^{(\mu_d)} \tag{23}$$

where $\mu = (\mu_1 \geq \cdots \geq \mu_d)$ is a partition with $\mu_1 \leq i$ and with at most d parts. Similarly, $\bigwedge^i(\mathrm{Sym}^{d+i-1} U)$ has a basis consisting of elements

$$s_\lambda(x) := x^{\lambda_1+i-1} \wedge x^{\lambda_2+i-2} \wedge \cdots \wedge x^{\lambda_i} \tag{24}$$

where $\lambda = (\lambda_1 \geq \cdots \geq \lambda_i)$ consists of partitions with $\lambda_1 \leq d$ and with at most i parts.

Consider the polynomial ring $\mathbf{k}[z_1, \cdots, z_i]$, and the subring consisting of symmetric polynomials. Examples of such are the **elementary symmetric polynomials**

$$e_r(z_1, \cdots, z_i) = \sum_{j_1 < \cdots < j_r} z_{j_1} \cdots z_{j_r},$$

which in turn can be used to define

$$e_\mu(z_1, \cdots, z_i) = e_{\mu_1}(z_1, \cdots, z_i) \cdots e_{\mu_d}(z_1, \cdots, z_i).$$

and the **Schur polynomials**

$$s_\lambda(z_1, \ldots, z_i) = \frac{\det(z_k^{\lambda_\ell+i-k})_{1\leq k,\ell\leq i}}{\det(z_k^{i-k})_{1\leq k,\ell\leq i}}.$$

Exercise 26. Show that the $\mathbf{k}$-vector space with basis

$$\{e_\mu(z_1, \cdots, z_i) : i \geq \mu_1 \geq \cdots \geq \mu_d\}$$

coincides with the $\mathbf{k}$-vector space with basis

$$\{s_\lambda(z_1, \cdots, z_i) : d \geq \lambda_1 \geq \cdots \geq \lambda_i\}.$$

If we let P_d^i denote the vector space of symmetric polynomials in Exercise 26 then the isomorphism Θ_d^i in (22) is constructed by composing the $\mathbf{k}$-linear isomorphisms

$$\mathrm{Sym}^d(\mathrm{D}^i U) \longrightarrow P_d^i, \quad e_\mu(x) \mapsto e_\mu(z_1, \cdots, z_i)$$

and

$$P_d^i \longrightarrow \bigwedge^i (\mathrm{Sym}^{d+i-1} U), \quad s_\lambda(z_1, \cdots, z_i) \mapsto s_\lambda(x).$$

4.1. **The coordinate independent description of Γ_g.** Let U denote a 2-dimensional $\mathbf{k}$-vector space, and let $\mathbb{P}U \simeq \mathbf{P}^1$ denote the corresponding projective line. More precisely, if we let

$$A = \mathrm{Sym}(U) = \mathbf{k} \oplus U \oplus \mathrm{Sym}^2 U \oplus \cdots$$

denote the symmetric algebra of U, then $\mathbb{P}U = \mathrm{Proj}(A)$. If we choose a basis $\{x, y\}$ for U, then we get an identification $A = \mathbf{k}[x, y]$, but we will avoid doing so in this section. The **Veronese subalgebra**

$$A^{(g)} = \bigoplus_{d \geq 0} \mathrm{Sym}^{dg} U,$$

is naturally a quotient of the symmetric algebra $S = \mathrm{Sym}(\mathrm{Sym}^g U)$, and this gives rise to a closed embedding

$$\Gamma_g \simeq \mathrm{Proj}(A^{(g)}) \hookrightarrow \mathrm{Proj}(S) \simeq \mathbf{P}^g.$$

The syzygy modules of Γ_g can be described in a coordinate-free fashion (using the Eagon–Northcott complex) by

$$B_{i,1}(\Gamma_g) = \mathrm{D}^{i-1} U \otimes \bigwedge^{i+1} (\mathrm{Sym}^{g-1} U) \text{ for } i = 1, \cdots, g-1.$$

This description is more refined than the earlier calculation of Betti numbers, which can be recovered by noting that $\dim(\mathrm{D}^{i-1} U) = i$ and $\dim(\mathrm{Sym}^{g-1} U) = g$. What is even more remarkable is that if we fix i and vary g, then it follows from Hermite reciprocity that

$$\mathfrak{B}_i = \bigoplus_{g \geq i+1} B_{i,1}(\Gamma_g)$$

has the structure of a free module over $\mathrm{Sym}(\mathrm{D}^i U)$, namely

$$\mathfrak{B}_i = \mathrm{D}^{i-1} U \otimes \mathrm{Sym}(\mathrm{D}^i U).$$

4.2. **The key insight for the proof of Theorem 22.** The proof of Theorem 22 is quite involved, but the key idea is to realize that, just as in the case of the syzygies of the rational normal curve, if we fix i and let g vary, then

$$\bigoplus_{g \geq i+3} B_{i,2}(\mathcal{T}_g) \tag{25}$$

has the structure of a module over a polynomial ring, namely over $\mathrm{Sym}(\mathrm{D}^{i+2} U)$. This module is no longer free, and the main technical part of the work is to translate via Hermite reciprocity the calculation of $B_{i,2}(\mathcal{T}_g)$, and show that the resulting module (25) is in fact the Weyman module $W^{(i+2)}$.

