1. Syzygies of the tangential variety to a rational normal curve

Consider the **rational normal curve of degree** g, denoted Γ_g and defined as the image of the **Veronese map**

$$\nu_g : \mathbf{P}^1 \longrightarrow \mathbf{P}^g, \quad [a:b] \longrightarrow [a^g : a^{g-1}b : \cdots : ab^{g-1} : b^g]. \tag{1}$$

A first natural question to ask is what are the equations vanishing along Γ_g. To answer it, we let $S = \mathbf{k}[z_0, \cdots, z_g]$ denote the homogeneous coordinate ring of $\mathbf{P}^g$, let $A = \mathbf{k}[x,y]$ denote the homogeneous coordinate ring of $\mathbf{P}^1$, and consider the pull-back homomorphism

$$\phi : S \longrightarrow A, \quad \phi(z_i) = x^{g-i}y^i, \text{ for } i = 0, \cdots, g.$$

The ideal $I(\Gamma_g)$ of polynomials vanishing along Γ_g is equal to $\ker(\phi)$. Since the matrix

$$\begin{bmatrix} x^g & x^{g-1}y & \cdots & xy^{g-1} \\ x^{g-1}y & x^{g-2}y^2 & \cdots & y^g \end{bmatrix}$$

has proportional rows (with ratio x/y), it follows that its 2×2 minors vanish identically. In other words, the 2×2 minors of

$$Z = \begin{bmatrix} z_0 & z_1 & \cdots & z_{g-1} \\ z_1 & z_2 & \cdots & z_g \end{bmatrix} \tag{2}$$

belong to $\ker(\phi) = I(\Gamma_g)$.

Exercise 1. Show that $I(\Gamma_g)$ is generated by the 2×2 minors of Z.

The next natural step is to investigate the minimal free resolution of the homogeneous coordinate ring $S/I(\Gamma_g)$ of Γ_g. We let

$$B_{i,j}(\Gamma_g) = \mathrm{Tor}_i^S(S/I(\Gamma_g), \mathbf{k})_{i+j}$$

denote the module of i-**syzygies of weight** j (or degree $i+j$), and define the **Betti numbers of** Γ_g as

$$b_{i,j}(\Gamma_g) = \dim_{\mathbf{k}} B_{i,j}(\Gamma_g).$$

The Betti numbers of Γ_g are recorded into the **Betti table** $\beta(\Gamma_g)$, where the columns account for the homological degree, and the rows for internal degree:

$$\begin{array}{c|ccc} & & i & \\ \hline & & \vdots & \\ j & \cdots & b_{i,j}(\Gamma_g) & \cdots \\ & & \vdots & \end{array}$$

For example, when $g = 3$, Γ_g is the **twisted cubic curve**, with Betti table

	0	1	2
0	1	–	–
1	–	3	2

where a dash indicates the vanishing of the corresponding Betti number. We say that the twisted cubic has a **linear minimal free resolution** (after the first step), since all (but the first) Betti numbers are concentrated in a single row. This is true more generally for any Γ_g, whose minimal free resolution is given by an Eagon–Northcott complex, and the corresponding Betti table takes the following shape:

	0	1	2	$\cdots$	i	$\cdots$	$g-1$
0	1	–	–	$\cdots$	–	$\cdots$	–
1	–	$\binom{g}{2}$	$2\cdot\binom{g}{3}$	$\cdots$	$i\cdot\binom{g}{i+1}$	$\cdots$	$(g-1)$

Remark 2. The description of the defining equations and that of the Betti table of Γ_g is independent on the characteristic of the field $\mathbf{k}$.

Let $\mathcal{T}_g$ denote the **tangential variety of** Γ_g, defined as the union of the tangent lines to Γ_g. One of the problems that we will be concerned with in these notes is the following.

Problem 3. Describe the defining equations and the Betti table of $\mathcal{T}_g$.

1.1. **The local description of Γ_g, $\mathcal{T}_g$.** Restricting the Veronese map ν_g in (1) to the affine chart $x = 1$ yields a local parametrization of Γ_g via

$$t \mapsto [1 : t : \cdots : t^g] = (t, t^2, \cdots, t^g),$$

where $[\cdots]$ represents projective notation, and $(\cdots)$ represents affine notation. The tangent directions to Γ_g are determined by differentiating with respect to t, so $\mathcal{T}_g$ is described in the affine chart $z_0 = 1$ by

$$(t, s) \longrightarrow (t, t^2, \cdots, t^g) + s \cdot (1, 2t, \cdots, gt^{g-1}) = (t + s, t^2 + 2ts, \cdots, t^g + gt^{g-1}s). \tag{3}$$

