I. A toy case

Fact: \( \exists \Gamma < \text{PSL}_2(\mathbb{R}) \) lattice such that there are many homomorphisms \( \Gamma \rightarrow \text{PSL}_2(\mathbb{R}) \).

Observation: \( \Gamma \times \Gamma < \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \) is a lattice. It has many homomorphisms to \( \text{PSL}_2(\mathbb{R}) \).

Definition: \( \Gamma \times \Gamma \) is an irreducible lattice if \( \pi_1(\Gamma) = S_3 \).

Example: \( \mathbb{Z}[T] \rightarrow \mathbb{R}^2/\mathbb{R}, \quad a + bT \rightarrow (a+bS, a-bT) \).

\( \text{PSL}_2(\mathbb{Z}[T]) \rightarrow \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) = \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \)

Remarks: the fact that the latter is a lattice is not obvious.
- It is an example of arithmetic lattice.
- All irreducible lattices in \( \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \) are arithmetic.

Thin (long case): \( \Gamma < \text{PSL}_2(\mathbb{R}) \times \text{PSL}_2(\mathbb{R}) \) irreducible lattice.

\( \rho : \Gamma \rightarrow \text{PSL}_2(\mathbb{R}) \) a homomorphism with non-trivial image.

Then \( \rho \) is the composition of an automorphism and projection.

Restriction of \( \rho \) to \( \Gamma \times \Gamma \) is irreducible lattice.

Convention: \( S \times S = G = \text{PSL}_2(\mathbb{R}) \).

The groups \( S, S, \Gamma \) are endowed with their Haar measure.

Observation: \( \Gamma \cap S_i \) is ergodic.

Def: Ergodic \( \mathcal{L}^2((S) = \{ \text{const} \}) \)

For \( f \in \mathcal{L}^2((S) \), \( A \in \text{SL}_2(\mathbb{R}) \),

\[ f \in \mathcal{L}^2((S) \rightarrow \mathcal{L}^2((S) \)

\( A \in \text{SL}_2(\mathbb{R}) \), \text{ dense} \Rightarrow \forall \Gamma \text{ ergodic} \mathcal{L}^2((S) = \{ \text{const} \}) \)

\( S \rightarrow \text{B}(\mathbb{L}(S)) \) is cont. \( \Gamma \times S \), dense, \text{ Haar}

\( \forall \Gamma, \forall \gamma \in \Gamma, \forall s \in \Gamma \times S \)

\[ s \rightarrow (s, \gamma s) \text{ cont} \Rightarrow \forall \Gamma, \forall \gamma \in \Gamma, \forall s \in \Gamma \times S \]

\( f \in \mathcal{L}^2((S) = \{ \text{const} \}) \)
Lemma: \( f : G \to \mathcal{S} \) homomorphism, \( \mathcal{S} \) dense.

\( \exists s, f \in C_0^* \), defined, measurable, \( \mathcal{R} \)-equivariant

\( \Rightarrow \exists \tilde{f}, \mathcal{S} \to \mathcal{G} \) cont. homomorphism s.t \( f = \tilde{f} \).

Claim: \( S \to G \) another such map \( \Rightarrow \exists \gamma \in G \) s.t \( \gamma(s) = f(s)g \).

Consider \( S \to \mathcal{G} \times \mathcal{G} \to \mathcal{G} / \mathcal{G} \) left action on point \( g \).

\[ G(x, y) = \mathcal{G}(x, y) = \{(x', y) : x', y \in \mathcal{G}\}, \quad g = \gamma(x) \]

(1) of Lemma: \( \exists g, x, \gamma(x) = g(x) \).

Claim: \( \mathcal{G} \to \mathcal{V} \) proper action with no stabs \( \exists s \to \mathcal{V} \) a.e. def. meas.

\( \Rightarrow \) same conclusion.

\[ S \to \mathcal{V} \to \mathcal{V} / \mathcal{G} \Rightarrow S \to \text{orbit} \]

Notation: \( A_1 < B_1 \leq S \), \( A = A_1 \times A_2 \), \( \mathcal{P} = \mathcal{P}_1 \times \mathcal{P}_2 \leq \mathcal{S} = S \times S \).

Lemma: \( \mathcal{G} \to \mathcal{V} \) proper \( \Rightarrow \) no \( \mathcal{P} \)-maps \( S / \mathcal{A} \to \mathcal{V} \).

