Problem sheet: Differential operators

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Lecture one exercises

1. If $\partial \in \operatorname{Der}_A(R)$, then $\partial(1) = 0$, and $\partial(a) = 0$ for all $a \in A$.

2. Use the definition to show that each $D^i_{R|A}$ is an R-module by the rule $r \cdot \delta = \bar{r} \circ \delta$.

3. Show that if $\alpha \in D^m_{R|A}$ and $\beta \in D^n_{R|A}$, then $\alpha \circ \beta \in D^{m+n}_{R|A}$. Conclude that $D_{R|A}$ is a (not-necessarily commutative) ring where the multiplication is composition.

4. Let $R = A[x]$. What is the difference between $\frac{\partial}{\partial x}\bar{x}$ and $\frac{\partial}{\partial x}(x)$?

5. Show that if $\alpha \in D^m_{R|A}$ and $\beta \in D^n_{R|A}$, then $\alpha\beta - \beta\alpha \in D^{m+n-1}_{R|A}$. Conclude that the graded ring $\bigoplus_{i\in\mathbb{N}} \frac{D^i_{R|A}}{D^{i-1}_{R|A}}$ is commutative.

6. Let $A \subseteq R \subseteq S$ be rings. Let $\iota : R \to S$ be the inclusion map, and $\pi : S \to R$ be an R-linear map. If $\delta \in D^i_{S|A}$, then $\pi \circ \delta \circ \iota \in D^i_{R|A}$.

7. Let $R = \dfrac{\mathbb{k}[x, y, z]}{(xy - z^2)}$. Compute $P^2_{R|\mathbb{k}}$ and compute $D^2_{R|\mathbb{k}}$.

8. Let $\mathbb{k}$ be a field, and $(R, \mathfrak{m}, \mathbb{k})$ be a finite length local $\mathbb{k}$-algebra. Show that $D_{R|\mathbb{k}} = \operatorname{Hom}_A(R, R)$.

9. Prove the two bullet points

   - $f(\underline{x} + \underline{z}) = \displaystyle\sum_{\lambda\in\mathbb{N}^n} \partial_\lambda(f(\underline{x}))z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ for all $f(\underline{x}) \in R$.
   - If $\dfrac{1}{\lambda_1! \cdots \lambda_n!} \in A$, then $\partial_\lambda(f(\underline{x})) = \dfrac{1}{\lambda_1! \cdots \lambda_n!} \dfrac{\partial^{|\lambda|} f}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}}$.

   about the general case of Taylor's formula.

10. Let $A \subseteq R$ be an inclusion of rings that is essentially of finite type, and $W \subseteq R$ be a multiplicative set.

    (a) Show that $W^{-1}P^i_{R|A} \cong P^i_{W^{-1}R|A}$. Note that in the LHS, the localization is over R. (Hint: a unit + a nilpotent is a unit.)

(b) Show that each $P^i_{R|A}$ is a finitely generated R-module.

(c) Show that $W^{-1}D^i_{R|A} \cong D^i_{W^{-1}R|A}$ for all i, as R-modules.

(d) Verify that the localization map $D^i_{R|A} \to W^{-1}D^i_{R|A} \cong D^i_{W^{-1}R|A}$ sends a differential operator δ on R to a differential operator $\widetilde{\delta}$ on $W^{-1}R$ such that $\widetilde{\delta}|_R = \delta$.

11. Use the description of $D_{R|\mathbb{k}}$ in the Bernstein-Gelfand-Gelfand example to show that this ring of differential operators is *not* finitely generated as a $\mathbb{k}$-algebra.

12. Show that the hypothesis $n > 1$ is necessary in the description of differential operators on Veronese rings.

Lecture two exercises

1. Let $R = \mathbb{F}_3[x]$. Check directly that $\frac{\partial}{\partial x}$ is R^3-linear. Write out a free R^3-basis for $\mathrm{Hom}_{R^3}(R, R)$, and express $\frac{\partial}{\partial x}$ in terms of this basis. Then, express the R-generator of $\mathrm{Hom}_{R^3}(R, R^3)$ as a function of $\frac{\partial}{\partial x}$.

