2

Conducting Composites

We begin the study of structural optimization by optimizing conducting media. This chapter introduces the subject of optimization. We describe the equations for equilibrium of conductivity in inhomogeneous media. We also discuss conducting composites and homogenization–the averaging of fields in micro-inhomogeneous media and the tensor of effective properties of a composite.

2.1 Conductivity of Inhomogeneous Media

2.1.1 Equations for Conductivity

Many physical processes are described by the conductivity or transport equations. The equilibria of electrical and thermal conduction are among them, where the electrical potential and temperature play the role of potentials, and various diffusion equilibria, where the concentration of the diffusive substance is the potential. Transport processes include chemical diffusion, flow in porous media, and the steady-state electrical field in a dielectric. The conductivity equations are derived from a few general conservation laws; they are applicable to various physical situations. In the text we often refer to thermal or electrical conduction when specific problems of structural optimization are discussed. However, the results can be equally well applied to other physical processes.

We consider the steady-state conductivity equilibrium. All variables are independent of time, and they depend only on the space coordinates $\mathbf{x} =$

(x_1, x_2, x_3). A detailed discussion of the conductivity equations can be found in standard textbooks on mathematical physics, such as (Courant and Hilbert, 1962) or physics such as (Landau and Lifshitz, 1984). Here we review the conductivity equations emphasizing the inhomogeneity of media. For definiteness, let us look at the electrical conductivity:

Current

Conductivity assumes that a current of particles passes through a medium. Let us denote the vector of the current by $\mathbf{j} = [j_1, j_2, j_3]$. The current satisfies a differential constraint (called the kinetic equation) that corresponds to the conservation of charge: The total number of particles that cross the boundary of any subdomain from inside and outside equals zero. By Green's theorem, $\mathbf{j}$ is divergencefree in Ω:

$$\nabla \cdot \mathbf{j} = 0 \quad \text{in } \Omega. \tag{2.1.1}$$

If we assume that sources or sinks with intensity $f(\mathbf{x})$ are present, then (2.1.1) takes a more general form:

$$\nabla \cdot \mathbf{j} = f \quad \text{in } \Omega. \tag{2.1.2}$$

It says that the difference between the number of particles that cross the boundary of a domain from inside and outside is equal to the density of the sources in that domain.

Field

The second equation of conductivity specifies the force field $\mathbf{e}$ that causes the motion of particles. We assume that the system is conservative. This implies the existence of a potential $w = w(\mathbf{x})$ for $\mathbf{e}$:

$$\mathbf{e} = \nabla w. \tag{2.1.3}$$

Constitutive Relations

The last equation $\mathbf{j} = \mathbf{j}(\mathbf{e})$ is the constitutive relation. It specifies the material properties mathematically as the dependence of $\mathbf{j}$ on $\mathbf{e}$. This dependence completely defines the conducting material.

Here we assume that this dependence is linear:

$$\mathbf{j} = \boldsymbol{\sigma}\mathbf{e}, \tag{2.1.4}$$

where $\boldsymbol{\sigma}$ is a positive definite symmetric tensor:

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^T, \quad \boldsymbol{\sigma} > 0.$$

We call $\boldsymbol{\sigma}$ the *conductivity tensor*. Formally, a linear conducting material is specified by its conductivity tensor.

Inhomogeneous Materials

The constitutive relations in isotropic materials express the proportionality between vectors $\mathbf{j}$ and $\mathbf{e}$. Inhomogeneous isotropic materials correspond to conductivity tensors of the form

$$\boldsymbol{\sigma}(\mathbf{x}) = \sigma(\mathbf{x}) I,$$

where $\sigma(\mathbf{x})$ is a scalar function and I is the identity matrix.[1]

In an inhomogeneous medium, the value of $\boldsymbol{\sigma}$ differs from one location to another ($\boldsymbol{\sigma} = \boldsymbol{\sigma}(\mathbf{x})$). We are especially interested in a description of a piecewise constant layout $\boldsymbol{\sigma}(\mathbf{x})$ that corresponds to a medium assembled from pieces of materials of different conductivities. Suppose that Ω is parted into several subdomains Ω_i, each of which contains a material with spatially constant properties $\boldsymbol{\sigma}_i$. The conductivity of the assembled medium is represented as

$$\boldsymbol{\sigma}(\mathbf{x}) = \sum_i \chi_i(\mathbf{x}) \boldsymbol{\sigma}_i$$

where χ_i is the characteristic function of the ith subdomain:

$$\chi_i = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in \Omega_i, \\ 0 & \text{if} \quad \mathbf{x} \notin \Omega_i. \end{cases} \tag{2.1.5}$$

The Second-Order Conductivity Equation

The system of equations (2.1.2), (2.1.3), and (2.1.4) allows us to determine the potential w from the sources f and the boundary conditions. This system is equivalent to the equation of second order,

$$\nabla \cdot \boldsymbol{\sigma} \nabla w = f \quad \text{in} \quad \Omega, \tag{2.1.6}$$

called the conductivity equation.

Remark 2.1.1 *Notice that $\nabla \cdot \mathbf{A} \nabla w \equiv 0$ if $\mathbf{A}$ is an antisymmetric tensor. This explains the symmetry of the conductivity tensor $\boldsymbol{\sigma}$: The solution to (2.1.6) does not depend on the antisymmetric part of $\boldsymbol{\sigma}$.*

The boundary conditions may have different forms. Generally, we consider the following mixed boundary value problem: The boundary $\partial\Omega$ of Ω consists of two components $\partial\Omega = \partial\Omega_1 \cup \partial\Omega_2$. The potential w is prescribed on $\partial\Omega_1$, and the normal component of the current is prescribed on $\partial\Omega_2$:

$$\begin{aligned} &w = \rho_1 && \text{on } \partial\Omega_1, \\ &\mathbf{n} \cdot \mathbf{j} = \mathbf{n} \cdot \boldsymbol{\sigma} \nabla w = \rho_2 && \text{on } \partial\Omega_2, \end{aligned} \tag{2.1.7}$$

[1] As a rule, we use the bold letters to denote vectors and tensors and plain letters to denote scalars. For example, $\boldsymbol{\sigma}$ means the conductivity tensor and σ means the scalar isotropic conductivity. However, the unit matrix is denoted by the plain italic I.

where ρ_1 and ρ_2 are given functions of the surface $\partial\Omega_i,\ i = 1, 2$.

If $\partial\Omega = \partial\Omega_1$, then the boundary value problem (2.1.6), (2.1.7) is called the *Dirichlet problem*, and if $\partial\Omega = \partial\Omega_2$, the problem is called the *Neumann problem*.

Note that passing from the system (2.1.2), (2.1.3), and (2.1.4) to the second-order equation (2.1.6) formally requires additional assumptions of smoothness of $\boldsymbol{\sigma}$ if (2.1.6) is considered in the classical sense. At the same time, the system (2.1.2), (2.1.3), and (2.1.4) does not require even the continuity of $\boldsymbol{\sigma}$. Naturally, we want to consider discontinuities in $\boldsymbol{\sigma}$ no matter what form of equation is used. Therefore, we understand the solution to (2.1.6) in the *weak sense* (Shilov, 1996): The integral equality

$$\int_\Omega (\nabla v \cdot \boldsymbol{\sigma} \nabla w + f v) + \oint_{\partial\Omega_1} \boldsymbol{\sigma} \nabla v \cdot \mathbf{n}(w - \rho_1) + \oint_{\partial\Omega_2} v(\mathbf{n} \cdot \boldsymbol{\sigma} \nabla w - \rho_2) = 0 \quad (2.1.8)$$

holds[2] for any test function $v \in H^1(\Omega)$.

Differential Constraints and Potentials

The system (2.1.2), (2.1.3), and (2.1.4) admits an equivalent representation called the dual form of (2.1.7). To derive this form we notice that the representation $\mathbf{e} = \nabla w$ implies a differential constraint on $\mathbf{e}$, because all components of $\mathbf{e}$ are determined by one scalar field w. The constraints have the form

$$\nabla \times \mathbf{e} \equiv 0. \quad (2.1.9)$$

Indeed, the vector $\nabla \times \mathbf{e} = \nabla \times \nabla w$ consists of components of the type $\frac{\partial^2 w}{\partial x_i \partial x_j} - \frac{\partial^2 w}{\partial x_j \partial x_i}$ which vanish identically due to integrability conditions.