Since $\mathcal{T}_g$ is irreducible, one can use the local description above to check when a homogeneous polynomial belongs to $I(\mathcal{T}_g)$. This observation can be applied to check that the quadratic equations constructed below vanish along $\mathcal{T}_g$. Let

$$\Delta_{i,j} = \det \begin{bmatrix} z_i & z_j \\ z_{i+1} & z_{j+1} \end{bmatrix}, \text{ for } 0 \le i < j \le g - 1,$$

and define

$$Q_{i,j} = \Delta_{i+2,j} - 2 \cdot \Delta_{i+1,j+1} + \Delta_{i,j+2}, \text{ for } 0 \le i < j \le g - 3. \tag{4}$$

Exercise 4. Check that $Q_{i,j} \in I(\mathcal{T}_g)$ for all $0 \le i < j \le g-3$.

Remark 5. As we'll see later, the quadrics constructed above will generate, in most cases, the ideal $I(\Gamma_g)$. The exceptions are when $g \le 4$, or when $\mathrm{char}(\mathbf{k}) \in \{2,3\}$.

1.2. **Characteristic zero interpretation of Γ_g, $\mathcal{T}_g$.** Suppose that $\mathrm{char}(\mathbf{k}) = 0$ (or more generally, that the binomial coefficients $\binom{g}{i}$, $i = 1, \cdots, g-1$, are invertible in $\mathbf{k}$), and consider the vector space of homogeneous forms of degree g in the variables X, Y. Since the binomial coefficients are invertible in $\mathbf{k}$, one can write each such form uniquely as

$$F(X,Y) = z_0 \cdot X^g + z_1 \cdot \binom{g}{1} \cdot X^{g-1}Y + \cdots + z_i \cdot \binom{g}{i} \cdot X^{g-i}Y^i + \cdots + z_g \cdot Y^g,$$

for $z_0, \cdots, z_g \in \mathbf{k}$. Notice that $F(X,Y)$ is a power of a linear form, namely $F(X,Y) = (aX+bY)^g$, if and only if

$$z_0 = a^g,\ z_1 = a^{g-1}b, \cdots,\ z_i = a^{g-i}b^i, \cdots,\ z_g = b^g.$$

It follows that Γ_g parametrizes (up to scaling) binary forms $F(X,Y)$ that factor as a power L^g of a linear form L in X, Y.

Exercise 6. Check that $\mathcal{T}_g$ parametrizes (up to scaling) binary forms $F(X,Y)$ that factor as a product $L_1^{g-1} \cdot L_2$, where L_1, L_2 are linear forms in X, Y.

In the case $g = 3$, it follows that $\mathcal{T}_3$ parametrizes cubic forms with a double root (in $\mathbf{P}^1$). This is a quartic surface in $\mathbf{P}^3$, defined by the vanishing of the discriminant of F:

$$-3z_1^2z_2^2 + 4z_0z_2^3 + 4z_1^3z_3 - 6z_0z_1z_2z_3 + z_0^2z_3^2 = 0. \tag{5}$$

If $\mathrm{char}(\mathbf{k}) = 2$, this equation becomes $(z_0z_3 - z_1z_2)^2 = 0$, $I(\mathcal{T}_3) = \langle z_0z_3 - z_1z_2 \rangle$, and $\mathcal{T}_3 \simeq \mathbf{P}^1 \times \mathbf{P}^1$. The following section describes more generally the situation in characteristic 2.

1.3. **The Betti table of $\mathcal{T}_g$ in characteristic 2.** Suppose $\mathrm{char}(\mathbf{k}) = 2$ and consider the matrix of linear forms

$$M = \begin{bmatrix} z_0 & z_1 & \cdots & z_{g-2} \\ z_2 & z_3 & \cdots & z_g \end{bmatrix}. \tag{6}$$

We claim that the 2×2 minors of M vanish on $\mathcal{T}_g$. As noted earlier, it suffices to check this assertion on the chart (3). Evaluating M on the point of $\mathcal{T}_g$ parametrized by (t,s) we obtain

$$M(t,s) = \begin{bmatrix} 1 & t & t^2 & t^3 & \cdots & t^{g-2} \\ t^2 & t^3 & t^4 & t^5 & \cdots & t^g \end{bmatrix} + s \cdot \begin{bmatrix} 0 & t & 0 & t^3 & \cdots & (g-2)\cdot t^{g-2} \\ 0 & t^3 & 0 & t^5 & \cdots & g \cdot t^g \end{bmatrix}$$

Since the second row of $M(t,s)$ is obtained from the first by multiplying by t^2, we get the desired conclusion. The 2×2 minors of M define a rational normal scroll $\mathcal{S}$ of dimension 2, so that $\mathcal{T}_g \subseteq \mathcal{S}$. Since $\dim(\mathcal{T}_g)=2$, it follows that $\mathcal{T}_g=\mathcal{S}$. It follows that the minimal resolution of $\mathcal{T}_g$ is given by an Eagon–Northcott complex, and the Betti table takes the form

