Topics in commutative algebra:

# Differential Operators

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## 0 What am I?

These are notes from two lectures given at the RTG Minicourse on Topics in Commutative Algebra that ran from May 7–11, 2018 at the University of Utah. The goal of these lectures is to introduce rings of differential operators on commutative rings in general (as opposed to just the polynomial / power series case), and to give some basic connections between

F-singularities, symbolic powers, and differential operators. The notes were much improved by suggestions of Eloísa Grifo, Luis Núñez-Betancourt, and the students in the workshop. I thank Srikanth Iyengar and Anurag Singh for organizing the workshop and inviting me to speak there. I also thank the NSF for funding the workshop by the grant DMS #1246989, and for funding me by the grant DMS #1606353.

# 1 Introduction to differential operators

## 1.1 Why differential operators

Associated to any pair of rings $A \subseteq R$, there is a ring of $A$-linear differential operators $D_{R|A}$ on $R$. These have proven to be a useful tool in many areas of commutative algebra (and in mathematics more generally!).

1. **Local cohomology:** Local cohomology modules $H^i_I(R)$ often fail to be finitely generated $R$-modules, which makes them difficult to study. However, if $A$ is a field and $R$ a polynomial ring over $A$, every local cohomology module $H^i_I(R)$ is not only finitely generated, but even *finite length* as a module over the bigger noncommutative ring $D_{R|A}$. The consequences of this are perhaps the best known application of differential operators in commutative algebra; there is no shortage of material on this topic, so I won't be saying any more about this.

2. **Singularities**

3. **Symbolic powers**

4. **And more:** Invariant theory, resolutions of singularities, multiplier ideals, Rees algebras, etc.

In harmony with the other topics, we will focus on the connections with singularities and symbolic powers here.

## 1.2 Derivations and differentials

Differential operators are a generalization of derivations, which you might be familiar with. Let us start by recalling some basics of derivations and differentials. These will be just for motivation, so don't worry if some of it is unfamiliar. We refer to [Eis95, Chapter 16] or [Mat89, Chapter 25].

**Definition 1.1** (Derivations)**.** *Let $A \subseteq R$ be a pair of rings, and $M$ an $R$-module. An $A$-linear derivation from $R$ to $M$ is an $A$-linear map $\partial : R \to M$ that satisfies the rule $\partial(ab) = a\partial(b) + b\partial(a)$ for all $a, b \in R$. The set of $A$-linear derivations from $R$ to $M$ is a module, denoted* $\mathrm{Der}_A(M)$.

For example, on $R = A[x]$, the map $\frac{\partial}{\partial x}$ (as we know it from calculus) is an $A$-linear derivation from $R$ to $R$: the rule above is just the Leibniz rule of calculus. This map has the appealing property of decreasing $(x)$-adic order, while being manageably structured.

Here is a nice warmup for those unfamiliar with the definition.

**Exercise 1.2.** If $\partial \in \mathrm{Der}_A(R)$, then $\partial(1) = 0$, and $\partial(a) = 0$ for all $a \in A$.

The functor of "derivations from $R$ to" can be represented. That is, there is a module $\Omega_{R|A}$ with the property that

$$\text{derivations from } R \text{ to } M \quad \longleftrightarrow \quad R\text{-linear maps from } \Omega_{R|A} \text{ to } M.$$

We will recall a construction of this module, and then state properly the assertion above.

First, we define a map $R \otimes_A R \xrightarrow{\text{mult}} R$ defined on simple tensors by $x \otimes y \mapsto xy$. The kernel of this map is the ideal of the diagonal

$$\Delta_{R|A} := \langle \{r \otimes 1 - 1 \otimes r \mid r \in R\} \rangle = \ker(R \otimes_A R \xrightarrow{\text{mult}} R).$$

**Definition 1.3** (Kähler differentials)**.** *Let $A \subseteq R$ be a pair of rings. The module of $A$-linear Kähler differentials on $R$ is*

$$\Omega_{R|A} := \Delta_{R|A}/\Delta_{R|A}^2.$$

There is a natural map $d : R \to \Omega_{R|A}$, call the *universal differential*, given by

$$d(r) = (r \otimes 1 - 1 \otimes r) + \Delta_{R|A}^2 \in \Omega_{R|A}.$$

**Proposition 1.4.** *Let $A \subseteq R$ be a pair of rings. For any $R$-module $M$, there is an isomorphism of $R$-modules*

$$\mathrm{Der}_A(M) \cong \mathrm{Hom}_R(\Omega_{R|A}, M).$$

*This isomorphism is functorial in $M$.*

Derivations and differentials have a close connection to singularities. The beloved Jacobian criterion for regularity is really a criterion on the Kähler differentials.

**Theorem 1.5** (Jacobian criterion)**.** *Let $\Bbbk$ be a* perfect *field, and $(R, \mathfrak{m})$ be a local ring essentially of finite type over $\Bbbk$: i.e., $R$ is a localization of a finitely generated $\Bbbk$-algebra.*

*Since $R$ is essentially of finite type over $\Bbbk$, we can write $R = \left(\frac{\Bbbk[x_1,\dots,x_n]}{(f_1,\dots,f_m)}\right)_{\mathfrak{p}}$. Then, the following are equivalent:*

1. *$R$ is regular;*
2. *The matrix $\left[\frac{\partial f_i}{\partial x_j}\right]_{ij}$ has rank $= \mathrm{ht}_{\Bbbk[\underline{x}]}((\underline{f}))$ when taken modulo $\mathfrak{p}$;*
3. *$\Omega_{R|\Bbbk}$ is free of rank $n - \mathrm{ht}_{\Bbbk[\underline{x}]}((\underline{f}))$.*

**Remark 1.6.** We may suspect that, when the conditions above hold, if $y_1, \dots, y_d$ generate $\mathfrak{m}$ modulo $\mathfrak{m}^2$, then $d(y_1), \dots, d(y_d)$ are a free basis for $\Omega_{R|\Bbbk}$. This is not true in general, as one can see from the ranks above. However, $d(y_1), \dots, d(y_d)$ are part of a free basis in this case.

The condition that $\Bbbk$ is perfect arises since the Jacobian criterion is really testing for *smoothness* over $\Bbbk$, a condition on the differentials that coincides with regularity in the case above. The Jacobian criterion holds more generally when $\Bbbk \subseteq R/\mathfrak{m}$ is separable.