Similarly, the differential constraint $\nabla \cdot \mathbf{j} = 0$ is identically satisfied if $\mathbf{j}$ corresponds to a vector potential $\mathbf{y}$:

$$\mathbf{j} = \nabla \times \mathbf{y}. \quad (2.1.10)$$

The vector potential $\mathbf{y} = [y_1, y_2, y_3]$ is determined up to the gradient of a scalar field ψ which can be chosen arbitrarily. Indeed,

$$\nabla \times \mathbf{y} = \nabla \times (\mathbf{y} + \nabla\psi).$$

Therefore, $\mathbf{y}$ depends on two arbitrary potentials: The number of independent functions (two) agrees with the number of components of a current vector $\mathbf{j}$ (three) reduced by one differential constraint $\nabla \cdot \mathbf{j} = 0$.

Table 2.1 summarizes the differential constraints and potentials in conductivity. In Chapter 14 (Table 14.1), we will observe similar duality of the potentials and constraints in elasticity equations.

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[2]The symbol "$d\mathbf{x}$" of the differential is omitted in the integrals like $\int_\Omega$ over the explicitly defined domain Ω of the independent variable $\mathbf{x}$.

Variable	Constraints	Potential
Field $\mathbf{e}$	$0=\nabla \times \mathbf{e}$	$\mathbf{e} = \nabla u$
Current $\mathbf{j}$	$0=\nabla \cdot \mathbf{j}$	$\mathbf{j} = \nabla \times \mathbf{y}$

TABLE 2.1. Differential constraints and potentials in conductivity.

Dual Form of Conductivity Equations

Equation (2.1.10) allows us to introduce the vector potential $\mathbf{y}$ in the conductivity problem:

$$\mathbf{j} = \nabla \times \mathbf{y} + \mathbf{j}_0, \quad \nabla \cdot \mathbf{j}_0 = f, \tag{2.1.11}$$

where $\mathbf{j}_0$ is a particular solution to (2.1.2). Vector field $\mathbf{j}_0$ is not uniquely defined and does not depend on the properties of the medium.

Equations (2.1.9), (2.1.11), and the inverse form of the constitutive relations

$$\mathbf{e} = \boldsymbol{\sigma}^{-1}\mathbf{j} \tag{2.1.12}$$

form a system of equations of conductivity that uses a vector potential $\mathbf{y}$ of currents instead of a scalar potential w of forces. The system (2.1.9), (2.1.11), and (2.1.12) is said to be *dual* to the system (2.1.2), (2.1.3), and (2.1.4) and conversely. These systems are equivalent.

The dual form of equation (2.1.6) is the vector equation

$$\nabla \times \boldsymbol{\sigma}^{-1}(\nabla \times \mathbf{y} + \mathbf{j}_0) = 0. \tag{2.1.13}$$

Its solution should also be understood in the weak sense, similar to (2.1.8).

2.1.2 Continuity Conditions in Inhomogeneous Materials

We have already mentioned that conductivity equations do not require the continuity of $\boldsymbol{\sigma}$. What happens to the fields $\mathbf{j}$ and $\mathbf{e}$ on the boundary Γ between the domains where $\boldsymbol{\sigma}$ takes different constant values $\boldsymbol{\sigma}^+$ and $\boldsymbol{\sigma}^-$? Denote the normal to Γ by $\mathbf{n}$ and the tangents – by $\mathbf{t}$ and $\mathbf{b}$.

1. The divergencefree nature of the current $\mathbf{j}$ (2.1.1) indicates that the normal component of $\mathbf{j}$ remains continuous (Figure 2.1):

   $$[\mathbf{j} \cdot \mathbf{n}] = 0, \tag{2.1.14}$$

   where $[z]$ is the jump of a variable z across Γ:

   $$[z] = z^+ - z^-.$$

   Physically, the normal component of the current is equal to the difference in the number of particles that cross the surface Γ from the

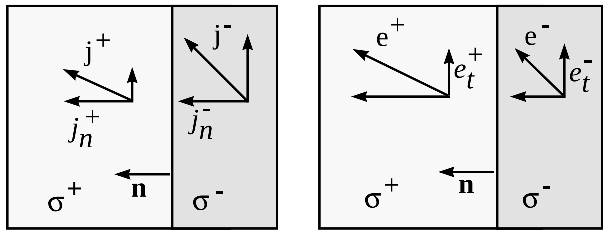

FIGURE 2.1. The refraction of the current and the field on the boundary between two isotropic conductors. The normal component of the current and the tangent component of the field are continuous on the boundary.

left and right (see the kinetic equation (2.1.1)). This number is zero, and therefore $[\mathbf{j} \cdot \mathbf{n}]$ is continuous.

Formally, we also could derive (2.1.14) from equation (2.1.2) written in local coordinates $(\mathbf{n}, \mathbf{t}, \mathbf{b})$:

$$\nabla \cdot \mathbf{j} = \frac{\partial j_n}{\partial n} + \frac{\partial j_t}{\partial t} + \frac{\partial j_b}{\partial b} = f.$$

It implies that the argument $j_n = \mathbf{j} \cdot \mathbf{n}$ of the normal derivative $\frac{\partial}{\partial n}$ is necessarily continuous on the surface Γ. Otherwise, the left-hand side in (2.1.14) would contain a δ-function that lacks its mate on the right-hand side.

Note that the finiteness of the tangent derivatives implies only the continuity of its argument *along* the boundary Γ of both sides, but it does not imply any smoothness of that argument when the boundary is crossed. Generally, we have

$$[\mathbf{j} \cdot \mathbf{t}] \neq 0, \quad [\mathbf{j} \cdot \mathbf{b}] \neq 0. \tag{2.1.15}$$

2. The tangent components of the field $\mathbf{e}$ are continuous due to the continuity of a potential w (Figure 2.1):

$$[\mathbf{e} \cdot \mathbf{t}] = [\mathbf{e} \cdot \mathbf{b}] = 0. \tag{2.1.16}$$

Indeed, the limiting values of w from the left (w^-) and right (w^+) of any point of surface Γ are equal; for two points $\mathbf{x}_1$ and $\mathbf{x}_2$ on Γ we have

$$w_1^+ = w_1^-, \quad w_2^+ = w_2^-;$$

the difference between the potentials at corresponding points is also equal. This implies

$$\frac{w_1^- - w_2^-}{|\mathbf{x}_1 - \mathbf{x}_2|} = \frac{w_1^+ - w_2^+}{|\mathbf{x}_1 - \mathbf{x}_2|},$$

where $w_i = w(\mathbf{x}_i)$. In the limit $|\mathbf{x}_1 - \mathbf{x}_2| \to 0$, the left-hand and right-hand side terms of the last equality represent a tangent derivative on the $(-)$ and the $(+)$ side of Γ. Equation (2.1.16) follows.

Another way to derive this condition is to examine the constraint $\nabla \times \mathbf{e} = 0$. The curl of $\mathbf{e}$ is represented as

$$\nabla \times \mathbf{e} = \left(\frac{\partial e_b}{\partial t} - \frac{\partial e_t}{\partial b}\right)\mathbf{n} - \left(\frac{\partial e_b}{\partial n} - \frac{\partial e_n}{\partial b}\right)\mathbf{t} + \left(\frac{\partial e_t}{\partial n} - \frac{\partial e_n}{\partial t}\right)\mathbf{b} = 0 \tag{2.1.17}$$

where e_n is the normal and e_t, e_b are tangent components of $\mathbf{e}$. The equality $\nabla \times \mathbf{e} = 0$ requires that the normal derivatives $\frac{\partial e_t}{\partial n}$ and $\frac{\partial e_b}{\partial n}$ be finite, hence e_t and e_b are continuous. Otherwise δ-functions occur in the left-hand side of (2.1.17).