The Maurey phenomenon: \( \mathcal{P} \to \text{Iso}(\mathcal{R}) \) cont. \( \times \) fixed \( \Rightarrow \times \) \( \mathcal{P} \)-fixed.

\[ \left( \begin{array}{c} x' \\ y' \end{array} \right) \left( \begin{array}{cc} 1 & b \\ 0 & 1 \end{array} \right) = \left( \begin{array}{c} x' \\ y' \end{array} \right) \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \]

\[ d(ux, x) = d(ux, x) = d(x, x) \to d(x, x) = 0. \]

Corollary: \( S \to \text{Iso}(\mathcal{R}) \) cont. \( \times \) \( \mathcal{A} \)-fixed \( \Rightarrow \times \) \( S \)-fixed.

\[ S \to \mathcal{A} \to S \]

Lemma: \( \mathcal{P} \to \text{Iso}(\mathcal{R}) \), \( S / \mathcal{A} \to \mathcal{P} \)-map \( \Rightarrow \mathcal{P} \) is a.e. constant.

Let \( \mathcal{P} \to \text{Iso}(\mathcal{R}) \); \( \mathcal{P} \)-invariant, \( \mathcal{S} \) \( \mathcal{P} \)-invariant.

For \( s, t \in \mathcal{S}(s, x) \), \( \mathcal{S} \to \mathcal{S}(s, t) \) is \( \mathcal{P} \)-inv. Assume \( d \)-bounded. Set

\[ D(s, t) = \left( \int d(s, t) \mathcal{P}(d) \, ds \right)^{1/2} \]

\( \mathcal{P} \) constant \( \Rightarrow \mathcal{P} \) constant.
Claim: no f-maps \( S/A \rightarrow G \).
- no f-maps \( S/A \rightarrow G/\langle \alpha \rangle \), if \( \alpha \) unbounded
  (but solvable \( \Rightarrow \) unbounded).

If of Lemma: \( S/A \rightarrow V \rightarrow V/G \Rightarrow S/A \rightarrow \text{orbit} \).

Example: \( G \ltimes \text{Prob}^{\mathcal{L}}(\text{IP}(\mathbb{R})) \) is a proper action.

Lemma: \( \exists \) f-map \( S/\rho \rightarrow \text{Prob}(\text{IP}(\mathbb{R})) \)

1) observe that \( \exists \) f-map \( S \rightarrow \text{Prob}(\text{IP}(\mathbb{R})) \).
2) use amenability of \( P \) to find a fixed point in
\[ L^0(S, \text{Prob}(\text{IP}(\mathbb{R}))) = L^0(S, C(\text{IP}(\mathbb{R}))) \]

Better Lemma*: \( \exists \) f-map \( S/\rho \rightarrow \text{IP}(\mathbb{R}) \)

by previous Lemma \( \exists \) \( S/A \rightarrow S/\rho \rightarrow \text{Prob}^{\mathcal{L}} \).
Assume \( \exists \) \( S/\rho \rightarrow \text{IP}(\mathbb{R}) \).
Note: \( S/A \), \( S/\rho \times S/\rho \), \( S/\rho \), \( S/A \) are all measurable.
\( S/A \rightarrow S/\rho \times S/\rho \rightarrow \text{Prob}^{\mathcal{L}}(\text{IP}(\mathbb{R})), \text{Prob}^{\mathcal{L}}(\text{IP}(\mathbb{R})) \rightarrow \text{Prob}^{\mathcal{L}}(\text{IP}(\mathbb{R})) \)
\( \mu_1, \mu_2 \quad \underset{\mathcal{L}}{\Rightarrow} \quad \mathcal{L}(\mu_1, \mu_2) \)

unless \( \mu_1, \mu_2 \) has always same support \( \Rightarrow \) \( \mu_1, \mu_2 \) fixed pair in \( \text{IP}(\mathbb{R}) \)
\( \Rightarrow \) \( \mathcal{L}(\mu_1, \mu_2) \) solvable.

