3. Weyman modules and the syzygies of $\mathcal{T}_g$

In this section we introduce a natural class of examples of Koszul modules that satisfy (in most characteristics) the hypotheses of Theorems 12 and 16. They are called **Weyman modules**, and provide a link with the Betti numbers of the tangential variety $\mathcal{T}_g$ studied in Section 1. We assume $\mathrm{char}(\mathbf{k}) \neq 2$.

3.1. **Weyman modules.** Let U be a $\mathbf{k}$-vector space of dimension two, and fix a basis $(1, x)$ so that $\mathrm{Sym}^d U$ can be identified with the space of polynomials of degree at most d in x. With this choice of basis we identify $\mathbf{k} \simeq \bigwedge^2 U$ via $1 \mapsto 1 \wedge x$. The perfect pairing

$$U \times U \longrightarrow \bigwedge^2 U \simeq \mathbf{k}$$

gives rise to an identification $U \simeq U^\vee$ which we will use freely. For instance we will identify $(\mathrm{Sym}^d U)^\vee \simeq \mathrm{D}^d U$, instead of the more natural isomorphism $(\mathrm{Sym}^d U)^\vee \simeq \mathrm{D}^d(U^\vee)$ (see the Appendix). Recall that in characteristic zero (or larger than d) we also have an isomorphism $\mathrm{D}^d U \simeq \mathrm{Sym}^d U$, so that $\mathrm{Sym}^d U$ is isomorphic to its dual!

For $d \geq 0$ we consider the map

$$\psi : \bigwedge^2 \mathrm{Sym}^d U \longrightarrow \mathrm{Sym}^{2d-2} U, \quad \psi(x^i \wedge x^j) = (i-j) \cdot x^{i+j-1} \text{ for } 0 \leq i, j \leq d. \tag{21}$$

Exercise 18. Show that if $\mathrm{char}(\mathbf{k}) \neq 2$ then ψ is surjective.

Exercise 19. Check that the map ψ is $\mathrm{SL}(U)$-equivariant, where $\mathrm{SL}(U)$ is the group of linear automorphisms of U with determinant one. All the identifications that we make in this section will be $\mathrm{SL}(U)$-equivariant!

We define a pair (V, K) by setting $V^\vee = \mathrm{Sym}^d U$ and $K^\perp = \ker(\psi)$. It follows that $V = \mathrm{D}^d U$, and since ψ is surjective (in characteristic $\neq 2$), we get

$$K^\vee = \bigwedge^2 V^\vee / K^\perp = \bigwedge^2 V^\vee / \ker(\psi) \simeq \mathrm{Im}(\psi) = \mathrm{Sym}^{2d-2} U,$$

so that $K = \mathrm{D}^{2d-2} U$. The Koszul module associated with this pair (V, K) is denoted

$$W^{(d)} := W(\mathrm{D}^d U, \mathrm{D}^{2d-2} U),$$

and is called a **Weyman module**. We will show that in most characteristics, Weyman modules satisfy the hypotheses of Theorems 12 and 16. This follows from the following.

Lemma 20. *Let $p = \mathrm{char}(\mathbf{k})$. If $p = 0$ or $p \geq n$, then $W^{(n-1)}$ has vanishing resonance (and therefore it has finite length).*

Proof. The vanishing of resonance is equivalent to the condition that $\ker(\psi)$ contains no non-zero decomposable form $f_1 \wedge f_2$. Suppose by contradiction that $\ker(\psi)$ contains a non-zero decomposable form $f_1 \wedge f_2$, and that $p = 0$ or $p \geq n$. By rescaling $f_1 \wedge f_2$, we may assume that f_1, f_2 are monic polynomials in x, say

$$f_1 = x^r + b_{r-1}x^{r-1} + \cdots + b_0, \quad f_2 = x^s + c_{s-1}x^{s-1} + \cdots + c_0.$$

If $r \neq s$, we may assume $r > s$ and we get $\psi(f_1 \wedge f_2) = (r - s)x^{r+s-1} + h(x)$, where $\deg(h) < r + s - 1$ and $0 < r - s \leq n - 1$. It follows that p cannot divide $r - s$, hence $\psi(f_1 \wedge f_2) \neq 0$, a contradiction. If $r = s$, then we can use the fact that $f_1 \wedge f_2 = f_1 \wedge (f_2 - f_1)$ to reduce to the case $r \neq s$. ☐

Exercise 21. With the notation in Lemma 20, check that if $3 \leq p \leq n - 1$ then $W^{(n-1)}$ has infinite length.

The relationship with the syzygies of $\mathcal{T}_g$ is given in the following theorem.

Theorem 22. *If* $\mathrm{char}(\mathbf{k}) \neq 2$, *then for each* $i = 1, \ldots, g - 3$, *we have a natural identification*

$$B_{i,2}(\mathcal{T}_g) = W^{(i+2)}_{g-3-i}.$$

Example 23. Let $g = 6$ and suppose that $\mathrm{char}(\mathbf{k}) = 0$. Combining Theorem 22 with (20) we get that the Betti table of $\mathcal{T}_6$ is

	0	1	2	3	4
0	1	–	–	–	–
1	–	6	5	–	–
2	–	–	5	6	–
3	–	–	–	–	1

Exercise 24. Write down the Betti table $\beta(\mathcal{T}_g)$ when $g = 7, 8, 9$ and $\mathrm{char}(\mathbf{k}) = 0$.

Based on Theorem 22, we can now finish the proof of Theorem 7.

Proof of Theorem 7 (the equivalence (9)). The implication "$\Longleftarrow$" in (9) was already discussed in Exercise 10. For the converse, we assume that $3 \leq p \leq \frac{g+1}{2}$, and using Theorem 22, we have to show that

$$W^{(i+2)}_{g-3-i} = 0 \text{ for } i \leq p - 3.$$

Let $n = i + 3$ and note that $p \geq n$, so Lemma 20 applies to show that $W^{(i+2)} = W^{(n-1)}$ has finite length. We also have that $p \geq n - 2$, so by Theorem 12 we conclude that $W^{(n-1)}_q = 0$ for $q \geq n - 3$. Since $g \geq 2p - 1$ and $i \leq p - 3$, it follows that if we let $q = g - 3 - i$ then $q \geq (2p - 1) - p = p - 1 \geq n - 3$. ☐

Exercise 25. Prove the equivalence (8) using Theorem 22.