The normal component e_n does not need to be continuous. Generally, we have $[\mathbf{e} \cdot \mathbf{n}] \neq 0$.

3. Let us compute the jumps of the discontinuous components of $\mathbf{e}$ and $\mathbf{j}$. The continuous components $\mathbf{e} \cdot \mathbf{t}$ and $\mathbf{e} \cdot \mathbf{b}$ of the field $\mathbf{e}$ correspond to the discontinuous components $\mathbf{j} \cdot \mathbf{t}$ and $\mathbf{j} \cdot \mathbf{b}$ of the current $\mathbf{j}$, and the discontinuous component $\mathbf{e} \cdot \mathbf{n}$ of the field corresponds to the continuous component $\mathbf{j} \cdot \mathbf{n}$ of the current. Together, the vectors of a current and a field have exactly three continuous and three discontinuous components. Let us denote by $\mathbf{d} = [e_n, j_t, j_b]$ the vector of discontinuous components, and by $\mathbf{c} = [j_n, e_t, e_b]$ – the vector of continuous components.

   To compute the jump of the components of $\mathbf{d}$, we solve the state equations (2.1.4) for $\mathbf{d}$:

$$\mathbf{d} = Z(\boldsymbol{\sigma})\mathbf{c},$$

   where the matrix Z is

$$Z(\boldsymbol{\sigma}) = \begin{pmatrix} \frac{1}{\sigma_{nn}} & -\frac{\sigma_{nt}}{\sigma_{nn}} & -\frac{\sigma_{nb}}{\sigma_{nn}} \\ -\frac{\sigma_{nt}}{\sigma_{nn}} & \sigma_{tt} - \frac{\sigma_{nt}^2}{\sigma_{nn}} & \sigma_{tb} - \frac{\sigma_{nt}\sigma_{nb}}{\sigma_{nn}} \\ -\frac{\sigma_{nb}}{\sigma_{nn}} & \sigma_{tb} - \frac{\sigma_{nt}\sigma_{nb}}{\sigma_{nn}} & \sigma_{bb} - \frac{\sigma_{nb}^2}{\sigma_{nn}} \end{pmatrix}$$

   and $\sigma_{nn}, \sigma_{nt}, \sigma_{nb}, \sigma_{bb}, \sigma_{bt}, \sigma_{tt}$ are the components of the tensor $\boldsymbol{\sigma}$ in the coordinates $\mathbf{n}, \mathbf{t}, \mathbf{b}$,

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{nn} & \sigma_{nt} & \sigma_{nb} \\ \sigma_{nt} & \sigma_{tt} & \sigma_{tb} \\ \sigma_{nb} & \sigma_{tb} & \sigma_{bb} \end{pmatrix}.$$

   Now we easily calculate the jump of $\mathbf{d}$ at two neighboring points that lie to the left and right of Γ. Using the continuity of $\mathbf{c}$, we compute

$$[\mathbf{d}] = \left(Z(\boldsymbol{\sigma}^+) - Z(\boldsymbol{\sigma}^-)\right)\mathbf{c}. \tag{2.1.18}$$

For isotropic materials, these relations become

$$\begin{aligned} [e_n] &= \left(\tfrac{1}{\sigma^+} - \tfrac{1}{\sigma^-}\right) j_n, \\ [j_t] &= (\sigma^+ - \sigma^-) e_t, \\ [j_b] &= (\sigma^+ - \sigma^-) e_b. \end{aligned} \tag{2.1.19}$$

The equations (2.1.18) enable us to determine $\mathbf{e}$ and $\mathbf{j}$ on one side of the boundary Γ if they are known on the other side. These formulas are used for calculation of the average fields of a composite. This technique was described in (Backus, 1962).

2.1.3 Energy, Variational Principles

Multidimensional Variational Problems

We can view the conductivity equation as the Euler equation that corresponds to a minimum of some multidimensional variational functional.

First, let us discuss minimizers of multidimensional variational problems. Consider the problem

$$\min_w \int_\Omega G(\mathbf{x}, w, \nabla w) - \oint_{\partial\Omega} g(\mathbf{x}, w) \tag{2.1.20}$$

where G is called the bulk Lagrangian and g is the surface Lagrangian. Suppose that w is a minimizer of (2.1.20). As in the one-dimensional problem, one can derive the necessary condition of optimality for w. The stationary solution to problem (2.1.20) is called the Euler–Lagrange equation. It is a direct multivariable analogue of the one-dimensional Euler equation (1.2.11), (1.2.12). The operator ∇ formally replaces the operator $\frac{d}{dx}$. The Euler–Lagrange equation has the form

$$S(G) = \nabla \cdot G_{\nabla w} - \frac{\partial G}{\partial w} = 0, \tag{2.1.21}$$

where $G_{\nabla w}$ is the vector

$$G_{\nabla w} = \frac{\partial G}{\partial \nabla w} = \left[\left(\frac{\partial G}{\partial (\frac{\partial w}{\partial x_1})} \right), \ \ldots, \ \left(\frac{\partial G}{\partial (\frac{\partial w}{\partial x_d})} \right) \right].$$

Any differentiable minimizer w of the problem (2.1.20) satisfies the Euler–Lagrange equation (2.1.21) and the boundary conditions

$$\delta w \left(\sum_{i=1}^d (G_{\nabla w})\, n_i - \frac{\partial g}{\partial w} \right) = 0, \quad \text{on } \partial\Omega, \tag{2.1.22}$$

where n_i is the ith component of the normal to the boundary $\partial\Omega$.

We do not derive these relations here. The derivation is analogous to the one-dimensional case and can be found in any standard course on calculus of variations (see, for example, (Fox, 1987)). However, we derive similar stationary equations in Section 5.2.

The Dirichlet Variational Principle

The steady-state equilibrium of a conducting body corresponds to the minimal solution to a variational problem called the Dirichlet variational principle (Courant and Hilbert, 1962):

$$I_e(\boldsymbol{\sigma}) = \min_{w \in \mathcal{W}} \int_\Omega (W_e(\nabla w, \boldsymbol{\sigma}) + f\, w) + \int_{\partial\Omega_2} w\, \rho_2, \tag{2.1.23}$$

where

$$\mathcal{W} = \left\{ w : \ w \in H^1(\Omega), \ w|_{\partial\Omega_1} = \rho_1 \right\},$$

ρ_2 is the normal component of applied boundary currents, and w is a potential ($\mathbf{e} = \nabla w$). The quadratic form

$$W_e(\nabla w, \boldsymbol{\sigma}) = \frac{1}{2} \nabla w \cdot \boldsymbol{\sigma} \nabla w$$

is called the *energy* of a conducting body. The Lagrangian $W_e(\nabla w, \boldsymbol{\sigma}) + f\, w$ is composed as a sum of the energy W_e and the work of the sources f in Ω.

The boundary condition

$$w|_{\partial\Omega_1} = \rho_1$$

is called the *main boundary condition*; all minimizers are subject to it. The condition

$$\left. \frac{\partial}{\partial n} \boldsymbol{\sigma} \nabla w \right|_{\partial\Omega_2} = \rho_2$$

is called the *natural* or *variational boundary condition*. It is satisfied at the minimum of $I_e(\boldsymbol{\sigma})$.

The Euler–Lagrange equations (2.1.22) for the Dirichlet variational principle coincide with the equilibrium equations (2.1.6) and (2.1.7). One can also check that the minimizer of the energy of an inhomogeneous medium jumps on the dividing surface between the materials, in accord with (2.1.19).