## 1.3 Differential operators

**Definition 1.7** (Differential operators)**.** *Let $A \subseteq R$ be a pair of rings. We define the $A$-linear differential operators on $R$ of order at most $i$, $D^i_{R|A}$, inductively in $i$.*

- $D^0_{R|A} = \mathrm{Hom}_R(R, R) \quad (= \{\bar{r} := \text{“multiplication by } r\text{”} \mid r \in R\})$
- $D^i_{R|A} = \{\delta \in \mathrm{Hom}_A(R, R) \mid \delta \circ \bar{r} - \bar{r} \circ \delta \in D^{i-1}_{R|A} \text{ for all } r \in R\ \}$.

*With no reference to order, the $A$-linear differential operators on $R$ are $D_{R|A} = \bigcup_{i \in \mathbb{N}} D^i_{R|A}$.*

**Exercise 1.8.** Use the definition to show that each $D^i_{R|A}$ is an $R$-module by the rule $r \cdot \delta = \bar{r} \circ \delta$. That is, $R$-linear combinations (in this sense) of differential operators of order at most $i$ are also differential operators of order at most $i$. In accordance with this, we will often write $R\delta$ to denote $\{\bar{r} \circ \delta \mid r \in R\}$.

As noted above, the operators of order 0 do not depend at all on $A$; they are just the multiplications by elements of $R$. Note that a function $\delta$ is $R$-linear if and only if $\delta \circ \bar{r} - \bar{r} \circ \delta = 0$ for all $r \in R$. Thus, we could have given the same definition by starting with the base case $D^{-1}_{R|A} = 0$ and stipulating the same inductive step. One way to think of the inductive step is as saying that differential operators of order at most $i$ are a little bit less $R$-linear that differential operators of order at most $i - 1$.

Let's understand the operators of order at most 1; let $\delta \in D^1_{R|A}$. Set $\delta' = \delta - \overline{\delta(1)}$. By Exercise 1.8, $\delta' \in D^1_{R|A}$. Note also that

$$\delta'(1) = \delta(1) - \overline{\delta(1)}(1) = \delta(1) - \delta(1) \cdot 1 = 0.$$

Then, for any $r \in R$, there is some $s_r \in R$ such that

$$\delta' \circ \bar{r} - \bar{r} \circ \delta' = \bar{s_r}.$$

To compute it, we plug in 1: $\bar{s_r}(1) = s_r$, while

$$(\delta' \circ \bar{r} - \bar{r} \circ \delta')(1) = (\delta' \circ \bar{r})(1) - (\bar{r} \circ \delta')(1) = \delta'(r) - r\delta'(1) = \delta'(r),$$

so

$$\delta' \circ \bar{r} - \bar{r} \circ \delta' = \overline{\delta'(r)}.$$

We then find that for any $r, s \in R$,

$$\delta'(rs) = (\delta' \circ \bar{r})(s) = (\bar{r} \circ \delta')(s) + \overline{\delta'(r)}(s) = r\delta'(s) + \delta'(r)s.$$

Thus $\delta'$ is a ($A$-linear) derivation! We see that we can write any element of $D^1_{R|A}$ as a sum of a “multiplication by” and a derivation; it is a consequence of Exercise 1.2 that this expression is unique. We summarize:

**Proposition 1.9.** *Let $A \subseteq R$ be rings. There is a direct sum decomposition $D^1_{R|A} \cong R \oplus \mathrm{Der}_A(R)$, where the copy of $R$ is the “multiplications by.”*

Differential operators admit another structure.

**Exercise 1.10.** Show that if $\alpha \in D^m_{R|A}$ and $\beta \in D^n_{R|A}$, then $\alpha \circ \beta \in D^{m+n}_{R|A}$. Conclude that $D_{R|A}$ is a (not-necessarily commutative) ring where the multiplication is composition.

In accordance with this structure, we often drop the composition circles: $\alpha\beta$ denotes $\alpha \circ \beta$ in the notation above. By its construction, $D_{R|A}$ is a subring of $\mathrm{End}_A(R)$, so $R$ is tautologically a left $D_{R|A}$-module.

**Remark 1.11.** It is worth making a note on the notation. Many authors use the same symbol for an element $r \in R$ and for the differential operator $\bar{r} \in D^0_{R|A}$. (They don't use the bars.) The cost for us is slightly clunkier notation and a few more words to talk about the module structure. The point is the following.

**Exercise 1.12.** Let $R = A[x]$. What is the difference between $\frac{\partial}{\partial x}\bar{x}$ and $\frac{\partial}{\partial x}(x)$?

While the ring $D_{R|A}$ is not necessarily commutative, it is noncommutative in a relatively mild way, as made precise by the following.

**Exercise 1.13.** Show that if $\alpha \in D^m_{R|A}$ and $\beta \in D^n_{R|A}$, then $\alpha\beta - \beta\alpha \in D^{m+n-1}_{R|A}$. Conclude that the graded ring $\bigoplus_{i\in\mathbb{N}} \frac{D^i_{R|A}}{D^{i-1}_{R|A}}$ is commutative.

Combining Proposition 1.9 and Exercise 1.10, we now have a recipe for a large number of differential operators.

**Proposition 1.14.** *Let $A \subseteq R$ be rings. Any $n$-fold composition of $A$-linear derivations is an element of $D^n_{R|A}$.*

*For example, if $R = A[x_1, \ldots, x_d]$, then any operator of the form*

$$\sum \overline{r_\alpha} \frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x^{\alpha_d}}$$

*is a differential operator (of order at most $|\alpha|$).*

There is another recipe for differential operators that is very useful.

**Exercise 1.15.** Let $A \subseteq R \subseteq S$ be rings. Let $\iota : R \to S$ be the inclusion map, and $\pi : S \to R$ be an $R$-linear map. If $\delta \in D^i_{S|A}$, then $\pi \circ \delta \circ \iota \in D^i_{R|A}$.

**Remark 1.16.** We can unpackage the inductive definition a bit as follows. For $r \in R$ and $\alpha \in \mathrm{Hom}_A(R, R)$, set $\mathrm{ad}(r)\alpha = \alpha\bar{r} - \bar{r}\alpha$. Then, $\alpha \in D^i_{R|A}$ if and only if for any $r_1, \ldots, r_{i+1} \in R$, $\mathrm{ad}(r_1)\,\mathrm{ad}(r_2)\cdots\mathrm{ad}(r_{i+1})\alpha = 0$. One also uses the notation $[\alpha, \bar{r}]$ for $\mathrm{ad}(r)\alpha$.

## 1.4 Modules of principal parts

There is an analogue to Kähler differentials that we can use to compute differential operators.