2. Let $\mathbb{k}$ be a field of characteristic zero. Show that $\mathbb{k}[x,y]$ and $\mathbb{k}[x^2, xy, y^2]$ are D-simple, while $\dfrac{\mathbb{k}[x,y,z]}{(x^3+y^3+z^3)}$ is not. Compute the ideal $\mathcal{J}$ in each case.

3. Let $\mathbb{k}$ be a field of characteristic $p \equiv 1 \pmod 3$. Is $\dfrac{\mathbb{k}[x,y,z]}{(x^3+y^3+z^3)}$ D-simple?

4. Show that if $(R, \mathfrak{m})$ is a direct summand of $(S, \mathfrak{n})$, and S is D-simple, then R is as well.

5. Let $\mathbb{k}$ be a field of characteristic $p \equiv 1 \pmod 3$, and $R = \dfrac{\mathbb{k}[x,y,z]}{(x^3+y^3+z^3)}$. Find a homogeneous differential operator of degree zero whose image (in R) is exactly R^p. Show that the operator you found *cannot* occur as the base change (by $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$) of a $\mathbb{Z}_p$-linear differential operator on $R = \dfrac{\mathbb{Z}_p[x,y,z]}{(x^3+y^3+z^3)}$.

6. Show that, for $I \subseteq R$, $I^{\langle n \rangle_R} = I^n$ for all n.

7. Show that if $(R, \mathfrak{m})$ is a regular local ring essentially of finite type over a perfect field $\mathbb{k}$, then $\mathfrak{m}^{\langle n \rangle_{\mathbb{k}}} = \mathfrak{m}^n$ for all n.

8. Let $\mathbb{k}$ be a field of characteristic zero, $S = \mathbb{k}[x_1, \dots, x_n]$ be a polynomial ring, and $R = S^{(d)}$ be the d-th Veronese subring of S. Compute the differential powers of the homogeneous maximal ideal of R.

9. Let R be essentially of finite type over a perfect field $\mathbb{k}$. Suppose that $\mathfrak{p}$ is a prime with $\mathfrak{p} = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R),\, \mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}$, and $R_{\mathfrak{p}}$ is regular. Show that $\mathfrak{p}^{(n)} = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R),\, \mathfrak{m} \supseteq \mathfrak{p}} \mathfrak{m}^n$.

10. Show that if R is essentially of finite type over a perfect field $\mathbb{k}$, and R is F-pure, then

$$I_e(R) = \{r \in R \mid \delta(r) \in \mathfrak{m} \text{ for all } \delta \in \mathrm{Hom}_{R^{p^e}}(R, R)\}.$$

11. Show that if R is F-pure and essentially of finite type over a perfect field $\Bbbk$, and $\mathfrak{p}$ is prime, if there exists a constant c such that for all n, $\mathfrak{p}^{\langle cn \rangle_{\Bbbk}} \subseteq \mathfrak{p}^{(n)}$, then $R_{\mathfrak{p}}$ is strongly F-regular.

Bonus problems

1. Let $A \subseteq R$ be rings, and $S = \dfrac{R[t]}{(f(t))}$, where $f(t) \in R[t]$ is a polynomial such that $\dfrac{\partial f}{\partial t} \in S$ is a unit. Show that $S \otimes_R P^i_{R|A} \cong P^i_{S|A}$ for all i. (Hint: follow the outline of our computation of differential operators on polynomial rings. Use a similar change of coordinates with t, and use Taylor's formula on $f(t)$.)

2. Let $\Bbbk$ be a field of characteristic zero, $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring, and $R = S^{(d)}$ be the d-th Veronese subring of S, with $n > 1$.

   (a) Show that $S_{x_j} \otimes_R P^i_{R_{x_j^d}|\Bbbk} \cong P^i_{S_{x_j}|\Bbbk}$ for all i, j.

   (b) Use the previous part to show that for each i, j, $D^i_{S_{x_j}|\Bbbk} \cong D^i_{R_{x_j^d}|\Bbbk}(S_{x_j})$.

   (c) Use the previous part to show that $D^i_{S|\Bbbk} \cong D^i_{R|\Bbbk}(S)$. (Hint: these are reflexive modules.)