Almost there:
\( S/A \rightarrow S/\rho \rightarrow S/\rho \times S/\rho \rightarrow S/\rho \rightarrow \text{IP}(\mathbb{R}) \)

\( \text{flip} \rightarrow \text{flip} \rightarrow \text{flip} \rightarrow \text{flip} \rightarrow \text{flip} \rightarrow \text{flip} \)
\( \mu_1 = \mu_2 \rightarrow \mathcal{L}(\mu_1, \mu_2) \)
\( \mathcal{W} = (\epsilon, \omega_1, \omega_2, \omega_3) \)
If of then: \( S/A \rightarrow \mathcal{P}(\mathbb{R})^4 \), \( \forall \sigma \in \mathcal{P}(\mathbb{R})^4 \times \mathcal{P}(\mathbb{R})^4 \times \mathcal{P}(\mathbb{R})^4 \times \mathcal{P}(\mathbb{R})^4 \) \( \sigma \otimes \sigma \otimes \sigma \otimes \sigma \).

but \( \mathcal{P}(\mathbb{R})^4 \cong G \), \( G \subset \mathcal{P}(\mathbb{R})^4 \) is proper \( \Rightarrow \exists \phi \in \mathcal{P}(\mathbb{R})^4 \) s.t. \( \phi \circ \sigma = \sigma \circ \sigma \).

\( \sigma = \sigma_0 \Rightarrow \sigma \) is a.e. const \( \Rightarrow \sigma \) fixed point in \( \mathcal{P}(\mathbb{R})^4 \).

\( \Rightarrow p(\sigma) \) solvable.

\( \sigma = \sigma_0 \) (WLOG) \( \Rightarrow \sigma \) does not depend on \( S^2/p_i \) coordinate

\( \Rightarrow \exists \sigma \in \mathcal{P}(\mathbb{R})^4 \), \( \sigma \) -map.

for \( s \in S \), \( \sigma \) is \( \sigma \) -map, by ergodicity \( \sigma = \sigma \circ \sigma \) a.e.

or \( \sigma = \sigma \circ \sigma \) a.e.

- \( \sigma \) const \( \Rightarrow \exists \sigma \) fixed point.

- \( \sigma \) not const \( \Rightarrow \exists \sigma \) s.t. \( \sigma \neq \sigma \circ \sigma \).

\( \Rightarrow \exists \psi : S \rightarrow \mathcal{P}(\mathbb{R})^4, \psi = \sigma \circ \sigma \).

reject the argument \( \Rightarrow \exists \psi : S, \psi(\mathcal{P}(\mathbb{R})^4)^{\mathcal{P}(\mathbb{R})^4} \) proper.

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Summary:
- amenability \( \Rightarrow \exists \) boundary map
- metric ergodicity \( \Rightarrow \) to points
- \( S/A \) has symmetries \( \Rightarrow \) through a factor
- \( \sigma \) dense in the factor \( \Rightarrow \) extends.
Lemma: \( \rho \in H \) dense, \( \rho \cdot \Gamma \rightarrow G \).
\[ \text{a.e. defined measurable n-map } H \rightarrow G \]
\[ \Rightarrow \exists \rho: I \rightarrow G \text{ s.t } \rho = \rho \cdot \Gamma. \]

Claim: \( \psi: H \rightarrow G \) another such \( \Rightarrow \exists g \in G \text{ s.t } \psi = \rho \cdot \varphi, \quad \rho \cdot \varphi(g) = \varphi(g) \).

Consider \( \psi: G \times G \rightarrow G \times G \) /left action
The group $S = SL_3(\mathbb{R})$ - combinatorial viewpoint.

$S \subset \mathbb{R}^3$. Other spaces on which it acts:
- $\mathbb{P}^2(\mathbb{R})$: $Gr(1,3) = \{ \lambda \in \mathbb{R}^3 | d\lambda(\lambda) = 0 \} \cong \langle e_1 \rangle$, $Q = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & 1 & 1 \end{pmatrix}$.
- $Gr(2,3) = \{ \pi \in \mathbb{R}^3 | d\pi(\pi) = 2 \} \cong \langle e_i, e_j \rangle$, $Q' = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$.
- Flag $\{ \lambda \in \mathbb{R}^3 \}$ $\circ \langle e_i \rangle \circ \langle e_i, e_j \rangle$, $P = \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
- $F = \text{point} = S/\mathbb{R}$.

Diagram of spaces: $S/\mathbb{R}, S/\mathbb{R}^2, S/\mathbb{R}^3$.

Remark: In $SL_n(\mathbb{R})$ we would get a hypercube diagram, vertices corresponding to subsets of $\{1, \ldots, n\}$.

