I: A toy case

Fact: $\exists\, \Gamma < PSL_2(\mathbb{R})$ lattic s.t $\Gamma \simeq F_2$. For such Γ there are many homomorphisms $\Gamma \longrightarrow PSL_2(\mathbb{R})$.

Observation: $\Gamma \times \Gamma < PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ is a lattice. It has many homomorphisms to $PGL_2(\mathbb{R})$.

Definition: $\Gamma < S_1 \times S_2$ is an irreducible lattice if $\overline{\pi_i(\Gamma)} = S_i$.

Example: $\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{R} \times \mathbb{R}$, $\quad a + b\sqrt{2} \longmapsto (a + b\sqrt{2}, a - b\sqrt{2})$.

$PSL_2(\mathbb{Z}[\sqrt{2}]) \hookrightarrow PSL_2(\mathbb{R} \times \mathbb{R}) \simeq PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$

Remarks:
- the fact that the latter is a lattice is not obvious.
- It is an example of Arithmetic lattice.
- All irred. lattices in $PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ are arithmetic.

Thm (toy case): $\Gamma < PSL_2(\mathbb{R}) \times PSL_2(\mathbb{R})$ irred. lattice.
$\rho : \Gamma \longrightarrow PSL_2(\mathbb{R})$ a homomorphism with non-solvable image.
Then ρ is the composition of an automorphism and projection.
restriction of a

Convention: $S_1 = S_2 = G = SL_2(\mathbb{R})$. $S = S_1 \times S_2$. $\Gamma < S$. $\rho : \Gamma \longrightarrow G$.
The groups S_1, S_2, S are endowed with their Haar measures.

Observation: $\Gamma \curvearrowright S_i$ is ergodic $\qquad$ Def: Ergodic $L^\infty(X)^S = \text{konst}$

[$f \in L^\infty(S_1)^\Gamma \Rightarrow \forall \varphi \in L^1(S_1),\ \forall \gamma \in \Gamma,\ \langle f, \varphi \rangle = \langle \gamma f, \varphi \rangle = \langle f, \gamma^{-1} \varphi \rangle$

$S_1 \hookrightarrow B(L^1(S_1))$ is cont. $\Gamma < S_1$ dense $\Rightarrow$]

$s \longmapsto s\varphi \longmapsto (f, s\varphi)$ cont $\Rightarrow \forall s,\ (f, s\varphi) = (f, \varphi) \Rightarrow$

$\forall \varphi, \forall s\ \langle sf, \varphi \rangle = \langle f, \varphi \rangle \Rightarrow f \in L^\infty(S)^S \Rightarrow f$ const.

__Lemma:__ $\Gamma \xrightarrow{\rho} G$ homomorphism, $\Gamma < S_1$ dense.

$\exists\, S_1 \xrightarrow{\varphi} G$ a.e defined, measurable, Γ-equivariant

$\Longrightarrow \exists\, \bar{\rho} : S_1 \to G$ cont. homomorphism s.t $\rho = \bar{\rho}|_\Gamma$.

__Claim:__ $S_1 \xrightarrow{\psi} G$ another such map $\Longrightarrow \exists\, g \in G$ s.t $\psi(s) = \varphi(s) g$.

[consider $S_1 \xrightarrow{\varphi \times \psi} G \times G \longrightarrow G \times G /$ ~~left~~ action

$\underset{\text{(orbit)}}{\longrightarrow}$ point

$G(x,y) = G(g,e) = \{(h, hg) \mid h \in G\}$, $g = y^{-1} x$)

etter lemma:

$S_1 \to \mathbb{P}^2(\mathbb{R})$

__Pf of Lemma:__ $s \in S$, $s' \xmapsto{\psi} \varphi(s's) = \varphi(s')\rho'(s)$. $g = \varphi(e)$, $\rho(\gamma) g = \rho(\gamma)\varphi(e) = \varphi(\gamma) = \varphi(e)\rho'(\gamma) = g\rho'(\gamma)$. $\bar{\rho}(s) = g\rho'(s)g^{-1}$.

