2nd Lecture

Ex: $I = (x,y) \cap (z,w) \subset k[x,y,z,w]$ is G_∞ but is not Linear type.

Sufficient condition for linear type

$\mathcal{M}$ acyclic $\Rightarrow I$ linear type

When is $\mathcal{M}$ acyclic?

$\boxed{\text{Since } I \text{ is linear type} \Rightarrow I\ G_\infty}$ We can assume G_∞. In this case the complex is short enough to apply the acyclicity Lemma. We also need the depth H_j to be big enough.

Acyclicity Lemma: $C_\cdot: 0 \to C_n \to \dots \to C_0$ a complex of finite module over a Noetherian local ring. Assume $\forall i > 0$

$$\left.\begin{array}{l} \cdot\ H_i(C_\cdot) = 0 \text{ or depth } H_i(C_\cdot) = 0 \\ \cdot\ \operatorname{depth} C_i \geq i \end{array}\right\} \Rightarrow C_\cdot \text{ is acyclic}$$

Since depth $C_i \geq i$ it forces the complex to have length $\leq \dim R$

(R, m) CM local or $*$-local

Def: The ideal I has sliding depth (SD) if

$$\operatorname{depth} H_i \geq d - n + i$$

where $H_i = H_i(f_1, \dots, f_n)$ ith homology of the Koszul homology of the chosen gens $f_1 \dots f_n$ but SD is indep. of the choice of gens

The way we apply AL we localize at the minimal primes of $\operatorname{Supp}(\bigoplus_{i>0} H_i)$.

In our case we localize the graded pieces of the $\mathcal{M}$. complex at one of these minimal primes so the homology is zero or finite length and we just need to check the second condition. But if I is G_∞, SD condition gives us globally and locally depth $H_i \geq i$, which is exactly what we need.

$\underline{\text{Ex}}$:

$$I \text{ SD}$$
$$\Uparrow$$
$$I \text{ SCM}: \ H_i \text{ CM } \ \forall i$$
$$\Uparrow \ \text{Huneke}$$
$$I \text{ licci}$$
$$\Uparrow \ \text{Apery-Gaeta, Watanabe}$$

perfect grade 2, perfect Gor grade 3

the reverse implications do not hold

$\underline{\text{Ex}}$ (M. Hw): $I = I_2 \begin{bmatrix} x_1 & x_2 & x_3 \\ x_2 & x_4 & x_5 \\ x_3 & x_5 & x_6 \end{bmatrix} \subseteq k[x_1, \dots, x_6]$ satisfies SD but not SCM

Thm: [Herzog - Simis - Vasconcelos] $\operatorname{ht} I > 0$

I G_∞ + SD $\Rightarrow$ $\mathcal{M}$. acyclic

In particular, I is linear type and blowup algebras are all CM

follow from the acyclicity lemma. Once $\mathcal{M}$ is acyclic we obtain linear type, in addition $\mathcal{M}$ acyclic gives $\mathcal{Z}$ acyclic and chasing depth on the $\mathcal{Z}$-complex gives us the CMness of the Rees ring.

Examples: The following ideals are of linear type with CM blowup algebras

① I complete intersection

$$\mathcal{R}(I) \cong \mathrm{Sym}(I) = \frac{R[y_1, \dots, y_n]}{I_2\begin{bmatrix} y_1 & \cdots & y_n \\ f_1 & \cdots & f_n \end{bmatrix}}$$

Clearly also any licci with G_∞, for example grade 2 perfect and grade 3 Gor perfect with G_∞

② Some determinantal ideals :

a. φ $n \times n{+}1$ generic matrix $\qquad I = I_n(\varphi)$

b. " $n \times n$ " $\qquad I = I_{n-1}(\varphi)$ (Huneke)

c. φ $n \times n$ generic symmetric matrix $\qquad I = I_{n-1}(\varphi)$ (Kotsev)

③ $X = \{\,:\,.\,\} \subseteq \mathbb{P}^2 \qquad I = I_X \subseteq R = k[x_0, x_1, x_2]$

$\mathrm{ht}\, I = 2 \le \mu(I) = 3 \le \dim R \qquad I = \sqrt{I} \Rightarrow G_\infty$

④ defining ideal of twisted cubic curve in $\mathbb{P}^3_k$

But $X = \{6 \text{ pts not on a conic}\} \subseteq \mathbb{P}^2_k \qquad I = I_X \subseteq R = k[x_0, x_1, x_2]$