The Thompson Variational Principle

Similarly, the dual system of conductivity equations (2.1.13) correspond to the Euler–Lagrange equations for the variational problem, called the Thomson variational principle:

$$I_{\mathbf{y}}(\boldsymbol{\sigma}) = \min_{\mathbf{y} \in \mathcal{Y}} \int_\Omega (W_j(\nabla \times \mathbf{y}, \boldsymbol{\sigma}) + \mathbf{j}_0 \cdot \nabla \times \mathbf{y}),$$

where

$$W_j(\nabla \times \mathbf{y}, \boldsymbol{\sigma}) = \frac{1}{2} (\nabla \times \mathbf{y}) \cdot \boldsymbol{\sigma}^{-1} (\nabla \times \mathbf{y}),$$

$\mathbf{j}_0$ is a particular solution to the equation $\nabla \cdot \mathbf{j} = f$, and

$$\mathcal{Y} = \left\{ \mathbf{y} : \ y_i \in H^1(\Omega), \quad (\nabla \times \mathbf{y} + \mathbf{j}_0) \cdot \mathbf{n}|_{\partial\Omega_2} = \rho_2 \right\}.$$

We also assume for simplicity that $\rho_1 = 0$. Recall that $\mathbf{j}_0$ is a particular solution to the equation $\nabla \cdot \mathbf{j} = f$.

Various Expressions for Energy

We have seen that the energy density W in a conducting medium can be written in various forms. It is equal to the scalar product of the current $\mathbf{j}$ and the field $\mathbf{e}$:

$$W(\mathbf{e}, \mathbf{j}) = \frac{1}{2}\mathbf{e} \cdot \mathbf{j}, \quad \text{where } \mathbf{e} = \nabla w, \ \mathbf{j} = \nabla \times \mathbf{y}.$$

Using the constitutive relations, the energy can also be represented either as a quadratic form of the field $\mathbf{e}$,

$$W_e(\mathbf{e}, \boldsymbol{\sigma}) = W(\mathbf{e}, \boldsymbol{\sigma}\mathbf{e}) = \frac{1}{2}\mathbf{e} \cdot \boldsymbol{\sigma}\mathbf{e}, \quad \text{where } \mathbf{e} = \nabla w,$$

or as the quadratic form of the current density $\mathbf{j}$,

$$W_j(\mathbf{j}, \boldsymbol{\sigma}) = W(\boldsymbol{\sigma}^{-1}\mathbf{j}, \mathbf{j}) = \frac{1}{2}\mathbf{j} \cdot \boldsymbol{\sigma}^{-1}\mathbf{j}, \quad \text{where } \mathbf{j} = \nabla \times \mathbf{y}.$$

Each of these forms corresponds to a variational principle; the Euler–Lagrange equations coincide with the equilibrium equations (2.1.2), (2.1.3), and (2.1.4).

Duality of Variational Principles

The Dirichlet and Thompson variational principles are related. Each of them is dual to the other. The duality of extremal problems was introduced in Chapter 1 for one-dimensional problems. The duality for the variational problems with multiple integrals is defined in the same fashion (see, for example, (Ekeland and Temam, 1976)).

Consider a multivariable Lagrangian $L(\mathbf{x}, u, \nabla u)$. Perform the Legendre transform of L, that is, find the dual vector variable $\mathbf{j}$ from the extremal problem (compare with (1.3.25)

$$L^{\text{dual}}(u, \mathbf{j}) = \min_{\nabla u} \left(\mathbf{j} \cdot \nabla u - L(\mathbf{x}, u, \nabla u)\right),$$

which gives $\mathbf{j} = \frac{\partial L}{\partial \nabla u}$. Solving the last equation for ∇u, we express ∇u as a function of $\mathbf{j}$:

$$\nabla u = \boldsymbol{\phi}(\mathbf{j}). \tag{2.1.24}$$

The dual energy is equal to

$$L^{\text{dual}}(u, \mathbf{j}) = \mathbf{j} \cdot \boldsymbol{\phi}(\mathbf{j}) - W(u, \phi(\mathbf{j})).$$

The Euler–Lagrange equation is expressed through $\mathbf{j}$ as:

$$\nabla \cdot \mathbf{j} - \frac{\partial L}{\partial u} = 0. \tag{2.1.25}$$

To satisfy the system (2.1.24), (2.1.25) we introduce a vector $\mathbf{j}_0$, that corresponds to a particular solution to the equation $\nabla \cdot \mathbf{j}_0 - \frac{\partial L}{\partial u} = 0$. The system (2.1.24) and (2.1.25) is equivalent to

$$\mathbf{j} + \mathbf{j}_0 = \nabla \times \mathbf{y}, \quad \nabla \times \boldsymbol{\phi}(\mathbf{j}) = 0. \tag{2.1.26}$$

The vector $\mathbf{y}$ is the dual to u potential. The system (2.1.26) or the equivalent second-order equation

$$\nabla \times \phi(\mathbf{j}_0 - \nabla \times \mathbf{y}) = 0$$

is the Euler–Lagrange equations in the dual variables $\mathbf{y}$.

Remark 2.1.2 *If the Lagrangian $L(u, \nabla u)$ is convex with respect to ∇u, then the Legendre transform is a convolution: The dual to the Lagrangian $L^* = L^{dual}$ coincides with L, that is, $L^{**}(u, \nabla u) = L(u, \nabla u)$. Otherwise, $L^{**}(u, \nabla u)$ is the convex envelope of $L(u, \nabla u)$ with respect to variable ∇u (see the discussion in Chapter 1 and (Ekeland and Temam, 1976)): $L^{**}(u, \nabla u) = \mathcal{C}L(u, \nabla u)$.*

Example 2.1.1 The Dirichlet and Thompson principles correspond to the dual Lagrangians. Indeed, the dual form for the Lagrangian $W = \frac{1}{2}\sigma(\nabla w)^2$ is

$$W^{\text{dual}} = \frac{1}{2\sigma}\mathbf{j}^2,$$

where the dual to $\mathbf{e}$ variable $\mathbf{j}$ is defined as

$$\mathbf{j} = (\sigma \nabla w), \quad \mathbf{j} = \nabla \times \mathbf{y}.$$

In this dual form, the conductivity σ is replaced with the resistivity $\frac{1}{\sigma}$.

2.2 Composites

To be prepared to deal with fine-scale oscillating solutions to structural optimization problems, we need to discuss the methods of homogenization. These methods replace oscillating sequences of property layouts with smooth layouts of the effective properties of composite media. The effective properties become controls in the homogenized problem.

We briefly discuss here micro-inhomogeneous media, which are also called media with microstructures or composites. A detailed exposition of micro-inhomogeneous media can be found in many books; we cite (Bensoussan et al., 1978; Sánchez-Palencia, 1980; Jikov et al., 1994; Bakhvalov and Panasenko, 1989).

2.2.1 Homogenization and Effective Tensor

Assumptions

A composite is viewed as a structure assembled from a very large number of fragments of given materials mixed in a prescribed way. Each fragment is assumed to be much smaller than the rate of varying of acting fields and than the size of a considered domain. At the same time, these domains are large enough to assume that the conductivity equation is valid in each fragment of material, which means that fragments are much larger than molecular sizes, the size of a free path, etc. The way of mixing is assumed to be regular in a sense: The microstructure is periodic, quasiperiodic, or statistically homogeneous. The behavior of a piece of the composite is representative of the behavior of neighboring pieces.

It is hopelessly difficult and often useless to describe fields at each point of the composite. For most purposes we do not need to know all the details. Instead, we simplify the problem by introducing an averaged description of a composite. The procedure that replaces the original problem by a simpler averaged problem is called *homogenization*.

In doing homogenization we replace the fine-scale oscillating inhomogeneous material with properties $\boldsymbol{\sigma}(\chi)$ by a homogeneous material with conductivity $\boldsymbol{\sigma}_*$. This material imitates some important features of the inhomogeneous system, and $\boldsymbol{\sigma}_*$ is called the *effective properties tensor* of the composite. In contrast with the rapidly oscillating layout $\boldsymbol{\sigma}$, the effective tensor $\boldsymbol{\sigma}_*$ is a constant or (in the quasiperiodic case) a smoothly varying tensor function of the point $\mathbf{x}$ of the domain.

Clearly, the simplified system cannot preserve all features of the original one, so we should choose the features we would like to preserve by homogenization. The general homogenization concept is to preserve the solution w of a boundary value problem (2.1.6). The requirement that the solution w in the homogenized system stay close to the solution to the initial system leads to an equation for the effective conductivity tensor $\boldsymbol{\sigma}_*$. The homogenized system also preserves the mean field $\mathbf{e}$ and the mean current $\mathbf{j}$.