**Definition 1.17** (Principal parts)**.** *Let $A \subseteq R$ be rings. The module of $i$-principal parts of $R$ over $A$ is*

$$P^i_{R|A} := \frac{R \otimes_A R}{\Delta^{i+1}_{R|A}}.$$

These modules naturally have the structure of a (cyclic) $R \otimes_A R$-module. We will often view them as $R$-modules; when we judge these by their $R$-module structures, it will be by the action of the left copy of $R$; i.e., by $r \cdot (a \otimes b + \Delta_{R|A}^{i+1}) = (ra \otimes b + \Delta_{R|A}^{i+1})$.

If $A \subseteq R$ is of finite type, we can write $R = \dfrac{A[x_1, \dots, x_n]}{(f_1, \dots, f_m)}$. Then, we can write

$$R \otimes_A R = \frac{A[x_1, \dots, x_n, \tilde{x}_1, \dots, \tilde{x}_n]}{(f_1(\underline{x}), \dots, f_m(\underline{x}), f_1(\underline{\tilde{x}}), \dots, f_m(\underline{\tilde{x}}))},$$

where $x_i$ stands for $x_i \otimes 1$, and $\tilde{x}_i$ stands for $1 \otimes x_i$. In this notation,

$$\Delta_{R|A} = \langle x_1 - \tilde{x}_1, \dots, x_n - \tilde{x}_n \rangle \subseteq R \otimes_A R.$$

If $A \subseteq R$ is essentially of finite type, then we can write $R \otimes_A R$ as a localization of the presentation above.

Let $d^i : R \to P_{R|A}^i$ be the map given by $d^i(r) = (1 \otimes r + \Delta_{R|A}^{i+1})$.

**Proposition 1.18.** *Let $A \subseteq R$ be rings. Then, there is an isomorphism*

$$\mathrm{Hom}_R(P_{R|A}^i, R) \cong D_{R|A}^i$$

*given by $\phi \mapsto \phi \circ d^i$.*

*Proof.* By Hom-tensor adjunction, there is an isomorphism

$$\mathrm{Hom}_R(R \otimes_A R, R) \cong \mathrm{Hom}_A(R, \mathrm{Hom}_R(R, R)) \cong \mathrm{Hom}_A(R, R),$$

given by

$$\phi(-) \mapsto (r \mapsto \phi(r \otimes -)) \mapsto \phi(1 \otimes -).$$

Both sides are $(R \otimes_A R)$-modules, with the action on the LHS given by $(a \otimes b) \cdot \phi = \phi((a \otimes b)-)$, and the action on the RHS by $(a \otimes b) \cdot \varphi = a\varphi(b-)$. The isomorphism above is $(R \otimes_A R)$-linear with respect to these structures.

Now, for $\alpha \in \mathrm{Hom}_A(R, R)$, we compute

$$\alpha \bar{r} - \bar{r}\alpha = (1 \otimes r)\alpha - (r \otimes 1)\alpha = (1 \otimes r - r \otimes 1)\alpha.$$

That is, in the notation of Remark 1.16, $\mathrm{ad}(r)\alpha = (1 \otimes r - r \otimes 1)\alpha$ via the $(R \otimes_A R)$-module structure on $\mathrm{Hom}_A(R, R)$ given above. Following Remark 1.16, $\alpha \in D_{R|A}^i$ if and only if for any $r_1, \dots, r_{i+1} \in R$,

$$(1 \otimes r_1 - r_1 \otimes 1) \cdots (1 \otimes r_{i+1} - r_{i+1} \otimes 1)\alpha = 0.$$

That is, $D_{R|A}^i = \left(0 :_{\mathrm{Hom}_A(R,R)} \Delta_{R|A}^{i+1}\right)$. Thus, we have

$$D_{R|A}^i \cong \left(0 :_{\mathrm{Hom}_R(R \otimes_A R, R)} \Delta_{R|A}^{i+1}\right) \cong \mathrm{Hom}_R\left(\frac{R \otimes_A R}{\Delta_{R|A}^{i+1}}, R\right).$$

We note finally that the Hom-tensor adjunction map agrees with the map given with precomposition by $d^i$. ∎

**Exercise 1.19.** Let $R = \dfrac{\Bbbk[x,y,z]}{(xy-z^2)}$. Compute $P^2_{R|\Bbbk}$ and use the proposition above (and M2 if you want) to compute $D^2_{R|\Bbbk}$.

We now want to use Proposition 1.18 to compute $D_{R|A}$ when $R = A[\underline{x}]$. As above, we can write

$$R \otimes_A R = A[x_1, \dots, x_n, \tilde{x}_1, \dots, \tilde{x}_n], \qquad \Delta_{R|A} = \langle x_1 - \tilde{x}_1, \dots, x_n - \tilde{x}_n \rangle,$$

where $x_i$ stands for $x_i \otimes 1$ and $\tilde{x}_i$ stands for $1 \otimes x_i$. Write $z_i = \tilde{x}_i - x_i$, so that

$$R \otimes_A R = A[x_1, \dots, x_n, z_1, \dots, z_n], \qquad \Delta_{R|A} = \langle z_1, \dots, z_n \rangle.$$

In these coordinates, we have $P^i_{R|A} = R[z_1, \dots, z_n]/(z_1, \dots, z_n)^{i+1}$. As an $R$-module, this is the free module

$$P^i_{R|A} = \bigoplus_{|\alpha| \le i} R z_1^{\alpha_1} \cdots z_n^{\alpha_n}.$$

Thus, as $R$-modules, we can describe $D^i_{R|A}$ as

$$\operatorname{Hom}_R(P^i_{R|A}, R) = \bigoplus_{|\alpha| \le i} R\, (z_1^{\alpha_1} \cdots z_n^{\alpha_n})^\star.$$

To describe $D^i_{R|A}$ as maps, we need to compute the maps $(z_1^{\alpha_1} \cdots z_n^{\alpha_n})^\star \circ d^i$. Given $f(\underline{x}) \in R$,

$$d^i(f(\underline{x})) = f(\underline{\tilde{x}}) = f(\underline{x} + \underline{z}).$$

At least in the setting of Calc III (over $\mathbb{R}$, let's say), we know how to expand the RHS using Taylor's formula:

$$f(\underline{x} + \underline{z}) = \sum_{\lambda \in \mathbb{N}^n} \frac{1}{\lambda_1! \cdots \lambda_n!} \frac{\partial^{|\lambda|} f}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}} z_1^{\lambda_1} \cdots z_n^{\lambda_n}.$$