$\mathrm{ht}\, I = 2 < \mu(I) = 4 > \dim R$

I is G_3 but NOT G_∞

Next GOAL Replace $G_\infty \rightsquigarrow G_d$

hence linear type $\rightsquigarrow$ linear type in the punctured spectrum

§ 3 NOT of linear type : grade $I = 2$

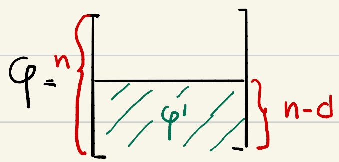
(R, m, k) or $*$-local Gor. , $|k| = \infty$ $\quad d = \dim R$

I perfect grade $I = 2$ $\quad I\ G_d$ $\quad n = \mu(I)$

$$0 \to R^{n-1} \xrightarrow{\varphi} R^n \longrightarrow I \to 0$$ with repect to a general generating set

$$y_i \mapsto f_i$$

May assume : $n > d$ $\quad$ otherwise $I\ G_\infty$



Thm [Ulrich] $\mathcal{R}(I)$ CM $\iff I_{n-d}(\varphi) = I_{n-d}(\varphi')$

Example : $R = k[x_1, \dots, x_d] \supset I$ $\quad \varphi$ linear $\Rightarrow \mathcal{R}(I)$ CM

What about the defining equations : there are not only the linear eqs so where are the other coming from? We will explain this in the simplified setting of the example.

In the setting of the exemple :

recall that the entries are the def. eqs of the sym Algebra.

Jacobian dual

$$[y_1, \dots, y_n] \cdot \varphi = [x_1, \dots, x_d]\, B \qquad B \in M_{d \times n-1}(k[y_1, \dots, y_n]) \text{ with}$$

linear entries in $\underline{y}$

$$\underline{y} \cdot \varphi = 0 \text{ in } \mathcal{R}(I) \Rightarrow I_d(B) \cdot x_i = 0 \quad \forall i \text{ in } \mathcal{R}(I)$$

$$\Rightarrow I_d(B) = 0 \text{ in } \mathcal{R}(I) \text{ as } x_i \text{ NZD in } \mathcal{R}(I)$$

$$\Rightarrow \underbrace{\frac{R[y_1, \dots, y_n]}{(\underline{x} \cdot B,\ I_d(B))}}_{\text{geometric residual } \cap} \twoheadrightarrow \mathcal{R}(I)$$

$\cong$ $\quad$ expected form

Thm [Morey - Ulrich] $\mathcal{R}(I)$ is of the expected form

the example of last time of 6 general pts or more generally

Ex $I = I_X$ $\quad X = \{ \binom{e}{2}$ points not on a curve of degree $e-2 \} \subseteq \mathbb{P}^2_k$

$\Rightarrow$ $\mathcal{R}(I)$ is CM of the expected form

For $d=3$ generalizations of Lan, Dona - Ramos - Simis

Graphs and Images of rational Maps

For a moment we don't assume grade 2

$R = k[x_1, \dots, x_d]$ $\quad k = \bar{k}$ $\quad I = (f_1, \dots, f_n)$ forms of degree δ

$$\begin{tikzcd}
\Phi : \mathbb{P}^{d-1}_k \arrow[r, dashed] \arrow[dr, two heads] & \mathbb{P}^{n-1}_k & \mathbb{P}^{d-1} \times \mathbb{P}^{n-1} \arrow[l, two heads] \\
& X = \operatorname{Im}\Phi \arrow[u, hook] & \operatorname{graph}\Phi \arrow[u, phantom, "\cup"] \arrow[ul, two heads]
\end{tikzcd}$$

Φ rational map defined off $V(I)$

$$P \longmapsto [f_1(P) : \dots : f_n(P)]$$

defining eqs of $\mathcal{Y}(I)$ $\qquad$ defining ideal of $\mathcal{R}(I)$

$$\begin{tikzcd}
& 0 \arrow[d] & & 0 \arrow[d] \\
& I(X) \arrow[d] & \subset & J \arrow[d] \\
y_i \arrow[d, mapsto] & T = k[\underline{y}] \arrow[d] & \subset & S = k[\underset{(1,0)}{\underline{x}}, \underset{(0,1)}{\underline{y}}] \arrow[d] \\
f_i t & \mathcal{Y}(I) = A(X) = k[f_1 t, \dots, f_n t] & \subset & \mathcal{R}(I)
\end{tikzcd}$$

$\operatorname{Im}\Phi$ $\qquad$ $\mathbb{P}^{d-1} \times \mathbb{P}^{n-1}$ $\qquad$ $\operatorname{graph}\Phi$

we obtain $I(X)$ immediately from J by setting the x's $= 0$.