Remark 2.2.1 *Much information about the system is lost. Particularly, the homogenization neglects processes determined by individual behavior of fine pieces of material: field concentration in the corner points of grains, cracks, fine-scale oscillations, percolation, etc. Still, the remaining problems are important: For example, we can obtain the temperature, electrical current density, and so on. In elasticity, homogenization preserves the displacement vector and the averaged tensors of stresses and strains.*

Homogenization is a local procedure: We replace the inhomogeneous medium in a small neighborhood Ω_ε by a uniform medium. The fields are replaced by their mean values, i.e., by averages over Ω_ε. It is assumed that the way of mixing materials in Ω_ε is repeated in the entire composite. This principle makes the homogenization procedure simple enough to be effectively applied.

Homogenization is an asymptotic procedure: Its result becomes better as the size of the representative volume of composite material tends to zero. The size of Ω_ε is compared with the size of the domain and the rate of varying of the external fields. The homogenization assumes that the size of the representative volume is much smaller than the other parameters of the system.

Various methods of homogenization were actively developed in the last decades. They were applied to various physical problems including periodic or random arrays of inclusions, suspensions, nonlinear inhomogeneous materials, diffusion in a stream, percolation problems, checkerboard structures, etc. A review of these problems is beyond the scope of this book; the reader is referred to the recent collections (Dal Maso and Dell'Antonio, 1991; Hornung, 1997; Markov and Inan, 1999; Berdichevsky, Jikov, and Papanicolaou, 1999; Markov and Preziosi, 1999) and the books (Christensen, 1979; Bakhvalov and Panasenko, 1989; Nemat-Nasser and Hori, 1993; Berdichevsky, 1997). The remarkable variety of problems and methods of homogenization are described in many papers that represent different aspects of the approach, such as: (Telega, 1990; Khruslov, 1991; Berlyand and Golden, 1994; Panasenko, 1994; Bourgeat, Kozlov, and Mikelić, 1995; Beliaev and Kozlov, 1996; Ryzhik, Papanicolaou, and Keller, 1996; Levin, 1999; Markov and Preziosi, 1999; Torquato, 1999). The papers (Cioranescu and Murat, 1982; Berdichevsky, Kunin, and Hussain, 1991; Milton, 1992; Cherkaev and Slepyan, 1995; Kozlov and Piatnitski, 1996; Zhikov, 1996; Balk, Cherkaev, and Slepyan, 1999) emphasize unexpected behavior of homogenized systems. The numerical aspects of homogenization were investigated in many papers, such as (Bakhvalov and Knyazev, 1994; Zohdi, Oden, and Rodin, 1996; Helsing, Milton, and Movchan, 1997; Sigmund and Torquato, 1997; Greengard and Rokhlin, 1997; Fu, Klimkowski, Rodin, Berger, Browne, Singer, van de Geijn, and Vemaganti, 1998; Greengard and Helsing, 1998).

Conductivity in an Inhomogeneous Body

To visualize a composite with infinitely small periodic elements we use an iterative process. Consider a periodic two-phase structure. Assume that the domain Ω consists of cubes Ω_i ($\Omega = \cup\Omega_i$) and that the material's layout is the same for all cubes of the size $\frac{1}{2^k}$. Each cube Ω^k is divided into two parts Ω_1^k and Ω_2^k, which are filled with the materials σ_1 and σ_2, respectively. The conductivity $\sigma(\mathbf{x})$ at a point of the cube Ω^k is equal to

$$\sigma(\chi) = \chi^k\sigma_1 + (1 - \chi^k)\sigma_2,$$

where $\chi = \chi^k(\mathbf{x})$ is the space-periodic characteristic function χ^k of the first material in the composite:

$$\chi^k(\mathbf{x}) = \begin{cases} 1 & \text{if} \quad \mathbf{x} \in \Omega_1^k, \\ 0 & \text{if} \quad \mathbf{x} \in \Omega_2^k. \end{cases}$$

Now consider a sequence of $\{\chi^k\}$ layouts. It is built by the following procedure. At each step, the representative cube of periodicity Ω^k is parted into eight cubes Ω^{k+1} half the linear size of each; each cube is filled with a geometrically identical layout of materials but in half the scale of that in the cube Ω^k.

Let us fix the size $\varepsilon = \frac{1}{2^k}$ of the periodicity cell and assume that these cells fill the domain Ω. Consider the conductivity equilibrium (2.1.6) in Ω. The solution w of the conductivity equation (2.1.6) can be represented in the form (Bensoussan et al., 1978)

$$w = w_0(\mathbf{x}) + \varepsilon w_\varepsilon \left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) + o(\varepsilon); \tag{2.2.1}$$

it consists of a smooth component $w_0(\mathbf{x}) = O(1)$ that is independent of the size ε and an almost-periodic oscillating component $\varepsilon w_\varepsilon \left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)$ that has zero mean value over the periodicity cell:

$$\int_\Omega w_\varepsilon \left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right) = 0.$$

The magnitude of w_ε is of order one.

The averaging of a process in a composite medium is done by applying the averaging operator $\langle \cdot \rangle(\mathbf{x})$:

$$\langle z \rangle(\mathbf{x}) = \frac{1}{|\Omega_\varepsilon|} \int_{\Omega_\varepsilon(\mathbf{x})} z, \tag{2.2.2}$$

where Ω_ε is a small rectangular domain with the point $\mathbf{x}$ in its center and $|\Omega_\varepsilon|$ is its volume. The operator (2.2.2) is the multidimensional analogue of (1.1.6). The size of the domain of averaging is assumed to be greater than the size ε of the cell of periodicity: $|\Omega_\varepsilon| \gg \varepsilon$. More exactly, we assume that the size of Ω_ε tends to zero, together with the diameter of fragments ω_i, but it remains much greater than the diameter. The one-dimensional averaging introduced in (1.1.9) agrees with the discussed definition.

The field $\mathbf{e} = \nabla w$ can be represented as $\mathbf{e} = \mathbf{e}_0 + \mathbf{e}_\varepsilon$ where

$$\mathbf{e}_0 = \nabla w_0, \quad \mathbf{e}_\varepsilon = \nabla w_\varepsilon, \quad \langle \mathbf{e}_\varepsilon \rangle = 0.$$

However, the magnitude of $\mathbf{e}_\varepsilon$ is at least of order of magnitude $|\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2|$ of the jump of $\mathbf{e}$ on the boundary between domains of materials $\boldsymbol{\sigma}_1$ and $\boldsymbol{\sigma}_2$. Formally, we also observe that ∇w, (2.2.1), is at least of order one because $\nabla \left(\varepsilon w_\varepsilon \left(\mathbf{x}, \frac{\mathbf{x}}{\varepsilon}\right)\right)$ is of order one.

The current $\mathbf{j}_\varepsilon$ in the medium has a similar representation

$$\mathbf{j} = \mathbf{j}_0 + \mathbf{j}_\varepsilon, \quad \langle \mathbf{j}_\varepsilon \rangle = 0;$$

where the magnitude of $\mathbf{j}_\varepsilon$ is at least of order $|\frac{1}{\boldsymbol{\sigma}_1} - \frac{1}{\boldsymbol{\sigma}_2}|$.

The current and field are subject to the constitutive relations

$$\mathbf{j}_0(\mathbf{x}) + \mathbf{j}_\varepsilon(\mathbf{x}) = \boldsymbol{\sigma}(\mathbf{x})(\mathbf{e}_0(\mathbf{x}) + \mathbf{e}_\varepsilon(\mathbf{x})).$$

We are interested in a description of the relation between the smooth components $\mathbf{e}_0$ and $\mathbf{j}_0$ when the size of the periodicity cell is much less than the other parameters of the system: $\varepsilon \to 0$.