Claim: What are the $S$-orbits in $S/\mathbb{R}^p \times S/\mathbb{R}^p$?
1. $\lambda_i, \pi_i = \lambda_i, \pi_i$
2. $\lambda = \lambda_i$, $\pi_i \neq \pi_i$
3. $\lambda + \lambda_i = \pi_i$
4. $\pi_i \cap \pi_i = \lambda_i = \lambda_i$
5. $\pi_i \cap \pi_i = \pi_i = \pi_i$

Claim: The generic orbit is $\mathbb{P}^2(\mathbb{R})^3$.

\[ (\lambda_1, \pi_{i_1}), (\lambda_2, \pi_{i_2}) \rightarrow (\lambda_1, \lambda_2, \pi_{i_1} \cap \pi_{i_2}) \]
\[ (\lambda_1, \lambda_2, \pi_{i_3}) \leftarrow (\lambda_1, \lambda_1, \lambda_3) \]

$S \times \mathbb{P}^2(\mathbb{R})^3$ transitively, $\text{cteb} (e_i, e_j, e_k) = A$, Orbit $= S/A$.

Note: $W = S_3 \times \mathbb{P}^2(\mathbb{R})^3 = S/A$ commuting with the $S$-action.
General fact: \[ N_0(H) \rightarrow Aut_0(G/H) \]
\[ n \rightarrow [gH \mapsto gng^{-1}H] \]
\[ \Rightarrow N_0(H)/H \cong Aut_0(G/H). \]

Def: \( N_0(A)/A \) is called the Weyl group.

Exercise: \( N_0(A) = \) monomial matrices.
\( N_0(A)/A \cong S_3 \).
Ergodic Theoretical nonsense

Given \((X, \mu) \overset{\pi}{\rightarrow} (Y, \nu)\) s.t. \([\pi_\ast \mu] = \nu\), we say that
Y is a factor of X.

We get \(L_0(Y) \overset{\pi_0}{\rightarrow} L_0(X)\).

Conversely, we also get \(L'(X) \overset{\pi'}{\rightarrow} L'(Y)\)
(think of these as spaces of densities, \(\pi' = (\pi_\ast)^\ast\).

\(L_0(Y)\) is a \(w^*\)-closed sub-algebra of \(L_0(X)\).

Conversely, every \(w^*\)-close \(s_0\) defines a factor.
we say that \(X \overset{s_0}{\rightarrow} Y \overset{\pi}{\rightarrow} X\), and get

\(Y \overset{\pi_0}{\rightarrow} X\)

Factors \(\sim w^*\)-closed sub-algebras:

\[
\begin{align*}
S_3 \times S_A & \rightarrow S_0 \\
S_0 \rightarrow S_0 \\
S_0 \rightarrow S/A \\
S_0 \rightarrow S/A_2 \\
S_0 \rightarrow Y \\
\text{maximal factor of } S_0 \\
\text{st } \pi \text{ is } v\text{-fixed} = L_0(S/A)^V \cap L_0(S_0)
\end{align*}
\]

Note: must be \(s\)-factor, hence one of \(S_0, S/A, S/A_2\).
Observations: \(\alpha, \beta\) are order reversing

\[ V < \alpha(s_p + x) \iff x_v \rightarrow x \]

Def: A Galois correspondence is a pair of posets \(A, B\) and order reversing maps \(\alpha: B \rightarrow A, \beta: A \rightarrow B\) such that:

\[ a \leq \alpha(b) \iff b \leq \beta(a) \]

It follows: \(a < \alpha^{-1}(a), b < \beta^{-1}(b)\).

\(\alpha, \beta\) are closure maps and \(\alpha, \beta\) defines a pair of closed subgroups of closed objects.

Closed factors are S-factors \(\Rightarrow\) closed subgroups are e, (e, e, w), (e, (e, e), )

Remark: called "parabolic" or "special subgroups" in the theory of Coxeter groups.
This: $\mathbb{F} < S$ lattice. Gram convergence

\[ p: \mathbb{F} \rightarrow \mathbb{C} \Rightarrow p(r) \text{ elementary or bounded.} \]

"pf": As discussed before we get a unique $S/p \cong M$.

$s.t. \; d(\mathbb{F}) < W$ is of index 2, but $d(\mathbb{F})$ is closed.

Remarks: to get $d_1$ we used that $\mathbb{P}$ is parabolic.
- we used the fact that $S/A$ is metrically ergodic for $\mathbb{P}$.

This follows from metric ergodicity for $S$, (we refer applied to root $s(g)$) and the passage to lattice as in the first lecture.
- Note also that $S/A \cong S/p \times S/p$.

Generalizations: Same holds for any Lie simple group.

Lemma: for simple groups closed $s(g)$ of $W$ (int $w$)

are never normal. (and can't be of index 2).