Even in rings where the rational numbers $\frac{1}{\lambda_1! \cdots \lambda_n!}$ don't make sense, Taylor's formula still holds. Define $\partial_\lambda$ to be the $A$-linear operator on $R$ such that

$$\partial_\lambda(x_1^{\beta_1} \cdots x_n^{\beta_n}) = \binom{\beta_1}{\lambda_1} \cdots \binom{\beta_n}{\lambda_n} x_1^{\beta_1 - \lambda_1} \cdots x_n^{\beta_n - \lambda_n}. \tag{$\star$}$$

Taylor's formula holds in full generality the following sense:

- $f(\underline{x} + \underline{z}) = \sum_{\lambda \in \mathbb{N}^n} \partial_\lambda(f(\underline{x})) z_1^{\lambda_1} \cdots z_n^{\lambda_n}$ for all $f(\underline{x}) \in R$.
- If $\dfrac{1}{\lambda_1! \cdots \lambda_n!} \in A$, then $\partial_\lambda(f(\underline{x})) = \dfrac{1}{\lambda_1! \cdots \lambda_n!} \dfrac{\partial^{|\lambda|} f}{\partial x_1^{\lambda_1} \cdots \partial x_n^{\lambda_n}}$.

Thus,

$$((z_1^{\alpha_1} \cdots z_n^{\alpha_n})^\star \circ d^i)(f(\underline{x})) = \partial_\alpha(f(\underline{x})).$$

We summarize this computation in the following.

**Proposition 1.20.** *Let $R = A[x_1, \ldots, x_n]$. Then $D^i_{R|A} = \bigoplus_{|\alpha| \leq i} R\partial_\alpha$. If $\mathbb{Q} \subseteq R$, then $D^i_{R|A} = \bigoplus_{|\alpha| \leq i} R\frac{\partial^{\alpha_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial^{\alpha_d}}{\partial x^{\alpha_d}}$.*

This computation generalizes to the affirmative situation of the Jacobian criterion. The difficult part is to show that the associated graded ring of $\Delta \subseteq R \otimes R$ is a polynomial ring; the rest goes almost exactly as above.

**Theorem 1.21.** *Let $\Bbbk$ be a perfect field, and $(R, \mathfrak{m})$ be a* regular *local ring essentially of finite type over $\Bbbk$. If $x_1, \ldots, x_t$ are elements of $R$ such that $d(x_1), \ldots, d(x_t)$ form a free basis for $\Omega_{R|\Bbbk}$, then there are $\Bbbk$-linear differential operators $\{\partial_\lambda \mid \lambda \in \mathbb{N}^t\}$ that satisfy the equation $(\star)$ for all $\beta \in \mathbb{N}^t$, and $D^i_{R|\Bbbk}$ is generated as an $R$-module by $\{\partial_\lambda \mid |\lambda| \leq i\}$.*

**Exercise 1.22.** Let $\Bbbk$ be a field, and $(R, \mathfrak{m}, \Bbbk)$ be a finite length local $\Bbbk$-algebra. Show that $D_{R|\Bbbk} = \operatorname{Hom}_A(R, R)$.

**Exercise 1.23.** Prove the two bullet points about the general case of Taylor's formula.

**Exercise 1.24.** Let $A \subseteq R$ be an inclusion of rings that is essentially of finite type, and $W \subseteq R$ be a multiplicative set.

1. Show that $W^{-1}P^i_{R|A} \cong P^i_{W^{-1}R|A}$. Note that in the LHS, the localization is over $R$. (Hint: a unit plus a nilpotent is a unit.)

2. Show that each $P^i_{R|A}$ is a finitely generated $R$-module.

3. Show that $W^{-1}D^i_{R|A} \cong D^i_{W^{-1}R|A}$ for all $i$, as $R$-modules.

4. Verify that the localization map $D^i_{R|A} \to W^{-1}D^i_{R|A} \cong D^i_{W^{-1}R|A}$ sends a differential operator $\delta$ on $R$ to a differential operator $\widetilde{\delta}$ on $W^{-1}R$ such that $\widetilde{\delta}|_R = \delta$.

**Remark 1.25.** One can define for $A \subseteq R$ be a pair of rings and an $R$-module $M$, $A$-linear differential operators from $R$ to $M$ inductively as

- $D^0_{R|A}(M) = \operatorname{Hom}_R(R, M)$
- $D^i_{R|A}(M) = \{\delta \in \operatorname{Hom}_A(R, M) \mid \delta \circ \bar{r} - \bar{r} \circ \delta \in D^{i-1}_{R|A}(M) \text{ for all } r \in R\ \}$.

In analogy with Proposition 1.18, and by essentially the same argument, one has the isomorphism $\operatorname{Hom}_R(P^i_{R|A}, M) \cong D^i_{R|A}(M)$.

## 1.5 More examples of differential operators

Even with the description given by Proposition 1.18, it is often difficult to compute differential operators. Here we will state two more classes of examples, and outline a proof for one of them.

**Example 1.26.** Let $\Bbbk$ be a field of characteristic zero, $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring of dimension $n > 1$, and $R = S^{(d)}$ be the $d$-th Veronese subring of $S$: the graded ring spanned by elements whose degrees are multiples of $d$.

As we computed above, the ring $D_{S|\Bbbk}$ is generated over $\Bbbk$ by $x_1, \dots, x_n, \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$. This is a graded noncommutative ring if we set $\deg(x_i) = 1$, $\deg(\frac{\partial}{\partial x_i}) = -1$, their degrees as maps $S \to S$.

Then, $D^i_{R|\Bbbk} = (D^i_{S|\Bbbk})^{(d)}$ for all $i$: the differential operators on $R$ are spanned by the differential operators on $S$ whose degrees are multiples of $d$.

For a specific example,

$$D^1_{\Bbbk[x^2,xy,y^2]\,|\,\Bbbk} = \Bbbk[x^2, xy, y^2] \left\langle \bar{1}, \bar{x}\frac{\partial}{\partial x}, \bar{y}\frac{\partial}{\partial x}, \bar{x}\frac{\partial}{\partial y}, \bar{y}\frac{\partial}{\partial y} \right\rangle$$

and

$$D^2_{\Bbbk[x^2,xy,y^2]\,|\,\Bbbk} = \Bbbk[x^2, xy, y^2] \left\langle \bar{1}, \bar{x}\frac{\partial}{\partial x}, \bar{y}\frac{\partial}{\partial x}, \bar{x}\frac{\partial}{\partial y}, \bar{y}\frac{\partial}{\partial y}, \frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial xy}, \frac{\partial^2}{\partial y^2} \right\rangle.$$

One can check that the module generators for $D^2_{\Bbbk[x^2,xy,y^2]\,|\,\Bbbk}$ specified above, along with $\overline{x^2}, \overline{xy}, \overline{y^2}$, actually generate $D_{\Bbbk[x^2,xy,y^2]\,|\,\Bbbk}$ as a $\Bbbk$-algebra.

