AMENABILITY
(RIGIDITY VIA ERGODIC METHODS)

Hereafter Γ is assumed to be a countable discrete group, but the definitions and proofs can be extended to locally compact secondly countable groups. The left regular representation $\lambda : \Gamma \to U(\ell^2\Gamma)$ by

$$(\lambda(g)f)(x) = f(g^{-1}x).$$

We shall use the same formula to define the linear isometric representation of Γ on $\ell^1(\Gamma)$, on $\ell^\infty\Gamma$, etc.

Theorem A. *For a group Γ the following conditions are equivalent:*

(1) *There exist a sequence $\{F_n\}$ of finite subsets of Γ so that for every $g \in \Gamma$*

$$\lim_{n\to\infty} \frac{|gF_n \triangle F_n|}{|F_n|} = 0.$$

(2) *There exist $f_n \in \ell^2\Gamma$ with $\|f_n\|_2 = 1$ so that for every $g \in \Gamma$*

$$\lim_{n\to\infty} \|\lambda(g)f_n - f_n\|_2 = 0.$$

(3) *There exist $f_n \in \ell^1\Gamma$ with $\|f_n\|_1 = 1$ so that for every $g \in \Gamma$*

$$\lim_{n\to\infty} \|\lambda(g)f_n - f_n\|_1 = 0.$$

(4) *There exists a Γ-invariant point in the space* $\mathrm{MEAN}(\Gamma) \subset (\ell^\infty\Gamma)^*$, *i.e. a linear functional M on $\ell^\infty\Gamma$ that is positive ($f \geq 0$ implies $M(f) \geq 0$), normalized ($M(\mathbf{1}) = 1$), and Γ-invariant ($M(\lambda(g)f - f) = 0$ for $g \in \Gamma$, $f \in \ell^\infty\Gamma$).*

(5) *For any convex compact subset $Q \subset V$ in a locally convex topological vector space V, and Γ-action on Q by continuous affine maps, there is a Γ-fixed point:*

$$\forall\, a : \Gamma \to \mathrm{Aff}(Q), \qquad Q^\Gamma \neq \emptyset.$$

(6) *For any action $\Gamma \curvearrowright X$ by homeomorphisms, on a compact metrizable space X, there is Γ-invariant probability measure μ on X:*

$$\forall\, \Gamma \to \mathrm{Homeo}(X), \qquad \mathrm{Prob}(X)^\Gamma \neq \emptyset.$$

*Groups satisfying these equivalent conditions are called **amenable**.*

Problem 1. Prove that if condition (1) above (called **Fölner condition**) is satisfied, one may replace the sequence $\{F_n\}$ by a sequence $\{F'_n\}$ of finite subsets that in addition to the almost invariance satisfies:

$$F'_n \subset F'_2 \subset \dots F'_n \;\to \Gamma.$$

Problem 2. Prove that in conditions (2) and (3) one may assume f_n to be positive.

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Problem 3. Prove: (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) $\implies$ (5) $\implies$ (6).
Suggestions:
For (2) use $f_n = |F_n|^{-1/2} \cdot 1_{F_n}$.
For (3) apply Cauchy-Schwarz

$$|\lambda(g)f_n^2 - f_n^2| = |\lambda(g)f_n - f_n| \cdot |\lambda(g)f_n + f_n|.$$

For (4) embed $\ell^1\Gamma \subset \ell^\infty(\Gamma)^*$ and use weak-* compactness of $\mathrm{MEAN}(\Gamma)$.
For (5) fix a point $q_0 \in Q$ and for $\phi \in V^*$ apply M to the function $f_\phi(g) := \phi(a(g).q_0)$ to find $q \in Q$ with $\phi(q) = M(f_\phi)$.
For (6) note that $\mathrm{Prob}(X)$ is an example of a convex compact with respect to the weak-* convergence (see below).

Let X be a metrizable compact space. Recall that any $\mu \in \mathrm{Prob}(X)$ defines linear functional on $C(X)$ by

$$\mu(f) := \int_X f\, d\mu$$

which is positive ($f \geq 0$ implies $\mu(f) \geq 0$) and normalized ($\mu(\mathbf{1}) = 1$). By Riesz representation theorem every positive, normalized, linear functional comes from a unique $\mu \in \mathrm{Prob}(X)$. The weak-* topology on $\mathrm{Prob}(X)$ is defined by sets

$$U(\mu, f_1, \ldots, f_k, \epsilon) = \left\{ \nu \in \mathrm{Prob}(X) \mid \max_{1\leq j\leq k} |\mu(f_j) - \nu(f_j)| < \epsilon \right\}$$

as a basis for the topology, where $\mu \in \mathrm{Prob}(X)$, $f_1, \ldots, f_k \in C(X)$, $\epsilon > 0$ are fixed.

