Definition 1. (1) A linear combination of \(v_1, \ldots, v_n \in \mathbb{R}^\ell\) is a vector \(\sum_{i=1}^{n} \lambda_i v_i, \lambda_i \in \mathbb{R}\). The linear hull of \(v_1, \ldots, v_n\) is the set of all linear combinations of \(v_1, \ldots, v_n\):

\[
S = \text{lin}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R} \right\}.
\]

\(S\) is called a subspace of \(\mathbb{R}^\ell\).

(2) An affine linear combination of \(v_1, \ldots, v_n \in \mathbb{R}^\ell\) is a vector \(\sum_{i=1}^{n} \lambda_i v_i, \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1\). The affine hull of \(v_1, \ldots, v_n\) is the set of all affine linear combinations of \(v_1, \ldots, v_n\):

\[
A = \text{aff}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \in \mathbb{R}, \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]

\(A\) is called an affine space in \(\mathbb{R}^\ell\).

(3) A cone combination of \(v_1, \ldots, v_n \in \mathbb{R}^\ell\) is a vector \(\sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0\). The conical hull of \(v_1, \ldots, v_n\) is the set of all cone combinations of \(v_1, \ldots, v_n\):

\[
K = \text{cone}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \geq 0 \right\}.
\]

\(K\) is called a cone in \(\mathbb{R}^\ell\).

(4) A convex combination of \(v_1, \ldots, v_n \in \mathbb{R}^\ell\) is a vector \(\sum_{i=1}^{n} \lambda_i v_i, \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1\). The convex hull of \(v_1, \ldots, v_n\) is the set of all convex combinations of \(v_1, \ldots, v_n\):

\[
P = \text{conv}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^{n} \lambda_i v_i : \lambda_i \geq 0, \sum_{i=1}^{n} \lambda_i = 1 \right\}.
\]

\(P\) is called a polytope in \(\mathbb{R}^\ell\).

Exercise 1. Compute the linear, affine, conical and convex hulls of the configurations

\[
A := \left\{ \left( \begin{array}{c} 1 \\ 0 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 3 \end{array} \right), \left( \begin{array}{c} 1 \\ 4 \end{array} \right) \right\} \quad \text{and} \quad B = \left\{ \left( \begin{array}{c} 0 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ 1 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ -1 \\ 1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ -1 \\ -1 \\ 1 \end{array} \right), \left( \begin{array}{c} 1 \\ -1 \\ 1 \\ 1 \end{array} \right) \right\}
\]

Definition 1 gives the generator representation of the various objects defined there. Each one of them also has a constraint/inequality representation.

(1) The subspace \(S = \text{lin}(v_1, \ldots, v_n) = \{ x \in \mathbb{R}^\ell : Cx = 0 \}\) for some matrix \(C\).

(2) The affine space \(A = \text{aff}(v_1, \ldots, v_n) = \{ x \in \mathbb{R}^\ell : Cx = b \}\) for some matrix \(C\) and vector \(b\).
(3) The cone $K = \text{cone}(v_1, \ldots, v_n) = \{ x \in \mathbb{R}^\ell : Cx \geq 0 \}$ for some matrix $C$.

(4) The polytope $P = \text{conv}(v_1, \ldots, v_n) = \{ x \in \mathbb{R}^\ell : Cx \geq b \}$ for some matrix $C$ and vector $b$.

**Exercise 2.**

1. Write the constraint representation of the cone $K = \text{cone}(A)$.
2. Write the constraint representation of the polytope $P = \text{conv}(B)$.

A **polyhedron** is a set of the form $\{ x \in \mathbb{R}^\ell : Cx \geq b \}$. Polytopes are bounded polyhedra. In particular, $K = \{ x : Cx \geq 0 \}$ is a special case of a polyhedron.

**Part II: Polyhedral Cones**

In general, a cone in $\mathbb{R}^\ell$ is any set that is closed under positive scalings; if $x \in K$ and $\lambda > 0$ then $\lambda x \in K$.

For example the circular cone in $\mathbb{R}^3$ $\{ (x, y, z)^T : x^2 + y^2 \leq z^2 \}$ with apex at the origin has this property. Notice that $K = \text{cone}(v_1, \ldots, v_n)$ also has this property. However, unlike the circular cone, $K$ is **finitely generated**. Notice also that if we were to write the circular cone as the set of cone combinations of a collection of vectors, then we would need infinitely many vectors (generators).

Similarly if we were to write the circular cone in terms of linear inequalities, we would need infinitely many inequalities. Finitely generated cones are finitely constrained, and are called **polyhedral cones**.

**Definition 2.** The polar of a cone $K$ is $K^\circ = \{ y \in \mathbb{R}^\ell : y^T x \geq 0 \ \forall \ x \in K \}$.

In words, $K^\circ$ consists of all vectors that make nonnegative inner product with all elements in $K$.

**Exercise 3.** Draw and write down both representations of $K^\circ$ for $K = \text{cone}(A)$.

**Exercise 4.**

1. Prove that $K^\circ$ is a cone.
2. Suppose $K = \text{cone}(v_1, \ldots, v_n)$ and $M = [v_1 \ v_2 \ \cdots \ v_n] \in \mathbb{R}^{\ell \times n}$ is a matrix with columns $v_1, v_2, \ldots, v_n$.