4.3. **Syzygies of curves.** Let X be a non-singular projective curve of genus g, that is,

$$\dim H^0(X, \Omega_X) = g.$$

We define the **gonality** of X to be the smallest degree of a map $X \longrightarrow \mathbf{P}^1$, so that $X \simeq \mathbf{P}^1$ if and only if $\mathrm{gon}(X) = 1$. If $\mathrm{gon}(X) = 2$ then we say that X is **hyperelliptic**. If $g \geq 1$ and we fix a basis $\omega_1, \cdots, \omega_g$ of $H^0(X, \Omega_X)$ then we can define a morphism

$$\phi : X \longrightarrow \mathbf{P}^{g-1},\ p \mapsto [\omega_1(p) : \cdots : \omega_g(p)],$$

and ϕ is a closed embedding if and only if X is not hyperelliptic. We call ϕ the **canonical map/embedding**. When X is hyperelliptic, ϕ maps X onto the rational normal curve Γ_{g-1} via a degree two map! When ϕ is an embedding, we can identify X with its image $\phi(X)$, which we refer to as a **canonical curve** (we note here that a canonical curve will have degree $2g-2$). An important problem in the projective geometry of curves is the following:

Problem 27. Describe the defining equations, and the Betti tables of canonical curves.

Example 28. When $g = 3$, a canonical curve is simply a quartic curve in $\mathbf{P}^2$, so the Betti table is

	0	1
0	1	–
1	–	–
2	–	–
3	–	1

When $g = 4$, a canonical curve is a complete intersection of a quadric and a cubic surface in $\mathbf{P}^3$:

	0	1	2
0	1	–	–
1	–	1	–
2	–	1	–
3	–	–	1

Theorem 29 (Petri)**.** *If X is general of genus $g \geq 5$ then the ideal $I(X)$ is generated by quadrics. The exceptions are the trigonal curves, and those isomorphic with plane quintics (which have genus 6).*

Example 30. When $g = 5$, the general canonical curves will have Betti table

	0	1	2	3
0	1	–	–	–
1	–	3	–	–
2	–	–	3	–
3	–	–	–	1

while for the trigonal ones the Betti table is

	0	1	2	3
0	1	–	–	–
1	–	3	2	–
2	–	2	3	–
3	–	–	–	1

Any canonical curve is Gorenstein, of Castelnuovo–Mumford regularity 3, and Green's Conjecture predicts that the vanishing behavior of the Betti numbers $b_{i,j}(X)$ characterizes a geometric invariant of X called the **Clifford index** (for most curves, this is equal to $\mathrm{gon}(X) - 2$, and for our discussion it is safe to assume that this is the case – see for instance Example 30).

Conjecture 31 (Generic Green)**.** *If X is a general curve over a field of characteristic zero then*

$$b_{i,2}(X) = 0 \text{ if and only if } i \leq \frac{g-3}{2}. \tag{26}$$

This conjecture was proved by Voisin, and work of Schreyer shows that the natural extension to positive characteristic p may fail (for instance when $p = 2$ and $g = 7$, or $p = 3$ and $g = 9$).

Conjecture 32 (Eisenbud–Schreyer)**.** *The Generic Green Conjecture holds in characteristic $p \geq \frac{g-1}{2}$.*

The general strategy for proving Generic Green's conjecture is to produce an example of a curve X_0 that satisfies (26), and then use the semicontinuity of the Betti numbers, as follows.

Theorem 33. *Generic Green's Conjecture holds over a field of characteristic $p \geq \frac{g+2}{2}$.*

Sketch of proof. Let X_0 denote a general hyperplane section of $\mathcal{T}_g$, so that

$$b_{i,j}(X_0) = b_{i,j}(\mathcal{T}_g) \text{ for all } i, j. \tag{27}$$

X_0 is a rational g-cuspidal curve, so it has (arithmetic) genus g. It can be realized as a limit of smooth curves of genus g, $(X_t)_t \longrightarrow X_0$. Using (27) and Theorem 7, it follows that

$$b_{i,2}(X_0) = 0 \text{ for } i \leq \frac{g-3}{2}.$$

Since Betti numbers are upper semicontinuous, it follows that $b_{i,2}(X_t) = 0$ for some (a general) t, so there is at least one smooth curve X_t satisfying the Generic Green Conjecture. Again by semicontinuity, the same is true for a curve X in a neighborhood of X_t in the (irreducible) moduli space of genus g curves. $\square$