   (d) Let $\pi : S \to R$ be the R-linear map given by sending a homogenous element to itself if d divides its degree, and to zero if d does not divide its degree. Show that the map
   $$\pi_* : D^i_{R|\Bbbk}(S) \to D^i_{R|\Bbbk} \qquad \delta \mapsto \pi \circ \delta$$
   gives a split surjection of R-modules for each i.

   (e) Show that $D^i_{R|\Bbbk}$ matches the given description.

3. Let $A \subseteq R \subseteq S$ be rings, where R and S are normal domains. Suppose that the inclusion of R into S is étale in codimension one. Show that every differential operator in $D^i_{R|A}$ extends to a differential operator in $D^i_{S|A}$. (Hint: Use the local structure theory of étale maps.)

4. Let $\Bbbk$ be a field, $S = \Bbbk[x_1, \dots, x_n]$, and $R = S/(f)$ for a homogenous form f.

   (a) Show that, up to a graded shift, there is a graded isomorphism
   $$D^i_{R|\Bbbk} \cong \operatorname{Hom}^{\mathrm{gr}}_{\Bbbk}(\omega_{P^i_{R|\Bbbk}}, \Bbbk),$$
   where $\operatorname{Hom}^{\mathrm{gr}}_{\Bbbk}$ is the module of graded $\Bbbk$-linear homomorphisms and $\omega_{P^i_{R|\Bbbk}}$ is a graded canonical module for the ring $P^i_{R|\Bbbk}$.

   (b) Show that, up to a graded shift, there is a graded isomorphism
   $$\operatorname{Ext}^{n-1}_{R \otimes_{\Bbbk} R}(P^i_{R|\Bbbk}, R \otimes_{\Bbbk} R) \cong \operatorname{Hom}^{\mathrm{gr}}_{\Bbbk}(\omega_{P^i_{R|\Bbbk}}, \Bbbk).$$

(c) Show that, up to a graded shift, there is a graded isomorphism

$$D_{R|\Bbbk} \cong \mathrm{H}^{n-1}_{\Delta_{R|\Bbbk}}(R \otimes_{\Bbbk} R).$$

5. Let $S = \mathbb{Z}_p[x_1, \dots, x_n]$, and $R = S/(f)$ for a homogeneous form f, where $\mathbb{Z}_p$ is the p-adic integers. Show that, up to a graded shift, there is a graded isomorphism

$$D_{R|\mathbb{Z}_p} \cong \mathrm{H}^{n-1}_{\Delta_{R|\mathbb{Z}_p}}(R \otimes_{\mathbb{Z}_p} R).$$

6. Let $S = \mathbb{Z}_p[x_1, \dots, x_n]$, and $R = S/(f)$ for a homogeneous form f, where $\mathbb{Z}_p$ is the p-adic integers. Let $\bar{r} = R/pR$. Show that there is an exact sequence:

$$D_{R|\mathbb{Z}_p} \xrightarrow{\cdot p} D_{R|\mathbb{Z}_p} \to D_{\bar{R}|\mathbb{F}_p} \to \mathrm{H}^{n}_{\Delta_{R|\mathbb{Z}_p}}(R \otimes_{\mathbb{Z}_p} R) \xrightarrow{\cdot p} \mathrm{H}^{n}_{\Delta_{R|\mathbb{Z}_p}}(R \otimes_{\mathbb{Z}_p} R).$$

7. Let $p \equiv 1 \pmod{3}$. Show that there is a nonzero p-torsion element in

$$\mathrm{H}^3_{(x-\tilde{x}, y-\tilde{y}, z-\tilde{z})}\left(\frac{\mathbb{Z}_p[x, y, z, \tilde{x}, \tilde{y}, \tilde{z}]}{(x^3 + y^3 + z^3, \tilde{x}^3 + \tilde{y}^3 + \tilde{z}^3)}\right).$$

Conclude that

$$\mathrm{H}^3_{(x-\tilde{x}, y-\tilde{y}, z-\tilde{z})}\left(\frac{\mathbb{Z}[x, y, z, \tilde{x}, \tilde{y}, \tilde{z}]}{(x^3 + y^3 + z^3, \tilde{x}^3 + \tilde{y}^3 + \tilde{z}^3)}\right)$$

has infinitely many associated primes.