**Example 1.27** (Bernstein–Gelfand–Gelfand [BGG72])**.** Let $\Bbbk$ be a field of characteristic zero, and $R = \dfrac{\Bbbk[x,y,z]}{(x^3+y^3+z^3)}$. Then, each $D^i_{R|\Bbbk}$ is graded, and one has

- $[D_{R|\Bbbk}]_{<0} = 0$; i.e., there are no differential operators of negative degree.
- $[D_{R|\Bbbk}]_0 = \Bbbk[x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}]$; i.e., every differential operator of degree zero is a polynomial in the "Euler operator" $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}$ which sends a homogeneous element $F$ of degree $d$ to $dF$.
- $\dfrac{[D^i_{R|\Bbbk}]_1}{[D^{i-1}_{R|\Bbbk}]_1 + (x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z})[D^{i-1}_{R|\Bbbk}]_1} \cong \Bbbk^3$ as vector spaces for each $i$.

**Exercise 1.28.** Use the description of $D_{R|\Bbbk}$ in the previous example to show that this ring of differential operators is *not* finitely generated as a $\Bbbk$-algebra.

**Exercise 1.29.** Show that the hypothesis $n > 1$ is necessary in Example 1.26.

The next pair of exercises outlines a proof of the first example.

**Exercise 1.30.** Let $A \subseteq R$ be rings, and $S = \dfrac{R[t]}{(f(t))}$, where $f(t) \in R[t]$ is a polynomial such that $\dfrac{\partial f}{\partial t} \in S$ is a unit. Show that $S \otimes_R P^i_{R|A} \cong P^i_{S|A}$ for all $i$. (Hint: follow the outline of our computation of differential operators on polynomial rings. Use a similar change of coordinates with $t$, and use Taylor's formula on $f(t)$.)

**Exercise 1.31.** Let $\Bbbk$ be a field of characteristic zero, $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring, and $R = S^{(d)}$ be the $d$-th Veronese subring of $S$, with $n > 1$.

1. Show that $S_{x_j} \otimes_R P^i_{R_{x_j^d}|\Bbbk} \cong P^i_{S_{x_j}|\Bbbk}$ for all $i, j$.

2. Use the previous part to show that for each $i, j$, $D^i_{S_{x_j}|\Bbbk} \cong D^i_{R_{x_j^d}|\Bbbk}(S_{x_j})$.

3. Use the previous part to show that $D^i_{S|\Bbbk} \cong D^i_{R|\Bbbk}(S)$. (Hint: these are reflexive modules.)

4. Let $\pi : S \to R$ be the $R$-linear map given by sending a homogenous element to itself if $d$ divides its degree, and to zero if $d$ does not divide its degree. Show that the map

$$\pi_* : D^i_{R|\Bbbk}(S) \to D^i_{R|\Bbbk} \qquad \delta \mapsto \pi \circ \delta$$

gives a split surjection of $R$-modules for each $i$.

5. Show that $D^i_{R|\Bbbk}$ matches the given description.

**Exercise 1.32.** Let $A \subseteq R \subseteq S$ be rings, where $R$ and $S$ are normal domains. Suppose that the inclusion of $R$ into $S$ is étale in codimension one. Show that every differential operator in $D^i_{R|A}$ extends to a differential operator in $D^i_{S|A}$. (Hint: Use the local structure theory of étale maps.)

We include here another consequence of Proposition 1.18 that allows us to give a different formula to describe differential operators.

**Exercise 1.33.** Let $\Bbbk$ be a field, $S = \Bbbk[x_1, \dots, x_n]$, and $R = S/(f)$ for a homogenous form $f$.

1. Show that, up to a graded shift, there is a graded isomorphism

$$D^i_{R|\Bbbk} \cong \operatorname{Hom}^{\mathrm{gr}}_{\Bbbk}(\omega_{P^i_{R|\Bbbk}}, \Bbbk),$$

where $\operatorname{Hom}^{\mathrm{gr}}_{\Bbbk}$ is the module of graded $\Bbbk$-linear homomorphisms and $\omega_{P^i_{R|\Bbbk}}$ is a graded canonical module for the ring $P^i_{R|\Bbbk}$.

2. Show that, up to a graded shift, there is a graded isomorphism

$$\operatorname{Ext}^{n-1}_{R\otimes_\Bbbk R}(P^i_{R|\Bbbk}, R \otimes_\Bbbk R) \cong \operatorname{Hom}^{\mathrm{gr}}_{\Bbbk}(\omega_{P^i_{R|\Bbbk}}, \Bbbk).$$

3. Show that, up to a graded shift, there is a graded isomorphism

$$D_{R|\Bbbk} \cong \mathrm{H}^{n-1}_{\Delta_{R|\Bbbk}}(R \otimes_\Bbbk R).$$

**Exercise 1.34.** Let $S = \mathbb{Z}_p[x_1, \dots, x_n]$, and $R = S/(f)$ for a homogeneous form $f$, where $\mathbb{Z}_p$ is the $p$-adic integers. Show that, up to a graded shift, there is a graded isomorphism

$$D_{R|\mathbb{Z}_p} \cong \mathrm{H}^{n-1}_{\Delta_{R|\mathbb{Z}_p}}(R \otimes_{\mathbb{Z}_p} R).$$

**Exercise 1.35.** Let $S = \mathbb{Z}_p[x_1, \dots, x_n]$, and $R = S/(f)$ for a homogeneous form $f$, where $\mathbb{Z}_p$ is the $p$-adic integers. Let $\bar{r} = R/pR$. Show that there is an exact sequence:

$$D_{R|\mathbb{Z}_p} \xrightarrow{\cdot p} D_{R|\mathbb{Z}_p} \to D_{\bar{R}|\mathbb{F}_p} \to \mathrm{H}^n_{\Delta_{R|\mathbb{Z}_p}}(R \otimes_{\mathbb{Z}_p} R) \xrightarrow{\cdot p} \mathrm{H}^n_{\Delta_{R|\mathbb{Z}_p}}(R \otimes_{\mathbb{Z}_p} R).$$

# 2 Differential operators, F-singularities, and symbolic powers

## 2.1 Differential operators in positive characteristic

In some sense, the information encoded by differential operators is closely related to the information captured by properties of the Frobenius map in positive characteristic. This is reflected to a great extent by the parallels between the theories of D-modules and F-modules in the study of local cohomology. One can make a more direct comparison, though.