Let us average the solution to the conductivity equations (2.1.2), (2.1.3), (2.1.4) by the operator (2.2.2). We find relations between the averaged current $\langle \mathbf{j} \rangle$, averaged field $\langle \mathbf{e} \rangle$, and averaged potential $\langle w \rangle$. The linear operators ∇ and $\langle \cdot \rangle$ commute (up to terms of order $O(\varepsilon)$) (see (Bensoussan et al., 1978; Jikov et al., 1994)); hence we can change their order. We obtain

$$\langle \nabla \cdot \mathbf{j} \rangle = \nabla \cdot \mathbf{j}_0 = f + O(\varepsilon), \quad \langle \mathbf{e} \rangle = \nabla \langle w \rangle + O(\varepsilon). \tag{2.2.3}$$

To complete the procedure we determine the connection between the average current $\langle \mathbf{j} \rangle = \langle \boldsymbol{\sigma}\, \mathbf{e} \rangle$ and the average field $\langle \mathbf{e} \rangle$. We introduce the tensor of *the effective properties of the composite* σ_* such that

$$\langle \mathbf{j} \rangle = \langle \boldsymbol{\sigma} \mathbf{e} \rangle = \boldsymbol{\sigma}_* \langle \mathbf{e} \rangle. \tag{2.2.4}$$

The effective tensor links vectors $\langle \mathbf{j} \rangle$ and $\langle \mathbf{e} \rangle$. Using the effective tensor we can formulate the homogenized equations of the medium. The system (2.2.3), (2.2.4) or the equivalent second-order equation

$$\nabla \cdot \boldsymbol{\sigma}_* \nabla \langle w \rangle = f + O(\varepsilon) \tag{2.2.5}$$

describes the conductivity in the medium with the conductivity tensor $\boldsymbol{\sigma}_*$ or the homogenized conductivity. The solution should be understood in the weak sense (Jikov et al., 1994).

Calculation of Effective Tensor

Here we find the ways to compute the effective tensor. We illustrate the idea of the approach with a simple example. Namely, we consider a two-dimensional problem of the conductivity of a composite of two isotropic materials.

Consider a periodic layout of two conducting materials. The element of periodicity Ω is the unit square

$$\Omega = \{x_1, x_2 : \; 0 \le x_1 \le 1, \quad 0 \le x_2 \le 1\}. \tag{2.2.6}$$

Suppose that Ω is parted into two rectangular parts Ω_1 and Ω_2 of the areas m_1 and m_2 respectively. They are occupied by two isotropic materials with conductivities σ_1 and σ_2 respectively.

Applied Fields

The following simple algorithm enables us to compute an effective tensor. Consider again the periodic structure (2.2.6). Suppose that a uniform external field $\mathbf{E} = \mathbf{E}_1$ is applied to the structure that is equal to $\mathbf{E}_1 = \mathbf{i}_1$, where $\mathbf{i}_1$ is a unit vector directed along the x_1-axis:

$$\mathbf{i}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We solve the boundary value problem

$$\nabla \cdot \boldsymbol{\sigma}(\mathbf{x})[\nabla w(\mathbf{x}) + \mathbf{E}_1] = 0, \quad w \text{ is periodic in } \Omega. \tag{2.2.7}$$

and compute the average current $\mathbf{j}_1 = \langle \boldsymbol{\sigma}(\mathbf{x})[\nabla w(\mathbf{x}) + \mathbf{E}_1] \rangle$.

The average current vector $\mathbf{j}_1 = \boldsymbol{\sigma}_* \mathbf{E}_1$ is equal to the first column of the effective tensor $\boldsymbol{\sigma}_*$:

$$\begin{pmatrix} j_1^1 \\ j_2^1 \end{pmatrix} = \begin{pmatrix} (\sigma_*)_{11}, & (\sigma_*)_{12} \\ (\sigma_*)_{21}, & (\sigma_*)_{22} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \tag{2.2.8}$$

or, in coordinate form,

$$j_1^1 = (\sigma_*)_{11}, \quad j_2^1 = (\sigma_*)_{21}.$$

This way we determine two elements of $\boldsymbol{\sigma}_*$ by measuring the vector of the averaged current.

The effective tensor cannot be completely determined by one "experiment," that is, by applying one external field. Indeed, the measured current $\mathbf{j}_1$ cannot depend on the conductivity in the direction orthogonal to the direction of the applied field. To determine the second column of the matrix of $\boldsymbol{\sigma}_*$ we solve boundary value problem (2.2.7) for a different external field $\mathbf{E}_2$. We can take as $\mathbf{E}_2$ a unit vector directed along the x_2-axis, $\mathbf{E}_2 = \mathbf{i}_2$, where

$$\mathbf{i}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Clearly, the average current $\mathbf{j}_2 = (j_1^2, j_2^2$ for this problem coincides with the second column of the effective tensor $\mathbf{j}_2 = \boldsymbol{\sigma}_* \cdot \mathbf{E}_2$ or

$$j_1^2 = (\sigma_*)_{12}, \quad j_2^2 = (\sigma_*)_{22}.$$

Remark 2.2.2 *Of course, one must expect symmetry $\sigma_{12} = \sigma_{21}$ in the coefficients of the effective tensor (Jikov et al., 1994). The symmetry can be shown in many ways, for example, by the symmetry of Green's function for the conductivity operator (2.1.5).*

The results are easily represented in matrix notation. Let us form the matrix E of external fields $\mathbf{E}_1, \mathbf{E}_2$ and the matrix J of corresponding currents $\mathbf{j}_1, \mathbf{j}_2$:

$$E = [\mathbf{E}_1, \mathbf{E}_2], \quad J = [\mathbf{j}_1, \mathbf{j}_2].$$

The effective properties can then be determined from the matrix equations

$$J = \boldsymbol{\sigma}_* E \quad \text{or} \quad \boldsymbol{\sigma}_* = J E^{-1}. \tag{2.2.9}$$

The inversion is possible if the external fields $\mathbf{E}_1$, $\mathbf{E}_2$ are linearly independent. Here we are considering the case where $E = I$; therefore, $\boldsymbol{\sigma}_* = J$.

Applied Currents

Alternatively, we may determine $\boldsymbol{\sigma}$ by applying the trial currents $\mathbf{J}_i$ instead of trial fields $\mathbf{E}_i$. This time we assume that the periodic composite is submerged into a uniform currents $\mathbf{J}_i$ instead of the uniform field $\mathbf{E}_i$. The problem for the periodicity cell becomes

$$\nabla \cdot [\boldsymbol{\sigma}(\mathbf{x}) \nabla w(\mathbf{x}) + \mathbf{J}_i] = 0 \text{ in } \Omega_\varepsilon, \quad i = 1, 2, \tag{2.2.10}$$

with periodic boundary conditions on w.

The external currents can be chosen as

$$\mathbf{J}_1 = \mathbf{i}_1, \quad \mathbf{J}_2 = \mathbf{i}_2.$$

Solving (2.2.10) for these currents, we obtain average the fields $\mathbf{e}_1$ and $\mathbf{e}_2$, respectively. Measuring the average fields, one measures the coefficients of the inverse tensors $\boldsymbol{\sigma}_*^{-1}$ due to the first equation of (2.2.9);

$$\boldsymbol{\sigma}_*^{-1} = E J^{-1},$$

where $J = [\mathbf{J}_1, \mathbf{J}_2]$ is the matrix of the applied currents and E is the matrix of the calculated mean fields. For linear media, these procedures lead to the same resulting effective tensor, because they describe the same linear relationship between averaged fields and currents.

2.2.2 Effective Properties of Laminates

As an example, let us compute the effective tensor of a laminate. The laminate geometry allows us to solve the partial differential problem (2.2.7) in closed form.

