Lecture 4

Bounding (bi) degrees of defining equations

[focal point ht 3 GOR]

Let $I = (f_1, \dots, f_n) \subset R = k[x_1, \dots, x_d]$ forms of degree δ

$$\phi : \mathbb{P}^{d-1} \xrightarrow{[f_1 : \dots : f_n]} \mathbb{P}^{n-1}, \qquad \mathbb{P}^{d-1} \twoheadrightarrow X = \operatorname{im}\phi \hookrightarrow \mathbb{P}^{n-1}$$

$$\begin{array}{ccccccc} 0 \to & \mathcal{J} & \to & S = k[\underline{x}, \underline{y}] & \twoheadrightarrow & R[It] = R[f_1 t, \dots, f_n t] \\ & \cup\!\!\!\mid & & \cup\!\!\!\mid & & \cup\!\!\!\mid \\ 0 \to & I(X) & \to & T = k[\underline{y}] & \twoheadrightarrow & A(X) = k[f_1 t, \dots, f_n t] \\ & & & y_i & \mapsto & f_i t \end{array}$$

$\deg x_i = (1,0)$ $\deg y_i = (0,1)$

$I(X) = \mathcal{J}_{(0,*)}$

Def I is of *fiber type* if $\mathcal{J}$ can be obtained by $I(X)$

$$0 \to \mathcal{A} \to \operatorname{Sym}(I) \to \mathcal{R}(I) \to 0$$
$$\mathcal{A} = \mathcal{J}/\mathcal{L}, \qquad \operatorname{Sym}(I) = S/(\mathcal{L})$$

I is of *fiber type* $\iff \mathcal{A} = I(X) \cdot \operatorname{Sym}(I)$

$\iff \mathcal{J} = \mathcal{L} + I(X)S$

$\iff \mathcal{J}$ is generated in degrees $(*,1)$ and $(0,*)$

$\mathcal{A}$ = R-torsion of $\operatorname{Sym}(I)$ $\qquad \mathfrak{m} = (\underline{x})$

$= H^0_{\mathfrak{m}}(\operatorname{Sym}(I))$

I_p of linear type $\forall p \neq \mathfrak{m}$

$\Leftarrow$ I G_d + SCM on the punctured spectrum

for instance

Assume $\mathcal{A} = H^0_{\mathfrak{m}}(\mathrm{Sym}(I))$

$$\boxed{\text{When is } I \text{ of fiber type?}}$$

I is of fiber type $\iff H^0_{\mathfrak{m}}(\mathrm{Sym}(I(\delta)))$ is generated in x-degree 0

$\iff H^0_{\mathfrak{m}}(\underbrace{\mathrm{Sym}_j(I(\delta))}_{\text{finite } R\text{-module}})$ " " $\forall j$

Fix j: If we had a minimal homogeneous free resolutions of $M = \mathrm{Sym}_j(I(\delta))$

$$\mathbb{F}_\bullet \twoheadrightarrow M$$

then $^*\mathrm{soc}(M) \cong k \otimes F_d(d)$

$\Rightarrow \mathrm{topdeg}\ \mathrm{soc}(M) = \underbrace{\mathrm{top\,gen\,deg}}_{b(F_d)}(F_d) - d$ topgen deg = largest generator degree

$\|$

$\therefore \Rightarrow \mathrm{Topdeg}\ H^0_{\mathfrak{m}}(M) = b(F_d) - d$

or use Serre duality and compute $H^0_{\mathfrak{m}}(M)$ via the resolution of M

*recall $\mathrm{soc}(M) = \underset{M}{0 :} \mathfrak{m} = \underset{\text{Koszul hom}}{H_d}(\underline{x}; M) = \mathrm{Tor}^d(k, M)$

Approximate Resolutions

Let $C_\bullet : \ \dots \ C_1 \to C_0 \to 0$

hom complex of finite R-modules with $H_0(C_\bullet) = M$

Assume $\dim H_j(C_\bullet) \le j \quad \forall j > 0$

Gruson - Lazarsfeld - Peskine

Prop [GLP, Chardin, KPU] If $\operatorname{depth} C_j \geq j+1 \quad 0 \le j \le d-1$

$\Rightarrow H^0_{\mathfrak{m}}(M)$ is <u>concentrated</u> in degrees $\le b(C_d) - d$

Thm [KPU] If $\operatorname{depth} C_j \geq j+1 \quad 0 \le j \le d-1$

$\Rightarrow H^0_{\mathfrak{m}}(M)$ is <u>generated</u> in degrees $\le b(C_{d-1}) - d + 1$

How do we get approximate resolutions for the symmetric powers?

Examples/Theorems:

1. $\operatorname{ht} I = 2 \quad R/I$ CM

[KPU]

$$0 \to \bigoplus_{j=1}^{n-1} R(-\varepsilon_j) \xrightarrow{\varphi} R^n \to I(\delta) \to 0 \qquad \varepsilon_1 \geq \varepsilon_2 \geq \dots$$

If I is G_d $\Rightarrow$ $\mathcal{A} = H^0_{\mathfrak{m}}(\operatorname{Sym}(I(\delta)))$ is concentrated

In x-degrees $\leq \sum_{j=1}^{d} \varepsilon_j - d$ and is concentrated in $\underline{x}$-degrees $\leq \sum_{j=1}^{d-1} \varepsilon_j - d + 1$

Approximate resolutions: are strands of the Koszul complex b/c on the punctured spectrum the symmetric algebra is generated by a regular sequence.