**Proposition 2.1.** *Let $R$ be essentially of finite type over a perfect field $\Bbbk$. Then*

$$D_{R|\Bbbk} = \bigcup_{e\in\mathbb{N}} \operatorname{Hom}_{R^{p^e}}(R,R).$$

*Proof.* First, we want to observe that both sides are subsets of $\operatorname{Hom}_{\Bbbk}(R,R)$. For the LHS, this was part of the definition. For the RHS, since $\Bbbk$ is perfect, $\Bbbk = \Bbbk^{p^e} \subseteq R^{p^e}$ for each $e$, so $R^{p^e}$-linear implies $\Bbbk$-linear.

From the presentation of $R\otimes_{\Bbbk} R$ we gave last time, we know that $R\otimes_{\Bbbk} R$ is also essentially of finite type, hence noetherian. In particular, there is a constant $a$ such that

$$\Delta^{ap^e}_{R|\Bbbk} \subseteq \Delta^{[p^e]}_{R|\Bbbk} \subseteq \Delta^{p^e}_{R|\Bbbk} \qquad \text{for all } e. \tag{‡}$$

Hence,

$$D_{R|\Bbbk} = \bigcup_{i\in\mathbb{N}} \left(0 :_{\operatorname{Hom}_{\Bbbk}(R,R)} \Delta^{i+1}_{R|\Bbbk}\right) = \bigcup_{i\in\mathbb{N}} \left(0 :_{\operatorname{Hom}_{\Bbbk}(R,R)} \Delta^{[p^e]}_{R|\Bbbk}\right).$$

We can write

$$\Delta^{[p^e]}_{R|\Bbbk} = \langle \{ r^{p^e} \otimes 1 - 1 \otimes r^{p^e} \mid r \in R\}\rangle.$$

Thus, for $\alpha \in \operatorname{Hom}_{\Bbbk}(R,R)$, we have $\Delta^{[p^e]}_{R|\Bbbk} \cdot \alpha = 0$ if and only if $r^{p^e} \otimes 1 \cdot \alpha = 1 \otimes r^{p^e} \cdot \alpha$ for all $r \in R$, which happens if and only if $r^{p^e}\alpha(-) = \alpha(r^{p^e}-)$ for all $r \in R$. That is, $\Delta^{[p^e]}_{R|\Bbbk} \cdot \alpha = 0$ if and only if $\alpha \in \operatorname{Hom}_{R^{p^e}}(R,R)$. $\square$

**Remark 2.2.** It follows from Equation (‡) that if $R$ is essentially of finite type over a perfect field $\Bbbk$, then there exists a constant $a$ such that, for all $e \in \mathbb{N}$, $\operatorname{Hom}_{R^{p^e}}(R,R) \subseteq D^{ap^e}_{R|\Bbbk}$.

**Exercise 2.3.** Let $R = \mathbb{F}_3[x]$. Check directly that $\frac{\partial}{\partial x}$ is $R^3$-linear. Write out a free $R^3$-basis for $\operatorname{Hom}_{R^3}(R,R)$, and express $\frac{\partial}{\partial x}$ in terms of this basis. Then, express the $R$-generator of $\operatorname{Hom}_{R^3}(R,R^3)$ as a function of $\frac{\partial}{\partial x}$.

## 2.2 D-simplicity and F-regularity

We have recalled already that, in a reasonably broad setting, derivations and Kähler differentials can be used to characterize regularity. With the full collection of differential operators, we can detect more subtle properties.

**Definition 2.4.** *Let $A \subseteq R$ be rings. We say that $R$ is D-simple, or D-simple over $A$, if $R$ is a simple $D_{R|A}$-module.*

**Lemma 2.5.** *Let $A \subseteq R$ be rings, with $(R, \mathfrak{m})$ local. The following are equivalent:*

1. *$R$ is D-simple;*
2. *For each nonzero $r \in R$, there is some $\delta \in D_{R|A}$ such that $\delta(r) = 1$;*
3. *The ideal $\mathcal{J} := \{r \in R \mid \delta(r) \in \mathfrak{m}$ for all $\delta \in D_{R|A}\}$ is zero.*

*Proof.* First we check (1)$\Leftrightarrow$(3). If $j \in \mathcal{J}$, then $\delta(j) \in \mathcal{J}$ for all $\delta \in D_{R|A}$, so $\mathcal{J}$ is a $D_{R|A}$-submodule of $D_{R|A}$. Thus, if $R$ is D-simple, $\mathcal{J} = 0$ (since $1 \notin \mathcal{J}$). Conversely, if $R$ is not D-simple, let $0 \neq I \subseteq \mathfrak{m}$ be a proper $D_{R|A}$-submodule of $R$. Then, for any $r \in I$, $\delta(r) \in I \subseteq \mathfrak{m}$ for all $\delta \in D_{R|A}$, so $I \subseteq \mathcal{J}$. Thus, if $R$ is not D-simple, $\mathcal{J} \neq 0$.

Second, to see (2)$\Leftrightarrow$(3), we observe that $r \notin \mathcal{J}$ if and only if some differential operator sends $r$ to a unit, which happens if an only if some differential operator sends $r$ to 1 (since we can postmultiply by the inverse of that unit). $\square$

**Exercise 2.6.** Prove the Lemma above without the hypothesis that $R$ is local.

**Exercise 2.7.** Let $\Bbbk$ be a field of characteristic zero. Show that $\Bbbk[x, y]$ and $\Bbbk[x^2, xy, y^2]$ are D-simple, while $\dfrac{\Bbbk[x, y, z]}{(x^3 + y^3 + z^3)}$ is not. Compute the ideal $\mathcal{J}$ for the last example.

We recall from the other lectures the following.

**Remark 2.8.** Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over a perfect field $\Bbbk$ of characteristic $p > 0$. Then $R$ is F-finite: $R$ is a finite $R^p$-module. In this case:

- $R$ is F-pure if, for each $e$, the inclusion $R^{p^e} \xrightarrow{\cdot 1} R$ splits as $R^{p^e}$-modules.
- $R$ is strongly F-regular if for every $c \neq 0$, there is some $e$ such that $R^{p^e} \xrightarrow{\cdot c} R$ splits as $R^{p^e}$-modules.

In general we have the implication

$$\text{strongly F-regular} \quad \Longrightarrow \quad \text{F-pure.}$$

**Theorem 2.9** (Smith)**.** *Let $(R, \mathfrak{m})$ be a local ring essentially of finite type over a perfect field $\Bbbk$. Suppose that $R$ is F-pure. Then $R$ is strongly F-regular if and only if $R$ is D-simple.*

*Proof.* For each $e$, let $\iota_e : R^{p^e} \to R$ be the inclusion map, and let $\theta_e : R \to R^{p^e}$ be an $R^{p^e}$-linear splitting of the inclusion.