Effective Tensor for Laminates of Two Conducting Materials

Consider the conductivity problem for a laminate composite in the plane. Let the periodicity cell be the unit square Ω. Assume that the laminates are oriented along the x_2-axis. The rectangles Ω_1 and Ω_2,

$$\Omega_1 : \{0 \le x_1 \le m,\ 0 \le x_2 \le 1\}, \quad \Omega_2 : \{m \le x_1 \le 1,\ 0 \le x_2 \le 1\},$$

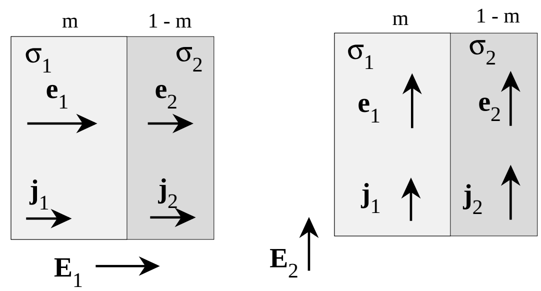

FIGURE 2.2. The fields and currents in a laminate. If the external field is applied across the layers (left), the current stays constant everywhere in the structure. If the external field is applied along the layers (right), the field stays constant everywhere in the structure.

are filled with isotropic conducting materials with conductivities σ_1 and σ_2, respectively; see Figure 2.2.

For physical reasons, we should expect that the laminates are equivalent to an *anisotropic* material, that is, characterized by a tensor of effective properties $\boldsymbol{\sigma}_*$.

Conductivity Across the Layers

First apply the unit external field $\mathbf{E}_1 = \mathbf{i}_1$ perpendicular to the layers (see Figure 2.2, left). The conductivity of the laminates is described by the boundary value problem

$$\nabla \cdot \sigma(\mathbf{x})(\nabla w(\mathbf{x}) + \mathbf{E}_1) = 0, \quad w \text{ is periodic in } \Omega, \tag{2.2.11}$$

where $\sigma(\mathbf{x}) = \sigma_i$ if $\mathbf{x} \in \Omega_i$. It has a simple analytical solution: The potential w is a continuous piecewise linear function of x_1:

$$w = w(x_1) = \begin{cases} \alpha_1 x_1 & \text{in } \Omega_1, \\ m\alpha_1 + \alpha_2 (x_1 - m) & \text{in } \Omega_2, \end{cases} \tag{2.2.12}$$

where α_1, α_2 are constants and we assume that $w(0) = 0$. The gradient ∇w is a piecewise constant vector directed along the x_1-axis:

$$\mathbf{e} = \nabla w = \begin{cases} \mathbf{e}_1 & \text{in } \Omega_1, \\ \mathbf{e}_2 & \text{in } \Omega_2, \end{cases}$$

where

$$\mathbf{e}_1 = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix} \alpha_2 \\ 0 \end{pmatrix}.$$

The constants $\alpha_1,\ \alpha_2$ are determined from two constraints. The periodicity of ∇w states that $\langle \nabla w \rangle = 0$, which yields to $m\alpha_1 + (1-m)\alpha_2 = 0$.

The second constraint comes from the jump condition $[\mathbf{J} \cdot \mathbf{n}]_{-}^{+} = 0$ on the line $x_1 = m$. It yields to

$$\sigma_1(\mathbf{e}_1 + \mathbf{E}_1) \cdot \mathbf{n} - \sigma_2(\mathbf{e}_2 + \mathbf{E}_1) \cdot \mathbf{n} = 0.$$

From these conditions, we compute $\alpha_1,\ \alpha_2$:

$$\alpha_1 = \frac{\sigma_2}{m\sigma_2 + (1-m)\sigma_1} - 1, \quad \alpha_2 = \frac{\sigma_1}{m\sigma_2 + (1-m)\sigma_1} - 1.$$

The solution of (2.2.12) satisfies the equation (2.2.11) and boundary conditions. The mean field in the cell is equal to $\mathbf{i}_1$, and the mean current is

$$\langle \mathbf{j} \rangle = m\sigma_1(\mathbf{e}_1 + \mathbf{E}_1) + (1-m)\sigma_2(\mathbf{e}_2 + \mathbf{E}_1) = \begin{pmatrix} \frac{\sigma_1\sigma_2}{m\sigma_2 + (1-m)\sigma_1} \\ 0 \end{pmatrix}.$$

Note that the current $\mathbf{j}$ is constant in the entire cell.

Thus we determine the two coefficients of the effective tensor by (2.2.8). The element σ_{11} is equal to the harmonic mean of the conductivities of the initial materials:

$$(Gs_*)_{11} = \sigma_h = \frac{\sigma_1\sigma_2}{m\sigma_2 + (1-m)\sigma_1} = \left(m\sigma_1^{-1} + (1-m)\sigma_2^{-1}\right)^{-1},$$

where σ_h is the harmonic mean of the conductivities.

We also find that $(Gs_*)_{12} = 0$. The symmetry of $\boldsymbol{\sigma}_*$ implies that $(Gs_*)_{21} = 0$. The normal to the layers is the eigenvector of the effective tensor; the effective conductivity across the layers is equal to one of the eigenvalues of this tensor.

Conductivity Along the Layers

Let us determine the effective conductivity in an orthogonal direction. We direct the applied field $\mathbf{E}_2 = \mathbf{i}_2$ along the layers (see Figure 2.2, right). The boundary value problem,

$$\nabla \cdot \sigma(\mathbf{x})(\nabla w(\mathbf{x}) + \mathbf{E}_2) = 0, \quad w \text{ is periodic in } \Omega,$$

where $\sigma(\mathbf{x}) = \sigma_i$ if $\mathbf{x} \in \Omega_i$, has the uniform solution, $w = x_2$. This solution satisfies the boundary conditions, the jump condition, and the differential equation. This solution implies that the field is constant everywhere, $\mathbf{e} = \nabla w = \mathbf{i}_2$, and the mean current $\langle \mathbf{j} \rangle$ is equal to

$$\langle \mathbf{j} \rangle = \langle \sigma \rangle \mathbf{e} = \begin{pmatrix} 0 \\ m\sigma_1 + (1-m)\sigma_2 \end{pmatrix}.$$

The eigenvalue of the effective conductivity tensor that has been determined is equal to the arithmetic mean σ_a of the conductivities:

$$(Gs_*)_{22} = \sigma_a, \quad \sigma_a = m\sigma_1 + (1-m)\sigma_2;$$

the eigenvector corresponds to the tangent to the layers.

The Effective Tensor

We have found that a laminate in a uniform external field behaves equivalently to a homogeneous but anisotropic medium. We denote the tensor of effective properties $\boldsymbol{\sigma}_*$ of a laminate structure by $\sigma_{\text{lam}} = \boldsymbol{\sigma}_*$. This tensor depends on the structural parameters: the volume fraction m of the materials and the orientation of the structure. The constitutive equation for this medium represents a relationship between the mean value of the current density and the mean value of the field; it depends on the normal $\mathbf{n}$ and the tangent $\mathbf{t}$ to the layers. In the coordinates $\mathbf{n}, \mathbf{t}$, it has the form

$$\begin{pmatrix} j_n \\ j_t \end{pmatrix} = \boldsymbol{\sigma}_{\text{lam}} \begin{pmatrix} e_n \\ e_t \end{pmatrix}; \quad \boldsymbol{\sigma}_{\text{lam}} = \begin{pmatrix} \sigma_h & 0 \\ 0 & \sigma_a \end{pmatrix},$$

where subindices $_n$ and $_t$ denote the normal and tangent components of the corresponding vectors.

Remark 2.2.3 *The asymptotic case of very different conductivities*

$$\sigma_1 \ll \sigma_2$$

corresponds to the asymptotics

$$\sigma_h \approx \frac{\sigma_1}{m_1}, \quad \sigma_a \approx \sigma_2 m_2.$$

This formula demonstrates that the conductivity along the layers is determined mainly by the conductivity of the best conductor σ_2 and the conductivity across the layers by the conductivity of the worst conductor σ_1. This remark emphasizes that composites can emphasize the property of each phase and possess new properties, such as anisotropy.

Generalizations

The obtained formulas permit a straightforward generalization to laminates made from more than two materials. In this case the arithmetic and harmonic means are expressed as

$$\sigma_a = \sum_{i=1}^{p} m_i \sigma_i, \quad \sigma_h = \left(\sum_{i=1}^{p} m_i \sigma_i^{-1} \right)^{-1}, \tag{2.2.13}$$

where p is the number of mixed materials, and σ_i and m_i are the conductivity and volume fraction of the ith material.

