PROPER+CONVERGENCE ACTIONS
(RIGIDITY VIA ERGODIC METHODS)

Problem 1. Let \( \phi : X \to Y \) be a continuous map between locally compact spaces. Prove that TFAE:

1. For every compact \( K \subset Y \) the preimage \( \phi^{-1}(K) \) is compact in \( X \).
2. The map \( \hat{\phi} : \hat{X} \to \hat{Y} \) between one point compactifications is continuous.
3. For every topological space \( Z \) the map \( X \times Z \stackrel{\phi \times 1}{\to} Y \times Z \) is closed.

Maps \( \phi : X \to Y \) with these properties are called proper.

Definition 2. A continuous action \( G \acts X \) of a lcsc group on a locally compact space is called proper action if the map
\[ G \times M \to M \times M, \quad (g, m) \mapsto (g \cdot m, m) \]
is a proper map.

Problem 3. A continuous action \( G \acts M \) on a locally compact space \( M \) is proper iff for any compact subset \( C \subset M \) the set
\[ \{ g \in G \mid gC \cap C \neq \emptyset \} \]
is precompact in \( G \).

Problem 4. Let \( G \acts M \) be a proper action. Prove:

1. Each \( G \)-orbit \( Gx \) is closed in \( M \).
2. For each \( x \in M \) the stabilizer \( G_x = \{ g \in G \mid g \cdot x = x \} \) is compact.
3. The space of orbits \( G \setminus M \) is Hausdorff.

Problem 5. Let \( K \subset G \) be a compact subgroup in a lcsc group. Prove that there is a \( G \)-invariant compatible metric on \( G/K \).

Problem 6. Prove that if \( G \acts M \) is a proper action on a locally compact secondly countable space, then the action \( G \acts \text{Prob}(M) \) is also proper.

Suggestion: Prove that for any compact subset \( Q \subset \text{Prob}(M) \) there is a compact set \( C \subset M \) so that
\[ \mu(C) > 1/2 \quad \forall \mu \in Q. \]

Definition 7. Let \( G \) be a lcsc, \( X \) a compact metrizable space, \( G \acts X \) a continuous action. The action \( G \acts X \) is called a convergence action if the diagonal \( G \)-action on the space of distinct triples
\[ X^{(3)} := \{(x_1, x_2, x_3) \in X^3 \mid x_i \neq x_j, \ 1 \leq i < j \leq 3 \} \]
is proper. A closed subgroup \( H \subset G \) in a convergence action is said to be elementary if it fixes a point \( x_0 \in X \), or a pair of points \( \{x_1, x_2\} \subset X \).

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Definition 8. Let $H$ be a discrete countable group, or a lcsc group, $Y$ a non-empty compact metrizable space, and $H \to \text{Homeo}(Y)$ a continuous homomorphism. The action $H \curvearrowright Y$ is said to be **minimal** if $Y$ has no proper $H$-invariant closed subsets.

Problem 9. Prove that any continuous action $H \curvearrowright Y$ on a compact space has a non-empty $H$-invariant subset $Z \subset Y$ the action on which is minimal.

Problem 10. Prove that the following conditions on continuous action $H \curvearrowright Y$ on a compact space are equivalent:

1. $H \curvearrowright Y$ is minimal.
2. For any $y \in Y$ the orbit $H.y$ is dense in $Y$.
3. For any non-empty open set $U \subset Y$ there is $n \in \mathbb{N}$ and $h_1, \ldots, h_n \in H$ so that $h_1U \cup \cdots \cup h_nU = Y$.

Problem 11. Let $G \curvearrowright X$ be a convergence action, $H < G$ a closed non-compact and non-elementary subgroup. Prove:

1. The action $H \curvearrowright X$ is a convergence action.
2. There is a unique $H$-invariant minimal closed subset $\Lambda_H \subset X$ (called the **limit set** of $H$).
3. The kernel $K_H := \ker(H \to \text{Homeo}(\Lambda_H))$ is a compact subgroup.

Hence the action of $H/K_H$ on $\Lambda_H$ is a non-elementary, faithful, minimal, convergence action.

Problem 12. Let $M$ be a proper $\delta$-hyperbolic space, and $H < \text{Isom}(M, d)$ a closed subgroup. Prove that the action $H \curvearrowright \partial M$ is a convergence action.

(The notions of elementary subgroups in the two contexts agree.)