Suppose that $R$ is strongly F-regular. Then for each $c \neq 0$, there is some $R^{p^e}$-linear $\phi : R \to R^{p^e}$ such that $\phi(c) = 1$. Then $(\iota_e \circ \phi) : R \to R$ is $R^{p^e}$-linear, hence an element of $\operatorname{Hom}_{R^{p^e}}(R, R) \subseteq D_{R|\Bbbk}$. We have $(\iota_e \circ \phi)(c) = 1$. Thus, $R$ is D-simple by Lemma 2.5.

Conversely, suppose that $R$ is F-pure and D-simple. Let $c \neq 0$ in $R$. By Lemma 2.5, there is a differential operator $\psi : R \to R$ such that $\psi(c) = 1$. By Proposition 2.1, $\psi$ is $R^{p^e}$-linear for some $e$. Then, $(\theta_e \circ \psi) : R \to R^{p^e}$ is $R^{p^e}$-linear, and $(\theta_e \circ \psi)(c) = 1$. That is, $(\theta_e \circ \psi)$ is an $R^{p^e}$-linear splitting of the map $R^{p^e} \xrightarrow{\cdot c} R$. Thus, $R$ is strongly F-regular. $\square$

The previous result can be thought of as expressing the difference between F-purity and strong F-regularity. Moreover, this "difference" is characterized in a characteristic free way!

**Exercise 2.10.** Let $\Bbbk$ be a field of characteristic $p \equiv 1 \pmod 3$. Is $\dfrac{\Bbbk[x,y,z]}{(x^3+y^3+z^3)}$ D-simple?

**Exercise 2.11.** Show that if $(R,\mathfrak{m})$ is a direct summand of $(S,\mathfrak{n})$, and $S$ is D-simple, then $R$ is as well.

**Exercise 2.12.** Let $\Bbbk$ be a field of characteristic $p \equiv 1 \pmod 3$, and $R = \dfrac{\Bbbk[x,y,z]}{(x^3+y^3+z^3)}$. Find a homogeneous differential operator of degree zero whose image (in $R$) is exactly $R^p$. Show that the operator you found *cannot* occur as the base change (by $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p$) of a $\mathbb{Z}_p$-linear differential operator on $R = \dfrac{\mathbb{Z}_p[x,y,z]}{(x^3+y^3+z^3)}$.

**Exercise 2.13.** Let $p \equiv 1 \pmod 3$. Show that there is a nonzero $p$-torsion element in

$$\mathrm{H}^3_{(x-\tilde{x},y-\tilde{y},z-\tilde{z})}\left(\frac{\mathbb{Z}_p[x,y,z,\tilde{x},\tilde{y},\tilde{z}]}{(x^3+y^3+z^3,\tilde{x}^3+\tilde{y}^3+\tilde{z}^3)}\right).$$

Conclude that

$$\mathrm{H}^3_{(x-\tilde{x},y-\tilde{y},z-\tilde{z})}\left(\frac{\mathbb{Z}[x,y,z,\tilde{x},\tilde{y},\tilde{z}]}{(x^3+y^3+z^3,\tilde{x}^3+\tilde{y}^3+\tilde{z}^3)}\right)$$

has infinitely many associated primes.

## 2.3 Differential powers

In our characterization of D-simplicity, it was useful to consider the ideal

$$\mathcal{J} := \{r \in R \mid \delta(r) \in \mathfrak{m} \text{ for all } \delta \in D_{R|A}\}.$$

The notion of differential powers is a refinement of this construction.

**Definition 2.14** (Differential powers [DDSG+17])**.** *Let $A \subseteq R$ be rings, and $I \subseteq R$ be an ideal. The $i$th $A$-linear differential power of $R$ is defined as*

$$I^{\langle i\rangle_A} := \{r \in R \mid \delta(r) \in I \textit{ for all } \delta \in D^{i-1}_{R|A}\}.$$

For a simple example, note that $I^{\langle 1\rangle_A} = I$: $D^0_{R|A} = R$, so the condition $D^0_{R|A}(r) \subseteq I$ is equivalent to $Rr \subseteq I$, and hence to $r \in I$.

**Exercise 2.15.** Show that, for $I \subseteq R$, $I^{\langle n\rangle_R} = I$ for all $n$.

Differential powers enjoy some nice basic properties.

**Proposition 2.16.** *Let $A \subseteq R$ be rings, and $I \subseteq R$ be an ideal.*

1. *$I^{\langle n\rangle_A}$ is an ideal.*

2. *$I(I^{\langle n-1\rangle_A}) \subseteq I^{\langle n\rangle_A}$, and hence $I^n \subseteq I^{\langle n\rangle_A}$.*

3. *If $I$ is prime, $I^{\langle n\rangle_A}$ is $I$-primary.*

*Proof.* We recall that for $\delta \in D^n_{R|A}$, $\mathrm{ad}(r)\delta = \delta\bar{r} - \bar{r}\delta \in D^{n-1}_{R|A}$; we often use this in the rearranged form $\delta\bar{r} = \bar{r}\delta + \mathrm{ad}(r)\delta$.

1. Let $a, b \in I^{\langle n\rangle_A}$, and $\delta \in D^{n-1}_{R|A}$. We have

$$\delta(a+b) = \underbrace{\delta(a)}_{\in I} + \underbrace{\delta(b)}_{\in I} \in I,$$

so $a + b \in I^{\langle n\rangle_A}$.

Let $a \in I^{\langle n\rangle_A}$, $r \in R$, and $\delta \in D^{n-1}_{R|A}$. We have

$$\delta(ra) = \delta\bar{r}(a) = \bar{r}\,\underbrace{\delta(a)}_{\in I} + \underbrace{(\mathrm{ad}(r)\delta)}_{\in D^{n-2}_{R|A}}(a) \in I,$$

so $ra \in I^{\langle n\rangle_A}$.

2. Let $a \in I$, $b \in I^{\langle n-1\rangle_A}$, and $\delta \in D^{n-1}_{R|A}$. We have

$$\delta(ab) = \delta\bar{a}(b) = \underbrace{\bar{a}}_{\in I}\delta(b) + \underbrace{(\mathrm{ad}(a)\delta)}_{\in D^{n-2}_{R|A}}(b) \in I,$$

so $ab \in I^{\langle n\rangle_A}$.