   Prove that
   
   (a) $K = \{ Mx : x \geq 0 \}$.
   
   (b) $K^\circ = \{ y : y^T M \geq 0 \}$.

**Theorem 3** (Farkas Lemma). For $M \in \mathbb{R}^{\ell \times n}$ and $b \in \mathbb{R}^\ell$, exactly one of the following is true:

1. either $Mx = b$, $x \geq 0$ has a solution

2. or there exists $y \in \mathbb{R}^\ell$ such that $y^T M \geq 0$ and $y^T b < 0$.

**Exercise 5.** Prove Farkas Lemma.

Hint: interpret the two statements in terms of a cone and its polar.

Farkas Lemma is the foundation of linear programming and is an example of a **theorem of alternatives**.

There are many such theorems in mathematics. For example, what would be the theorem of alternatives for linear systems:

For $M \in \mathbb{R}^{\ell \times n}$ and $b \in \mathbb{R}^\ell$ either $Mx = b$ has a solution or ..... ??

**Theorem 4** (Carathéodory’s theorem). Let $K = \text{conv}(v_1, \ldots, v_n) \subseteq \mathbb{R}^\ell$ be a cone. Then for any $b \in K$ there exists a linearly independent subset $\mathcal{U}$ of $\{v_1, \ldots, v_n\}$ such that $b \in \text{cone}(\mathcal{U})$. In particular, if $M = [v_1 \ v_1 \ \cdots \ v_n]$, then $Mx = b, x \geq 0$ has a solution $x$ whose support has size at most $\ell$.

**Exercise 6.** Prove Carathéodory’s theorem.
**PART III: DESIGNS AND EIGENPOLYTOPES**

Let $G = ([n], E)$ be a connected, undirected regular graph with normalized adjacency matrix $AD^{-1}$. Recall that both $A$ and $D$ are $n \times n$ matrices:

- $A_{ij} = 1$ if $ij \in E$ and 0 otherwise.
- $D$ is a diagonal matrix with $D_{ii} = \text{deg}(i)$ for all $i \in [n]$.

Suppose $AD^{-1}$ has $m$ eigenspaces ordered as $\Lambda_1 = \text{span}\{1\} < \ldots < \Lambda_m$. A subset $W \subset [n]$ with nonnegative weights $a_w : w \in W$ is a $k$-design of $G$ if

$$\sum_{w \in W} a_w \varphi(w) = \frac{1}{|V|} \sum_{v \in V} \varphi(v), \quad \forall \varphi \in \Lambda_1, \ldots, \Lambda_k.$$

**Exercise 7.**

1. Prove that every subset of $V = [n]$ will average the first eigenvector $1 = \varphi_1$ of $AD^{-1}$.

2. Prove that there is no proper subset of $V = [n]$ (with positive weights) that can average all eigenvectors of $AD^{-1}$.

Let $U$ denote a matrix whose rows are an eigenbasis of $AD^{-1}$, and let $I \subset [m]$ index a proper subset of eigenspaces of $AD^{-1}$.

1. Let $U_I$ denote the submatrix of $U$ consisting of all rows corresponding to the eigenspaces $\Lambda_i$ for $i \in I$.

2. Let $U_I$ denote the collection of columns of $U_I$, which we note may occur with repetition.

**Definition 5** (C. Godsil, 1978). The polytope $P_I = \text{conv}(U_I)$ is an *eigenpolytope* of $G$ for the eigenvalues indexed by $I$.

**Exercise 8.** The Truncated Tetrahedral graph is shown below, along with an orthogonal basis of eigenvectors for $AD^{-1}$. The horizontal lines divide the eigenspaces.

```
\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
-1 & -1.5 & -0.5 & 1.5 & 2 & 1.5 & -0.5 & -1.5 & -1 & 1 & 0 & 0 \\
2 & 1.5 & 1.5 & -0.5 & -1 & -1.5 & -1.5 & -0.5 & -1 & 0 & 1 & 0 \\
-1 & -0.5 & -1.5 & -1.5 & -1 & -0.5 & 1.5 & 1.5 & 2 & 0 & 0 & 1 \\
0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 \\
-1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 1 & 0 \\
1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 \\
0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 \\
\end{bmatrix}
```

In this exercise we will use eigenpolytopes to compute designs.

1. Compute the eigenpolytopes $P_3$ and $P_4$.

2. Compute the minimal designs of $G$ that average $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_5$ using $P_4$.

3. Compute the minimal designs of $G$ that average $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ using $P_3$. 


Exercise 9. Let $G$ be the graph of the 3-dimensional cube in $\mathbb{R}^3$. For this graph,

$$A = \begin{pmatrix}
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
\end{pmatrix}.$$

The matrix $AD^{-1} = \frac{1}{3}A$ has eigenvalues

$$1, -1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3},$$

with eigenvectors:

$$\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\
0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\
0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\
0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \\
\end{pmatrix}.$$

Compute various types of designs of the cube.