The generalization to the three-dimensional case is also straightforward. The effective properties tensor $\boldsymbol{\sigma}_{\text{lam}}$ is equal to

$$\boldsymbol{\sigma}_{\text{lam}} = \begin{pmatrix} \sigma_h & 0 & 0 \\ 0 & \sigma_a & 0 \\ 0 & 0 & \sigma_a \end{pmatrix};$$

the normal direction of laminates corresponds to the eigenvalue σ_h.

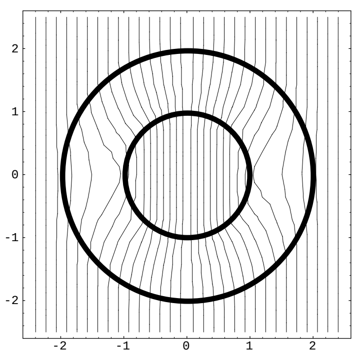

FIGURE 2.3. The geometry of coated circles. The field outside the external disk is homogeneous. The inner disk has higher conductivity than the effective medium, and the exterior annulus has lower conductivity than the effective medium. Observe the complete mutual compensation of the inclusion. The inclusion is "invisible" in a uniform external field.

2.2.3 Effective Medium Theory: Coated Circles

Here we describe a way to calculate effective properties for media with located symmetric inclusions. The approach leads to exact formulas for the effective conductivity. It was originated and discussed in (Bruggemann, 1935; Bruggemann, 1937; Hashin and Shtrikman, 1962a; Christensen, 1979) and others. Specifically we discuss the structure of the "coated spheres" suggested in (Hashin and Shtrikman, 1962a).

Consider a homogeneous material with isotropic conductivity σ_*. Suppose we replace the medium in a disk of unit radius with the following two-phase configuration. The inner disk of radius $r_0 < 1$ is filled with material σ_1, and the annulus $r_0 < r < 1$ is filled with material σ_2. This configuration (see Figure 2.3) is called the *coated circles* (or, in three dimensions, the *coated spheres*).

Assume that the configuration is submerged into a uniform external field $\mathbf{e}(r, \theta) \to \cos\theta$, when $r \to \infty$. The corresponding potential w tends to the affine function $w \to r\cos\theta$.

Suppose we manage to define the conductivity σ_* so that the field everywhere outside of the inclusion is a constant vector: $\mathbf{e} = \mathbf{i}_1$. In polar coordinates, this condition takes the form

$$\mathbf{e}(r, \theta) = [\cos\theta, \sin\theta] \quad \forall r > 1. \tag{2.2.14}$$

In this case, we cannot detect the presence of the inclusion by observing the fields anywhere outside of the inclusion. Hence, we cannot distinguish the homogeneous configuration with conductivity σ_* from a configuration with one, or several, or even infinitely many circular inclusions of the described type; see Figure 2.3. In this case we call σ_* the effective conductivity of a composite made of coated circles.

To find σ_* we explicitly calculate the field everywhere in the configuration. The field satisfies the boundary value problem

$$\nabla^2 w = 0 \text{ in } R^2, \quad \lim_{r\to\infty} \frac{\partial w(r,\theta)}{\partial r} = \cos\theta,$$

and satisfies the jump conditions on the circles and the effective medium condition (2.2.14).

This problem permits separation of variables; the solution w has the form $w = R(r)\cos\theta$. The function $R(r)$ must satisfy the ordinary differential equation

$$r\frac{d}{d\,r}\left(r\frac{d}{d\,r}R\right) - R = 0 \tag{2.2.15}$$

the conditions

$$\begin{array}{rlrl} R(0) &= 0, & R'(0) &= 0, \\ [R(r_0)] &= 0, & [\sigma(r)R'(r_0)] &= 0, \\ [R(1)] &= 0, & [\sigma(r)R'(1)] &= 0, \\ \lim_{r\to\infty} R &= r, & & \end{array} \tag{2.2.16}$$

where $[x]$ means the jump of x, and the condition (2.2.14). The conductivity $\sigma(r)$ is

$$\sigma(r) = \begin{cases} \sigma_1 & \text{if} \quad r \in [0, r_0), \\ \sigma_2 & \text{if} \quad r \in [r_0, 1), \\ \sigma_* & \text{if} \quad r \in [1, \infty). \end{cases}$$

We assume that the potential is zero at $r = 0$ (we can always assume this, because the potential is defined up to a constant), and we require the continuity of the field at $r = 0$. The last condition in (2.2.16) says that the field in the system with the inclusion tends to a homogeneous field when $r \to \infty$. The remaining conditions express the continuity of the potential and of the normal current on the circles $r = r_0$ and $r = 1$.

The solution to (2.2.15) that satisfies the conditions (2.2.16) has the form

$$w = \begin{cases} A_0 r & \text{if} \quad 0 < r < r_0, \\ A_1 r + \frac{B_1}{r} & \text{if} \quad r_0 < r < 1, \\ r + \frac{B_2}{r} & \text{if} \quad 1 < r. \end{cases} \tag{2.2.17}$$

To define the four constants $A_0,\ A_1,\ B_1$, and B_2 we use conditions (2.2.16).

The key point of the scheme is the following: We assign the constant σ_* in such a way that $B_2 = 0$ or that the field is homogeneous if $r > 1$. This way, (2.2.14) is satisfied.

Accounting for the constants, we have

$$\begin{array}{l} A_0 = \frac{2\,\sigma_2}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}, \\ A_1 = \frac{\sigma_1 + \sigma_2}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}, \\ B_1 = \frac{m_1\,(-\sigma_1 + \sigma_2)}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}, \end{array} \tag{2.2.18}$$

and

$$\sigma_* = \sigma_{HS} = \sigma_1 \frac{(1+m_1)\,\sigma_1 + m_2\,\sigma_2}{m_2\,\sigma_1 + (1+m_1)\,\sigma_2}. \tag{2.2.19}$$

Formula (2.2.19) shows the effective conductivity of the configuration. The conductivity was calculated in (Hashin and Shtrikman, 1962a), where it was also proven that σ_{HS} is the *extreme* isotropic conductivity that one can achieve by arbitrary mixing of two isotropic materials in the prescribed proportion.

Remark 2.2.4 *A generalization of the procedure was suggested in (Milton, 1980), which considered the geometry of "coated ellipses" (one inscribed into another) and found the explicit description of their effective properties. This time, the effective medium is anisotropic. The idea of the calculation is the same: We consider one "coated elliptical inclusion," i.e., two ellipses in an unbounded domain and a homogeneous field applied at infinity.*

2.3 Conclusion and Problems

This chapter introduced the main objects for the structural optimization of conducting composites.

- We described the conductivity of an inhomogeneous medium, the differential constraints and potentials for fields and currents, and the jump conditions on the boundary between different materials. The corresponding pair of dual variational principles was introduced.

- We described the properties of composites and the homogenization procedure. An algorithm has been presented to compute the tensor of effective properties of a composite. We have analytically computed the effective properties of laminates and of coated circles.

Problems

1. Consider the function

$$f(c_1, \ldots, c_n, x) = \sum_{i=1}^{n} \chi_i(x) c_i,$$

where χ_i are the characteristic functions of nonoverlapping domains of x, and a function $G(z)$. Prove the superposition rule

$$G(f(c_1, \ldots, c_n, x)) = f(G(c_1), \ldots G(c_n), x).$$

2. Consider a conducting composite made of two anisotropic materials. Define the magnitude of the jump of discontinuous components of $\mathbf{e}$ and $\mathbf{j}$ through the tensors of conductivity.

3. How many external fields are needed to compute all coefficients of two- and three-dimensional conductivity tensors by calculating the energy? Suggest an algebraic procedure to calculate the eigenvalues and eigenvectors of an effective tensor.

4. Derive the effective properties using an external current instead of the external field. Prove that the resulting effective tensor remains the same.

5. Derive the effective properties for the three-dimensional geometry of "coated spheres."