Blowup Algebras

Lectuc 1

§ Motivations

Why study them? Where do they play a role?

1. Resolution of singularities : The Blowup Algebras are the algebras associated to the blowup of a variety along a subvariety

2. Macauleyfication : (a mild version of resolution of singularities) when one does not require the new scheme to be smooth or regular but only CM, solved by Kawasaki

3. Multiplicity theory : If a ring is graded one has the definition of HF as the vector space dimension of each component. One can mimic this using the Blowup algebras in a local ring.

4. Generalized multiplicity : which takes care of ideals and modules that are not necessarely of finite colength in a free module

5. Study of asymptotic properties of ideals These algebras allow one to study all powers of an ideal at once or all the members of a filtration at once. For instance we can investigate the number of generators and the regularity of all large powers

6. Integral Dependence : the blowup algebras are the environment in which integral dependence of ideals and modules takes place. For instance a question like: How many powers of an ideal need to be tested in order to guarantee that all powers are integelly closed is a problem about Rees algebras.

7. Briançon-Skoda Thms, cores, and multiplier ideals
8. Equisingularity Theory : The goal is to devise criteria for analytic sets that occur in a flat family to be "alike" Most of these criteria are multiplicity based which are expressed and worked out via Rees algebras of ideals and modules.
9. Symbolic powers and set theoretic complete intersections The Noetherianess of the symbolic Rees algebras implies that a 1-dimensional prime ideal is a stci
10. Images and Graphs of rational maps between projective spaces are given algebraically by special fiber rings and Rees rings.
11. Computing defining equations or syzygies of Segre products, Veronese embeddings, Gauss images, tangential and secant varieties, projections ...
12. Applications to applied mathematics for instance geometric modeling and philogenetics
13. Study of singularities of images of rational maps
14. Rees like algebras were used to disprove the Eisenbud-Goto conjecture

These are just some of the motivations to study Blowup Algebras

§1 Definitions

R Noetherian   sometimes local (R, m) sometimes graded

I ideal (if R is graded I is homogeneous)

Symmetric algebra $\mathrm{Sym}(I) = \bigoplus_{j=0}^{\infty} \mathrm{Sym}_j(I)$

$\downarrow$

where t is a variable

Rees ring $\mathcal{R}(I) = R[It] = \bigoplus_{j=0}^{\infty} I^j t^j \subset R[t]$ $\mathrm{Proj}(\mathcal{R}(I))$ = blowup of $\mathrm{Spec}(R)$ along $V(I)$

graded epimorphism $\downarrow$

associated graded ring $gr_I(R) = \mathcal{R}(I) \otimes_R R/I = \bigoplus_{j=0}^{\infty} I^j/I^{j+1}$ $\mathrm{Proj}(gr_I(R))$ = exceptional fiber

$\downarrow$

special fiber ring $\mathcal{F}(I) = \mathcal{R}(I) \otimes_R R/m = \bigoplus_{j=0}^{\infty} I^j/mI^j$ $\mathrm{Proj}(\mathcal{F}(I))$ = special fiber

I is proper

they are all called blowup algebras.

§2 Questions

- relations between the blowup algebras
- CMness of blowup algebras
- Gorenstein, normality, multiplicity
- canonical module
- defining equations
- bounds on (bi) degree of defining equations
- connection with singularities of images of rational maps

$d = \dim R$

Forster #

Thm • $\dim Sym(I) = \sup\{\mu(I_p) + \dim R/p \mid p \in Spec(R)\}$
(Huneke - Rossi)

(HW) • $ht\, I > 0 \Rightarrow \dim \mathcal{R}(I) = d+1$

• (R,m) $I \neq R \Rightarrow \dim gr_I R = d$

$\Rightarrow \ell(I) := \dim \mathcal{F}(I) \leq d$ analytic spread

§3 Linear type

One studies the Rees algebra via the epimorphism from the symmetric algebra. One reason to do this is that the defining equations of the symmetric algebras are well known:

① If F is a free module with basis $y_1, \dots, y_n$ then

(HW) $$\mathrm{Sym}(F) \cong R[y_1, \dots, y_n]$$ is a polynomial ring in the variables $y_1, \dots, y_n$

② If E is a finite R-module we look at a presentation

(HW) $$0 \to \underset{\substack{\text{cycles} \\ \text{syzygy module}}}{Z} \to F = Ry_1 \oplus \dots \oplus Ry_n \to E \to 0$$

$\Rightarrow$ applying the symmetric algebra functor to the last map

$$0 \to Z \cdot \mathrm{Sym}(F) \to \mathrm{Sym}(F) = R[y_1, \dots, y_n] \to \mathrm{Sym}(E) \to 0$$

Concretely: $$R^m \xrightarrow{\varphi} F \to E \to 0$$

$$\Rightarrow \quad \mathrm{Sym}(E) = \frac{R[y_1, \dots, y_n]}{([y_1 \cdots y_n] \cdot \varphi)}$$

$$(\underline{y} \cdot \varphi) := \mathcal{L}$$

the converse is also true, any quotient of a polynomial ring by an ideal generated by linear forms in the y's is the symmetric algebra of a module

Since we know the defining eqs of the symmetric algebra, we also know the defining eqs of the Rees algebra if the two are $\cong$ and in this case we say that the ideal is of linear type

Definition I is of *linear type* if $\quad \operatorname{Sym}(I) \underset{\text{nat}}{\overset{\sim}{\twoheadrightarrow}} \mathcal{R}(I)$

this is the best case scenario, when does it happen?