2. ht $I = 3$ $\quad$ R/I Gorenstein

[KPU]

$$R^n(\varepsilon) \to R^n \to I(\delta) \to 0$$

If I is $G_d \Rightarrow \mathcal{A} = H^0_m(\mathrm{Sym}(I(\delta))$ is concentrated in x-degrees

$$\leq \begin{cases} d(\varepsilon-1) & \text{if } d \text{ is odd} \\ d(\varepsilon-1) + \frac{n-d-1}{2}\varepsilon & \text{if } d \text{ is even} \end{cases}$$

and is generated in x-degrees $\leq (d-1)(\varepsilon-1)$

<u>In particular</u> : $\varepsilon = 1$ (i.e. I is linearly presented)

$\Rightarrow$ I is of fiber type

Approximate resolutions: complexes provided by Kustin-Ulrich

In this case the condition is either vacuous or is I g.c.i.

3. Assume $\dim R/I \leq 1$. If I has $G_d \Rightarrow \mathcal{A}$ is concentrated in degrees $\leq d(\mathrm{reg}\, I - \delta)$

[PU]

and is generated in degrees $\leq (d-1)(\mathrm{reg}\, I - \delta)$

Approximate resolutions: use the Weyman construction they do not satisfy the depth assumptions but for concentration degree a gen. of the prop works

Also using as approximate resolution the Z. complexes then Chardin shows that $\mathcal{A}$ is concentrated in deg $\leq (d-1)(\delta-1)-1$

With a different approach estimating the regularity of Tor

Theorem. • [Eisenbud–Huneke–Ulrich] $\dim R/I = 0$ and I has a linear resolution for $\lceil \frac{d}{2} \rceil$ steps

$\Rightarrow$ $\mathcal{A}$ is concentrated in degree 0, namely I is fiber type and $J = \mathcal{L} : m_0$

• [P-Ulrich] $\dim R/I \leq 1$ and I has a linear resolution for $\lceil \frac{d}{2} \rceil$ steps

$\Rightarrow$ $\mathcal{A}$ is generated in degree 0 (that is I is of fiber type)

Example of ideals of fiber type (not with these techniques)

• [Bruns–Conca–Varbaro] Maximal minors of generic matrices

Conjecture: Let φ be a $m \times n$ matrix of variables then $I_t(\varphi)$ is of fiber type

Expected form (up to radical)

Assume I is linearly presented

$$\bigoplus R(-1) \xrightarrow{\varphi} R^n \xrightarrow{[f_1 \dots f_n]} I(\delta) \to 0$$

In this case $I_1(\varphi) = (x_1, \dots, x_d)$. Recall the definition of Jacobian dual

$$\underbrace{[y_1 \dots y_n] \cdot \varphi = [x_1 \dots x_d] \cdot B}_{\text{equations defining } \mathrm{Sym}(I)}$$

$\Rightarrow I_d(B) \subset J$ and if

$J = (I_d(B), \mathcal{L})$ then J is of the expected form

Rmk Expected form $\overset{I \text{ linearly presented}}{\Rightarrow}$ fiber type

Thm:

- Assume $\mathcal{A} = H^0_m(\mathrm{Sym}(I(\delta)))$. If $\mathrm{Sym}_t(I(\delta))$, for some $t \gg 0$, has an approximate free resolution that is linear for the first d-steps then
$$J = \sqrt{\mathcal{L} + I_d(B)} \quad \text{and} \quad I(X) = \sqrt{I_d(B)}$$
- Assume $\dim R/I = 0$ and I has a linear resolution for $\lceil \frac{d}{2} \rceil$ steps then
$$J = \sqrt{\mathcal{L} + I_d(B)} \quad \text{and} \quad I(X) = \sqrt{I_d(B)}$$

Pf of the second part:

[EHU] $\Rightarrow J = \mathcal{L} : m \Rightarrow J$ is the annihilator of the S-module $mS/\mathcal{L}$

A presentation of $mS/\mathcal{L}$ is $[\,\pi \mid B\,]$

π: presentation matrix of m; B: jacobian dual

because $\underset{\substack{\shortparallel \\ \text{generate } \mathcal{L}}}{y \cdot \varphi} = \underline{x} \cdot B$

$$\Rightarrow J = \mathrm{ann}_S\, mS/\mathcal{L} \subset \sqrt{\mathrm{Fitt}_0(mS/\mathcal{L})} \subset \sqrt{(m, I_d(B))}$$

$$\Rightarrow I(X) = J_0 = \left(\sqrt{m, I_d(B)}\right)_0 \subset \sqrt{I_d(B)} \subset I(X)$$

$$\Rightarrow I(X) = \sqrt{I_d(B)} \underset{\substack{I \text{ is of} \\ \text{fiber type by EHU}}}{\Longrightarrow} J = \sqrt{(\mathcal{L}, I_d(B))}$$

$\square$

Theorem [Kustin-P-Ulrich] I grade 3 Gorenstein linearly presented with G_d

d odd $\Rightarrow$ $\mathcal{J}$ has the expected form

d even $\Rightarrow$ $\mathcal{J}$ is $\mathcal{L} + I_d(B)$ + content ideal of a polynomial obtained from φ and B

Ex: 5 general points in $\mathbb{P}^3$

Thm [P-Ulrich] (R, m, k) local or $*$-local Gorenstein, $d = \dim R$, $|k| = \infty$

I perfect Gor grade 3 with G_d, $n = \mu(I) > d$

may assume otherwise is linear type and $\mathcal{R}(I)$ is CM

$\mathcal{R}(I)$ CM $\iff$ $n = d+1$ and $I_1(\varphi)$ is generated by the entries of a single generalized row

serious restriction on the # of gens of I NOT present for grade 2

$\Downarrow$ this forces b/c BE

$I_1(\varphi)$ to be a c.i.

FLAT DESCENT reduce to $\Downarrow$ if I is generated by forms of the same degree

φ linearly presented