PROPER+CONVERGENCE ACTIONS
(RIGIDITY VIA ERGODIC METHODS)

Problem 1. Let $\phi : X \longrightarrow Y$ be a continuous map between locally compact spaces. Prove that TFAE:

(1) For every compact $K \subset Y$ the preimage $\phi^{-1}(K)$ is compact in X.
(2) The map $\hat{\phi} : \hat{X} \longrightarrow \hat{Y}$ between one point compactifications is continuous.
(3) For every topological space Z the map $X \times Z \overset{\phi\times 1}{\longrightarrow} Y \times Z$ is closed.

Maps $\phi : X \longrightarrow Y$ with these properties are called **proper**.

Definition 2. A continuous action $G \curvearrowright X$ of a lcsc group on a locally compact space is called **proper action** if the map

$$G \times M \longrightarrow M \times M, \qquad (g, m) \mapsto (g.m, m)$$

is a proper map.

Problem 3. A continuous action $G \curvearrowright M$ on a locally compact space M is **proper** iff for any compact subset $C \subset M$ the set

$$\{g \in G \mid gC \cap C \neq \emptyset\}$$

is precompact in G.

Problem 4. Let $G \curvearrowright M$ be a proper action. Prove:

(1) Each G-orbit $G.x$ is closed in M.
(2) For each $x \in M$ the stabilizer $G_x = \{g \in G \mid g.x = x\}$ is compact.
(3) The space of orbits $G\backslash M$ is Hausdorff.

Problem 5. Let $K < G$ be a compact subgroup in a lcsc group. Prove that there is a G-invariant compatible metric on G/K.

Problem 6. Prove that if $G \curvearrowright M$ is a proper action on a locally compact secondly countable space, then the action $G \curvearrowright \mathrm{Prob}(M)$ is also proper.
Suggestion: Prove that for any compact subset $Q \subset \mathrm{Prob}(M)$ there is a compact set $C \subset M$ so that

$$\mu(C) > 1/2 \qquad \forall \mu \in Q.$$

Definition 7. Let G be a lcsc, X a compact metrizable space, $G \curvearrowright X$ a continuous action. The action $G \curvearrowright X$ is called a **convergence action** if the diagonal G-action on the space of distinct triples

$$X^{(3)} := \left\{(x_1, x_2, x_3) \in X^3 \mid x_i \neq x_j,\ 1 \leq i < j \leq 3\right\}$$

is proper. A closed subgroup $H < G$ in a convergence action is said to be **elementary** if it fixes a point $x_0 \in X$, or a pair of points $\{x_1, x_2\} \subset X$.

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Definition 8. Let H be a discrete countable group, or a lcsc group, Y a non-empty compact metrizable space, and $H \to \mathrm{Homeo}(Y)$ a continuous homomorphism. The action $H \curvearrowright Y$ is said to be **minimal** if Y has no proper H-invariant closed subsets.

Problem 9. Prove that any continuous action $H \curvearrowright Y$ on a compact space has a non-empty H-invariant subset $Z \subset Y$ the action on which is minimal.

Problem 10. Prove that the following conditions on continuous action $H \curvearrowright Y$ on a compact space are equivalent:

(1) $H \curvearrowright Y$ is minimal.
(2) For any $y \in Y$ the orbit $H.y$ is dense in Y.
(3) For any non-empty open set $U \subset Y$ there is $n \in \mathbb{N}$ and $h_1, \ldots, h_n \in H$ so that $h_1 U \cup \cdots \cup h_n U = Y$.

Problem 11. Let $G \curvearrowright X$ be a convergence action, $H < G$ a closed non-compact and non-elementary subgroup. Prove:

(1) The action $H \curvearrowright X$ is a convergence action.
(2) There is a unique H-invariant minimal closed subset $\Lambda_H \subset X$ (called the **limit set** of H).
(3) The kernel $K_H := \mathrm{Ker}(H \to \mathrm{Homeo}(\Lambda_H))$ is a compact subgroup.

Hence the action of H/K_H on Λ_H is a non-elementary, faithful, minimal, convergence action.

Problem 12. Let M be a proper δ-hyperbolic space, and $H < \mathrm{Isom}(M, d)$ a closed subgroup. Prove that the action $H \curvearrowright \partial M$ is a convergence action.
(The notions of elementary subgroups in the two contexts agree.)