Problem 4. Prove that $\mathrm{Prob}(X)$ with the weak-* topology is a convex compact metrizable space, the map $X \to \mathrm{Prob}(X)$, $x \mapsto \delta_x$, is a homeomorphic embedding of X as the set of **extremal points**, i.e. probability measures μ that can be written as $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ with $\mu_1, \mu_2 \in \mathrm{Prob}(X)$ only for $\mu = \mu_1 = \mu_2$.
Suggestion: To show that $\mathrm{Prob}(X)$ is a metrizable compact, fix a sequence $\{f_j\}_{j=1}^\infty$ of continuous functions $f_j : X \to [0,1]$ that span a dense subspace in $C(X)$ (prove that such a sequence exists, first), and show that the map $\mathrm{Prob}(X) \to [0,1]^{\mathbb{N}}$, $\mu \mapsto \{\mu(f_j)\}_{j=1}^\infty$, is an embedding with a closed image.

Problem 5. Prove that finite groups and the integers $\mathbb{Z}$ are amenable groups by verifying as many of the properties (1)-(6) in Theorem A as possible.

Problem 6. Prove that the free group $F_2 = \mathbb{Z} * \mathbb{Z}$ is not amenable by observing the failure of as of the properties (1)-(6) in Theorem A as possible.

A finitely generated group is said to have **sub-exponential growth** if

$$\limsup_{n\to\infty} n^{-1} \log |B_n| = 0,$$

where B_n denotes the ball of radius n with respect to a fixed word metric on the group (check that this property is independent of the choice of a metric).

Problem 7. Prove that any finitely generated group of sub-exponential growth is amenable.
Suggestion: Prove that some sub-sequence of balls $\{B_{n_j}\}_{j=1}^\infty$ forms a Fölner sequence.

Theorem B. *Prove that the class Amen of amenable groups is closed under the following operations:*

(1) *Forming extensions: if* $1 \to A \to B \to C \to 1$ *is an exact sequence of groups with* A *and* C *amenable, then* B *is amenable.*
(2) *Taking subgroups: if* $A < B$ *and* B *is amenable, then also* A *is amenable.*
(3) *Taking quotients: if* $B \to C$ *is a surjective homomorphism and* B *is amenable, then* C *is amenable.*
(4) *Forming direct limits: in particular, if* $A_1 < A_2 < \dots$ *is an increasing sequence of amenable groups then their union is also amenable.*

Problem 8. Prove properties (1), (3), (4) in Theorem B using the fixed point characterization of amenability as in Theorem A.(5).

Problem 9 (Important)**.** Prove that a finite extension of a solvable group has the fixed point characterization of amenability as in Theorem A.(5).

Theorem C. *Let* G *be a lcsc group,* $P < G$ *a closed amenable subgroup,* $\Gamma <_L G$ *a lattice,* X *and a compact metrizable space and* $\rho : \Gamma \to \mathrm{Homeo}(X)$ *a homomorphism.*
Prove that there exists a measurable map

$$\phi : G/P \to \mathrm{Prob}(X) \qquad \textit{satisfying} \quad \textit{a.e.} \qquad \phi \circ \gamma = \rho(\gamma) \circ \phi \quad (\gamma \in \Gamma).$$

Proof. Consider the collection of equivalence classes (up to agreement on co-null sets) of measurable maps

$$Q = \{ f : G \to \mathrm{Prob}(X) \mid f(\gamma g) = \rho(\gamma)\phi(g) \quad \gamma \in \Gamma, \text{ a.e. } g \in G \} \;/\; \sim$$

This set is non-empty, convex, and compact w.r.to convergence in measure on finite measure subsets. One can also view Q as a subset of the unit ball in

$$L^\infty(\Gamma\backslash G, \mathrm{Prob}(X)) \subset L^\infty(\Gamma\backslash G, C(X)^*) = L^1(\Gamma\backslash G, C(X))^*.$$

G acts by left translations on Q; this action is affine and continuous in the above topology. The restriction of this action has a P-fixed point $\Phi : G \to \mathrm{Prob}(X)$ in Q, that gives a measurable Γ-equivariant map $\phi : G/P \to \mathrm{Prob}(X)$. □

Let Γ be a group. A function $\rho : \Gamma \to \mathbb{R}$ is called a **quasi-morphism** if

$$\sup_{g,h \in \Gamma} |\rho(xy) - \rho(x) - \rho(y)| < +\infty.$$

The set of all quasi-morphisms forms a vector space w.r.to pointwise operations of addition and multiplication by a scalar. It contains all bounded functions $\ell^\infty\Gamma$ and the space of characters $H^1(\Gamma, \mathbb{R})$ as subspaces.

Problem 10. Prove that if Γ is amenable than any quasi-morphism $\rho : \Gamma \to \mathbb{R}$ is a sum of a character and a bounded function.