Necessary condition which shows that the linear type property is rather restrictive $\quad (R, m, k) \quad I \subset m$

Assume first local, then linear type $\Rightarrow$

$$\mathcal{F}(I) = \mathcal{R}(I) \otimes_R k = \operatorname{Sym}(I) \otimes_R k$$
$$= \operatorname{Sym}(I \otimes_R k)$$
$$= \operatorname{Sym}_k(k^{\mu(I)})$$

Compute dimension we obtain

$$d \geq \ell(I) = \mu(I)$$

Hence in any Noetherian ring Apply this locally at prime in $V(I)$ and we obtain

$$\mu(I_p) \leq \dim R_p \qquad \forall p \in V(I) \quad := \quad G_\infty$$

$\therefore$ I linear type $\Rightarrow$ I G_∞

(Hw) Assume $\operatorname{ht} I > 0$: I is G_∞ $\iff$

$$\operatorname{ht} \operatorname{Fitt}_j(I) \geq j+1 \qquad \forall j \geq 1$$

how about sufficient conditions? At least if the grade $I>0$ linear type is equivalent to the symmetric algebra being torsion free over R. To check this it would be useful to have a resolution of the symmetric algebra. In general such resolutions are hard to find, however a substitute is the approximation complex introduced by HSV

$$y_i \longmapsto f_i$$

$$0 \to \underset{\text{syzygies of } I}{Z} \xrightarrow{\varphi} F = Ry_1 \oplus \dots \oplus Ry_n \to I \to 0$$

$S = \mathrm{Sym}(F)$ we had already seen

$$0 \to S\cdot Z \to S \to \mathrm{Sym}(I) \to 0 \implies S \otimes Z(-1) \xrightarrow{\tilde{\varphi}} S \to \mathrm{Sym}(I) \to 0$$

Consider the koszul complex of $\tilde{\varphi}$: $K_\cdot(\tilde{\varphi})$

$$K(\tilde{\varphi}): \dots \to S \otimes_R \overset{i}{\Lambda} Z(-i) \to S \otimes \overset{i-1}{\Lambda} Z(-i+1) \to \dots \to S \otimes Z(-1) \to S$$

with the usual differentials (this is the def of Koszul complex of BH or Eisenbud, if you are used to notation is the generalization there $S^n \xrightarrow{\varphi} S$ $K_\cdot(\varphi)$.)

this is a homogeneous complex of S-modules

Rmk $H_0(K_\cdot(\tilde{\varphi})) = \mathrm{Sym}(I)$

This complex is too long to satisfy the acyclicity lemma we get rid of torsion (notice that for $j \geq n$ $\overset{j}{\Lambda} Z$ is torsion) by passing to the double dual then

Each module in the complex is the same as $S \otimes (\Lambda Z)^{**}$ into R

If grade $I \geq 2$ $\implies$ $(\overset{i}{\Lambda} Z)^{**} \xrightarrow[\text{nat}]{\sim} Z_i$

Z is the module of 1-cycle of the koszul complex of $f_1, \dots, f_n$

Since the koszul complex has an algebra structure there is a natural map from $\overset{i}{\Lambda} Z$ to the ith cycle

when grade $I \geq 2$ by taking double dual this becomes an $\simeq$

For any ideal we obtain Z-complex

we consider this complex even if the grade I is not at least 2.

$$Z_\bullet : \quad 0 \to S \otimes_R Z_n(-n) \to \dots \to S \otimes_R Z_i(-i) \to \dots \to S \otimes Z(-1) \to S$$

Rmk: $H_0(Z_\bullet) = \mathrm{Sym}(I)$

the homology does not change since $K(\varphi)^{**}$ and Z are the same in degree 1 and 0.

The graded pieces of Z are

$$Z_{j\bullet} : \quad 0 \to S_0(F) \otimes Z_j \to \dots \to S_{j-i}(F) \otimes Z_i \to \dots \to S_{j-1}(F) \otimes Z_1 \to S_j(F) \to \underbrace{S_j(I) \to 0}$$

"resolve" the j-th symmetric power of I if it is exact.

with $H_0(Z_{j\bullet}) = \mathrm{Sym}_j(I) = S_j(I)$

FACT One considers the double complex of graded S-modules with rows $K_\bullet(y_1, \dots, y_n; S)$ and cols $K_\bullet(f_1, \dots, f_n) \otimes_R S$

Taking vertical cycles one gets the Z-complex

Taking vertical homology one gets the $\mathcal{M}_\bullet$ complex

Rmk $H_0(\mathcal{M}_\bullet) = \mathrm{Sym}_{R/I}(I/I^2)$

these complexes depend on the choice of the gens but their homology does not

HW

HW:

Prop ① $\exists$ long exact sequence of homology

$$\to H_i(Z_{j+1\bullet}) \to H_i(Z_{j\bullet}) \to H_i(\mathcal{M}_{j\bullet}) \to H_{i-1}(Z_{j+1\bullet}) \to \dots \quad \text{for } \begin{cases} i > 0 \text{ or} \\ j \neq -1 \end{cases}$$

② [for $i=0$] $\quad S_{j+1}(I) \xrightarrow{\lambda_j} S_j(I) \to S_j(I/I^2) \to 0 \quad$ for $j \neq -1$

where $\lambda_j(\underbrace{a_1 \cdot \dots \cdot a_{j+1}}_{\in S_{j+1}}) = a_1 \lambda_j(\underbrace{a_2 \cdot \dots \cdot a_{j+1}}_{\in S_j})$ is the downgrading homomorphism

Thus $\mathcal{M}_\bullet$ is acyclic $\overset{\text{HW}}{\Longleftrightarrow}$ $\mathcal{Z}_\bullet$ is acyclic and λ is injective

$$\Updownarrow \text{HW}$$

I is of linear type

Q: When is $\mathcal{M}_\bullet$ acyclic?