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & \cdots & i & \cdots & g-2 \\ \hline 0 & 1 & - & - & \cdots & - & \cdots & - \\ 1 & - & \binom{g-1}{2} & 2\cdot\binom{g-1}{3} & \cdots & i\cdot\binom{g-1}{i+1} & \cdots & (g-2) \end{array}$$

Note that $I(\mathcal{T}_g)$ is generated in this case by $\binom{g-1}{2}$ quadrics, so that the $\binom{g-2}{2}$ quadrics defined in (4) are not enough to generate $I(\mathcal{T}_g)$.

1.4. **The Betti numbers of $\mathcal{T}_g$ for $\mathrm{char}(\mathbf{k})\neq 2$.** If $\mathrm{char}(\mathbf{k})\neq 2$, we will see that the homogeneous coordinate ring of $\mathcal{T}_g$ is Gorenstein of Castelnuovo–Mumford regularity 3, that is, the Betti table of $\mathcal{T}_g$ has the following shape

$$\begin{array}{c|ccccccc} & 0 & 1 & 2 & \cdots & g-4 & g-3 & g-2 \\ \hline 0 & 1 & - & - & \cdots & - & - & - \\ 1 & - & b_{1,1} & b_{2,1} & \cdots & b_{g-4,1} & b_{g-3,1} & - \\ 2 & - & b_{1,2} & b_{2,2} & \cdots & b_{g-4,2} & b_{g-3,2} & - \\ 3 & - & - & - & \cdots & - & - & 1 \end{array} \tag{7}$$

with $b_{i,1}=b_{g-2-i,2}$ for $i=1,\cdots,g-3$. In order to understand the Betti table, it is then sufficient to study the second row (that is, $b_{\bullet,2}$). The following theorem, whose proof we'll discuss later, completely characterizes the (non-)vanishing behavior of the Betti numbers of $\mathcal{T}_g$.

Theorem 7. *Suppose that $p=\mathrm{char}(\mathbf{k})\neq 2$. If $p=0$ or $p\geq \frac{g+2}{2}$, then*

$$b_{i,2}(\mathcal{T}_g)\neq 0 \quad\Longleftrightarrow\quad \frac{g-2}{2}\leq i\leq g-3. \tag{8}$$

If $3\leq p\leq \frac{g+1}{2}$, then

$$b_{i,2}(\mathcal{T}_g)\neq 0 \quad\Longleftrightarrow\quad p-2\leq i\leq g-3. \tag{9}$$

Notice that for $p=3$ and $g\geq 5$, we have $b_{1,2}(\mathcal{T}_g)\neq 0$, so the ideal $I(\mathcal{T}_g)$ requires cubic minimal generators!

Exercise 8. Show that if $p\neq 2$ then $b_{1,1}(\mathcal{T}_g)=\binom{g-2}{2}$, and (4) gives all the quadrics vanishing on $\mathcal{T}_g$.

Example 9. To illustrate the dependence of the Betti numbers of $\mathcal{T}_g$ on the characteristic, consider the case $g = 5$. If $\mathrm{char}(\mathbf{k}) \neq 2, 3$, it follows from the Exercise above and Theorem 7 that $\beta(\mathcal{T}_5)$ is

	0	1	2	3
0	1	–	–	–
1	–	3	–	–
2	–	–	3	–
3	–	–	–	1

so that $\mathcal{T}_5$ is a complete intersection of three quadrics. If $\mathrm{char}(\mathbf{k}) = 3$ then it can be checked that the Betti table is

	0	1	2	3
0	1	–	–	–
1	–	3	2	–
2	–	2	3	–
3	–	–	–	1

while for $\mathrm{char}(\mathbf{k}) = 2$, we have seen that $\beta(\mathcal{T}_5)$ is

	0	1	2	3
0	1	–	–	–
1	–	6	8	3

There is one part of Theorem 7 that can be checked using ideas from Section 1.3:

Exercise 10. Show that if $p < g$ then the 2×2 minors of

$$\begin{bmatrix} z_0 & z_1 & \cdots & z_{g-p} \\ z_p & z_{p+1} & \cdots & z_g \end{bmatrix} \tag{10}$$

vanish on $\mathcal{T}_g$. Show that these minors cut out a scroll $\mathcal{S}$ of dimension p, and conclude that $b_{i,1}(\mathcal{S}) \neq 0$ for $1 \leq i \leq g - p$. Deduce the corresponding non-vanishing for $b_{i,1}(\mathcal{T}_g)$, and derive (using the Gorenstein property) the implication "$\Longleftarrow$" in (9).