__Claim:__ $G \curvearrowright V$ proper action with no stabs, $\exists\, S_1 \longrightarrow V$ a.e def, meas Γ-eq.

$\Longrightarrow$ same conclusion.

[$S_1 \longrightarrow V \longrightarrow V/G \Longrightarrow S_1 \longrightarrow$ Orbit]

__Notation:__ $A_i < P_i < S_i$. $A = A_1 \times A_2$, $P = P_1 \times P_2 < S = S_1 \times S_2$.

__Lemma:__ $G \curvearrowright V$ proper $\Longrightarrow$ no Γ-maps $S/A \longrightarrow V$.

__The Mautner pheomenon:__ $P_1 \to \mathrm{Iso}(X)$ cont. x A_1-fixed $\Rightarrow$ x P-fixed.

$$\underset{a}{\underset{\shortparallel}{\begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}}} \underset{u}{\underset{\shortparallel}{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}}} \begin{pmatrix} \lambda^{-1} & \\ & \lambda \end{pmatrix} = \begin{pmatrix} 1 & \lambda^2 t \\ 0 & 1 \end{pmatrix} \xrightarrow{\lambda \to 0} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$d(ux, x) = d(ua^{-1}x, a^{-1}x) = d(aua^{-1}x, x) \longrightarrow d(x,x) = 0$.

__Cor:__ $S_1 \to \mathrm{Iso}(X)$ cont. x A_1-fixed $\Rightarrow$ x S_1-fixed.

$S \to$ $\qquad A \qquad S$

__Lemma:__ $\Gamma \to \mathrm{Iso}(X)$. $S/A \xrightarrow{\varphi} X$ Γ-map $\Rightarrow \varphi$ is a.e constant.

__Pf:__ $\varphi : S \to S/A \to X \in L_\Gamma(S, X)$ is A-fixed. $S \curvearrowright L(S,X)$ pre-comp.

For $f_1, f_2 \in L_\Gamma(S,X)$, $s \longmapsto d(f_1(s), f_2(s))$ is Γ-inv. Assume d-bounded. Set

$D(f_1, f_2) = \left(\int_{S/\Gamma} d(f_1(s), f_2(s))^2 \, ds \right)^{1/2}$. $\quad \bar{\varphi}$ constant $\Rightarrow \varphi$ const.

Cor: $\Gamma \curvearrowright S/A$ is ergodic.

Def: Metric Ergodicity.

Claim: no Γ-maps $S/A \to G$.
no Γ-maps $S/A \to G/K$, if $\rho(\Gamma)$ unbounded
(not solvable $\Rightarrow$ unbounded).

Pf of lemma: $S/A \to V \to V/G \Rightarrow S/A \to$ orbit.

Example: $G \curvearrowright \mathrm{Prob}^{\geq 3}(\mathbb{P}^1(\mathbb{R}))$ is a proper action.

Lemma: $\exists$ Γ-map $S/P \to \mathrm{Prob}(\mathbb{P}^1(\mathbb{R}))$
1) observe that $\exists$ Γ-map $S \to \mathrm{Prob}(\mathbb{P}^1(\mathbb{R}))$.
2) use amenability of P to find a fixed point in
$L_\Gamma(S, \mathrm{Prob}(\mathbb{P}^1(\mathbb{R}))) \subset L^\infty(S, C(\mathbb{P}^1(\mathbb{R}))^*) = L^1(S, C(\mathbb{P}^1(\mathbb{R})))^*$.

