GALOIS CORRESPONDENCE
(RIGIDITY VIA ERGODIC METHODS)

Definition 1. Let $\mathcal{A}$ and $\mathcal{B}$ be two **partially ordered sets** (hereafter POSets). A **Galois Correspondence** between these POSets is a pair of order-reversing maps

$$\alpha : \mathcal{B} \longrightarrow \mathcal{A}, \qquad \beta \mathcal{A} \longrightarrow \mathcal{B}$$

such that for any $A \in \mathcal{A}$ and $B \in \mathcal{B}$ one has

$$A \leq \alpha(B) \qquad \Longleftrightarrow \qquad \beta(A) \geq B.$$

Hereafter we fix a Galois Correspondence as above.

Problem 2. Consider the maps

$$\alpha \circ \beta : \mathcal{A} \to \mathcal{A}, \qquad \beta \circ \alpha : \mathcal{B} \to \mathcal{B}.$$

and denote $\bar{A} := \alpha \circ \beta(A)$ and $\bar{B} := \beta \circ \alpha(B)$ for $A \in \mathcal{A}$, $B \in \mathcal{B}$. Prove

(1) $A \mapsto \bar{A}$ and $B \mapsto \bar{B}$ are order preserving.
(2) $A \leq \bar{A}$ and $B \leq \bar{B}$ for every $A \in \mathcal{A}$, $B \in \mathcal{B}$.
(3) $\bar{\bar{A}} = \bar{A}$ and $\bar{\bar{B}} = \bar{B}$ for every $A \in \mathcal{A}$, $B \in \mathcal{B}$.

Problem 3. Prove that in a Galois Correspondence each one of the maps $\alpha : \mathcal{B} \to \mathcal{A}$, $\beta : \mathcal{A} \to \mathcal{B}$, is determined by the other by proving that

$$\beta(A) = \max \left\{ B' \in \mathcal{B} \mid A \leq \alpha(B') \right\}$$

and similarly for α in terms of β.

The operations $A \leq \bar{A}$ are called the **closure** operation, and elements with

$$A = \bar{A}$$

are called **closed** in $\mathcal{A}$. Similarly for $\mathcal{B}$. Denote the collections of closed objects by a bar:

$$\bar{\mathcal{A}} := \left\{ \bar{A} \mid A \in \mathcal{A} \right\}, \qquad \bar{\mathcal{B}} := \left\{ \bar{B} \mid B \in \mathcal{B} \right\}.$$

Problem 4. Prove that $\bar{\mathcal{A}}$ is a POSet which is a "retract" of $\mathcal{A}$: it is a sub-POSet and the image under $A \mapsto \bar{A}$ idempotent. Prove that α and β induce and anti-isomorphism of POSets

$$\bar{\beta} = \bar{\alpha}^{-1} : \bar{\mathcal{A}} \xrightarrow{\cong} \bar{\mathcal{B}}.$$

Date: Aug, 2017.