better Lemma$^{\mathrm{or}}$: $\exists$ Γ-map $S/P \to \mathbb{P}^1(\mathbb{R})$
by previous lemma $\nexists$ $S/A \to S/P \to \mathrm{Prob}^{\geq 3}$.
Assume $\exists$ $S/P \to$ ~~$\mathbb{P}^1(\mathbb{R})$~~ $\mathrm{prob}^{=2}$.
Note: ~~$S/A \times S$~~ $S_1/P_1 \times S_1/P_1 \supseteq \mathrm{diag} \sqcup S/A$
$S/P \times S/P \Leftrightarrow S/A$ measured isomorphism
$S/A \hookrightarrow S/P \times S/A \to \mathrm{Prob}^2(\mathbb{P}^1(\mathbb{R})) \times \mathrm{Prob}^2(\mathbb{P}^1(\mathbb{R})) \longrightarrow \mathrm{Prob}^{\geq 3}(\mathbb{P}^1(\mathbb{R}))$
$\mu_1, \mu_2 \longmapsto \frac{1}{2}(\mu_1 + \mu_2)$
unless μ_1, μ_2 has always same support $\Rightarrow \rho(\Gamma)$ fixed pair in $\mathbb{P}^1(\mathbb{R})$
$\Rightarrow \rho(\Gamma)$ solvable.

Almost there: S/A

$S_1/A \times S_1/P_1 \times S_2/P_2 \times S_2/P_2 \simeq S/P \times S/P \longrightarrow S/P \longrightarrow \mathbb{P}^1(\mathbb{R})$

flip$_1$ flip$_2$ flip

flip $=$ flip$_1$ $\cdot$ flip$_2$ $\qquad W = (e, w_1, w_2, w_0)$

Pf of thm: $S/A \longrightarrow \mathbb{P}^1(\mathbb{R})^4$, $\varphi \times \varphi \circ \omega_1 \times \varphi \circ \omega_2 \times \varphi \circ \omega_3$

but $\mathbb{P}^1(\mathbb{R})^{(3)} \simeq G$, $G \curvearrowright \mathbb{P}^1(\mathbb{R})^{(4)}$ is proper $\Rightarrow \exists\, \omega \neq \omega'$ s.t $\varphi \circ \omega = \varphi \circ \omega'$

$\Rightarrow \exists\, \omega \neq e$ $\varphi = \varphi \circ \omega$.

$\omega = \omega_0 \Rightarrow \varphi$ is a.e const $\Rightarrow \rho(\Gamma)$ fixed point in $\mathbb{P}^1(\mathbb{R})$

$\Rightarrow \rho(\Gamma)$ solvable.

$\omega = \omega_2$ (WLOG) $\Rightarrow \varphi$ does not depend on S_2/P_2 coordinate

$\Rightarrow \exists\ S_1 \to S_1/P_1 \longrightarrow \mathbb{P}^1(\mathbb{R})$. Γ-map.

for $s \in S_1$, $\varphi \circ s$ is Γ-map. by ergodicity $\varphi = \varphi \circ s$ a.e

or $\varphi \neq \varphi \circ s$ a.e.

- φ const $\Rightarrow \exists\ \rho(\Gamma)$ fixed point
- φ not const $\Rightarrow \exists\, s$ s.t $\varphi \neq \varphi \circ s$.

$\Rightarrow$ $\exists\, \psi$: $S_1 \longrightarrow \mathbb{P}^1(\mathbb{R})^{(2)}$, $\psi = \varphi \times \varphi \circ s$.

repeat the argument $\Rightarrow$ $\exists\, \psi: S_1 \longrightarrow \left(\mathbb{P}^1(\mathbb{R})^{(2)}\right)^{(2)} \curvearrowleft G$ proper.

Summary: amenability $\Rightarrow \exists$ boundary map.

metric ergodicity $\Rightarrow$ to points.

S/A has symmetries $\Rightarrow$ through a factor.

Γ dense in the factor $\Rightarrow$ extends.

<u>Lemma</u>: $\Gamma < H$ dense, $\rho: \Gamma \to G$.

$\exists$ a.e defined measurable Γ-map $H \xrightarrow{\varphi} G$

$\Rightarrow \exists \bar{\rho}: H \to G$ s.t $\rho = \bar{\rho}|_\Gamma$.

<u>Claim</u>: $\psi: H \to G$ another such $\Rightarrow \exists g \in G$ s.t $\psi = R_g \circ \varphi$, $\cancel{\forall g \in G}$ $\forall h \in H$ $\psi(h) = \varphi(h) g$

[consider $H \xrightarrow{\varphi \times \psi} G \times G \longrightarrow G \times G /$left action

III: The group $S = SL_3(\mathbb{R})$ - combinatorial view point

$S \curvearrowright \mathbb{R}^3$. other spaces on which it acts:

- $\mathbb{P}^2(\mathbb{R}) = Gr(1,3) = \{\lambda < \mathbb{R}^3 \mid \dim(\lambda) = 1\} \ni \langle e_1 \rangle$. $Q_1 = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & * & * \end{bmatrix}$
- $Gr(2,3) = \{\Pi < \mathbb{R}^3 \mid \dim(\Pi) = 2\} \ni \langle e_1, e_2 \rangle$. $Q_2 = \begin{bmatrix} * & * & * \\ * & * & * \\ 0 & 0 & * \end{bmatrix}$
- $\text{Flag} = \{\lambda < \Pi < \mathbb{R}^3 \mid \ldots\} \ni (\langle e_1 \rangle < \langle e_1, e_2 \rangle)$. $P = \begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}$
- $* = \text{point} = S/S$.

Diagram of spaces:

$$\begin{tikzcd} & S/P \arrow[dl] \arrow[dr] & \\ S/Q_1 \arrow[dr] & & S/Q_2 \arrow[dl] \\ & * & \end{tikzcd}$$

Remark: In $SL_n(\mathbb{R})$ we would get a hypercube diagram, vertices corresponding to subsets of $\{1, \ldots, n-1\}$.

~~Claim:~~ **Q:** What are the S-orbits in $S/P \times S/P$?

1) $(\lambda_1, \Pi_1) = (\lambda_2, \Pi_2)$
2) $\lambda_1 = \lambda_2$, $\Pi_1 \neq \Pi_2$
3) $\lambda_1 \neq \lambda_2$, $\Pi_1 = \Pi_2$
4) $\Pi_1 \cap \Pi_2 = \lambda_1$, $\lambda_1 \neq \lambda_2$
5) $\Pi_1 \cap \Pi_2 = \lambda_2$, $\lambda_1 \neq \lambda_2$
6) generic.

3 lines in general position

Claim: The generic orbit $\simeq \mathbb{P}^2(\mathbb{R})^{\{3\}}$

$$(\lambda_1, \Pi_1), (\lambda_2, \Pi_2) \longmapsto (\lambda_1, \lambda_2, \Pi_1 \cap \Pi_2)$$

$$(\lambda_1, \lambda_1 \oplus \lambda_3), (\lambda_2, \lambda_1 \oplus \lambda_3) \longleftarrow (\lambda_1, \lambda_2, \lambda_3)$$

$S \curvearrowright \mathbb{P}^2(\mathbb{R})^{\{3\}}$ transitively, $\text{stab}(\langle e_1 \rangle, \langle e_2 \rangle, \langle e_3 \rangle) = A$. Orbit $\simeq S/A$.

Note: $W = S_3 \curvearrowright \mathbb{P}^2(\mathbb{R})^{\{3\}} \simeq S/A$ commuting with the S-action

General fact: $N_G(H) \longrightarrow Aut_G(G/H)$

$n \longmapsto [gH \longmapsto gn^{-1}H]$

$\Rightarrow \quad N_G(H)/H \simeq Aut_G(G/H)$.

Def: $N_S(A)/A$ is called the Weyl group.

Exercise: $N_S(A)$ = monomial matrices.

$N_S(A)/A \simeq S_3$

<u>Ergodic Theoretical nonsense</u>

Given $(X,\mu) \xrightarrow{\pi} (Y,\nu)$ s.t $[\pi_* \mu] = [\nu]$ we say that Y is a factor of X.

We get $L^\infty(Y) \xhookrightarrow{\pi^*} L^\infty(X)$.

We also get $L^1(X) \xrightarrow{\pi_*} L^1(Y)$

(think of these as spaces of densities). $\pi^* = (\pi_*)^*$.

$L^\infty(Y)$ is a w^*-closed sub-algebra of $L^\infty(X)$.

Conversely, every w^*-close s.a defines a factor.

we say that

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & Y_1 \arrow[d] \\ & Y_2 \end{tikzcd}$$

$Y_1 \geq Y_2$. and get

Factors $\simeq$ W^*-closed sub-algebras.

(up to equivalence)

$\lambda_1, \lambda_1 \oplus \lambda_2$

$(\lambda_1, \lambda_2, \lambda_3) \longmapsto$ [illegible]

$$\begin{tikzcd} S_3 \curvearrowright S/A \arrow[r] \arrow[rr, bend right] & S/P \arrow[r, "q"] & X \end{tikzcd}$$

, $\bar{q} \in L(S/A, X) \curvearrowleft S_3 = W$

<u>Factors of S/P</u> — <u>Subgroups of S_3</u>

$S/P \xrightarrow{q} X$ $\overset{\alpha}{\rightsquigarrow}$ $\mathrm{Stab}_W(\bar{q})$.

$S/P \to S/P$ — $\{e\}$

$S/P \to S/Q_1$ — $\{e, (23)\}$

$S/P \to S/Q_2$ — $\{e, (12)\}$

$S/P \to *$ — W

$S/P \xrightarrow{q_V} X_V$ $\overset{\beta}{\leftsquigarrow}$ V

maximal factor of S/P s.t $\bar{q}$ is V-fixed $\simeq L^\infty(S/A)^V \cap L^\infty(S/P)$

<u>Note</u>: must be S-factor, hence one of S/P, S/Q_1, S/Q_2, $*$

<u>Observations</u>: - α, β are order <u>reversing</u>

- $V < \alpha(S/_p \twoheadrightarrow X) \iff X_V \to X$

<u>Def</u>: A Galois correspondence is a pair of posets A, B and order reversing maps $\alpha: B \to A$, $\beta: A \to B$ s.t.: $a \le \alpha(b) \iff b \le \alpha(a)$

<u>it follows</u>: $a \le \alpha\beta(a)$, $b \le \beta\alpha(b)$. ~~theorem~~
$\alpha\beta, \beta\alpha$ are closure maps and α, β defines iso of ~~closed~~ the subposets of closed objects.

Closed factors are S-factors $\implies$ closed subgroups are
$e, \{e,(12)\}, \{e,(23)\}, W$

<u>Remark</u>: called "parabolic" or "special subgroups" in the theory of Coxeter groups.

Thm: $\Gamma < S$ lattice, $G \curvearrowright M$ convergence.

$\rho: \Gamma \to G \implies \rho(\Gamma)$ elementary or bounded.

"pf": As discussed before (assuming $\rho(\Gamma)$ not) get a unique $S/P \xrightarrow{\varphi_0} M$.

s.t $\alpha(\varphi_0) < W$ is of index 2, but $\alpha(\varphi_0)$ is closed.

Remarks: - to get φ_0 we used that P is parabolic.

- we used the fact that S/A is metrically ergodic for Γ.

This follow from metric ergodicity for S (wantner applied to root s.g) and the passge to lattice as in the first lecture.

- Note also that $S/A \simeq S/P \times S/P$.

Generalizations: Same holds for any h.r simple group.

Lemma: ~~this~~ for simple groups closed sg of W (but e, w) are never normal. (and can't be of index 2).