3. It follows from the previous part that $\sqrt{I^{\langle n\rangle_A}} = I$. We induce on $n$ to show that $r \notin I$ and $ar \in I^{\langle n\rangle_A}$ imply that $a \in I^{\langle n\rangle_A}$. The base case $n = 1$ is trivial. Let $r \notin I$, $ar \in I^{\langle n\rangle_A}$, and $\delta \in D^{n-1}_{R|A}$. By the inductive hypothesis, we can assume that $I^{\langle n-1\rangle_A}$ is $I$-primary, and hence that $a \in I^{\langle n-1\rangle_A}$ (since $ar \in I^{\langle n\rangle_A} \subseteq I^{\langle n-1\rangle_A}$). We have

$$r\delta(a) = \bar{r}\delta(a) = \delta\bar{r}(a) - (\mathrm{ad}(r)\delta)(a) = \underbrace{\delta(ra)}_{\in I} - \underbrace{(\mathrm{ad}(r)\delta)}_{\in D^{n-2}_{R|A}}(a) \in I,$$

so $\delta(a) \in I$, and hence, $a \in I^{\langle n\rangle_A}$. □

**Example 2.17.** Let $R = \Bbbk[x_1, \ldots, x_d]$, and $\mathfrak{m} = (x_1, \ldots, x_d)$. Let's show that $\mathfrak{m}^{\langle n\rangle_\Bbbk} = \mathfrak{m}^n$ for all $n$. We know already that $\mathfrak{m}^n \subseteq \mathfrak{m}^{\langle n\rangle_\Bbbk}$, so it suffices to show the other containment. Let $f(\underline{x}) \in R \smallsetminus \mathfrak{m}^n$. Write $f = \sum_\lambda c_\lambda \underline{x}^\lambda$ as a sum of monomials, and let $\alpha$ be such that $c_\alpha \underline{x}^\alpha$ is a nonzero monomial of minimal degree; we know that $|\alpha| \leq n-1$. Then,

$$\partial_\alpha(f) = \sum_{\lambda \geq \alpha} c_\lambda \partial_\alpha(\underline{x}^\lambda) \equiv c_\lambda \pmod{\mathfrak{m}}.$$

Since $\partial_\alpha \in D^{n-1}_{R|\Bbbk}$, we see that $f(\underline{x}) \notin \mathfrak{m}^{\langle n\rangle_\Bbbk}$.

**Exercise 2.18.** Show that if $(R, \mathfrak{m})$ is a regular local ring essentially of finite type over a perfect field $\Bbbk$, then $\mathfrak{m}^{\langle n \rangle_{\Bbbk}} = \mathfrak{m}^n$ for all $n$.

Even still for maximal ideals, differential powers can be a bit more subtle.

**Example 2.19.** Let $\Bbbk$ be a field of characteristic zero, $R = \dfrac{\Bbbk[x,y,z]}{(x^3+y^3+z^3)}$, and $\mathfrak{m} = (x,y,z)$. Then $\mathfrak{m}^{\langle n \rangle_{\Bbbk}} = \mathfrak{m}$ for all $n$.

Indeed, let $\delta \in D^{n-1}_{R|\Bbbk}$. By Example 1.27, we can write $\delta = \sum_i \delta_i$, where $\delta_i$ of homogeneous of degree $i$, and each $i \geq 0$. Then, $f \in \mathfrak{m}$, $f$ is a sum of positive degree forms, and $\delta(f)$ is then a sum of positive degree forms too, hence in $\mathfrak{m}$. Thus, $\mathfrak{m} \subseteq \mathfrak{m}^{\langle n \rangle_{\Bbbk}}$. The other containment follows since $\mathfrak{m}^{\langle n \rangle_{\Bbbk}}$ is $\mathfrak{m}$-primary.

**Exercise 2.20.** Let $\Bbbk$ be a field of characteristic zero, $S = \Bbbk[x_1, \dots, x_n]$ be a polynomial ring, and $R = S^{(d)}$ be the $d$-th Veronese subring of $S$. Compute the differential powers of the homogeneous maximal ideal of $R$.

There is one more property of differential powers we will use later.

**Proposition 2.21.** *Let $R$ be essentially of finite type over $A$, and $\mathfrak{p} \subset R$ be a prime ideal. Then, $\mathfrak{p}^{\langle n \rangle_A} R_{\mathfrak{p}} \subseteq (\mathfrak{p} R_{\mathfrak{p}})^{\langle n \rangle_A}$.*

*Proof.* Let $f \in \mathfrak{p}^{\langle n \rangle_A}$, and $\delta \in D^{n-1}_{R_{\mathfrak{p}}|A}$. By Exercise 1.24, we can write $\delta = \overline{1/w}\,\eta$, where $\eta|_R \in D^{n-1}_{R|A}$. Then, $\delta(f) = \frac{1}{w}\eta(f) \in \frac{1}{w}\mathfrak{p}R_{\mathfrak{p}} = \mathfrak{p}R_{\mathfrak{p}}$, as required. $\square$

## 2.4 Differential operators and symbolic powers

With the basic facts on differential powers we collected above, we can relate differential powers to symbolic powers.

**Proposition 2.22.** *Let $A \subseteq R$ be rings, and $\mathfrak{p} \subset R$ be prime. Then $\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{\langle n \rangle_A}$.*

*Proof.* By Proposition 2.16, $\mathfrak{p}^{\langle n \rangle_A}$ is a $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$. The symbolic power $\mathfrak{p}^{(n)}$ is the smallest $\mathfrak{p}$-primary ideal containing $\mathfrak{p}^n$. $\square$

**Theorem 2.23** (Zariski-Nagata theorem)**.** *Let $\Bbbk$ be a perfect field, $R$ be essentially of finite type over $\Bbbk$, and $\mathfrak{p} \in \operatorname{Spec}(R)$. If $R_{\mathfrak{p}}$ is regular, then $\mathfrak{p}^{(n)} = \mathfrak{p}^{\langle n \rangle_{\Bbbk}}$ for all $n$.*

*Proof.* We have the containment $\mathfrak{p}^{(n)} \subseteq \mathfrak{p}^{\langle n \rangle_{\Bbbk}}$ already from the previous proposition. For the other containment, since $\mathfrak{p}^{\langle n \rangle_{\Bbbk}}$ is $\mathfrak{p}$-primary by Proposition 2.16 (3), it suffices to check the other containment after localization at $\mathfrak{p}$. We have

$$\mathfrak{p}^{\langle n \rangle_{\Bbbk}} R_{\mathfrak{p}} \underset{\text{Prop 2.21}}{\subseteq} (\mathfrak{p}R_{\mathfrak{p}})^{\langle n \rangle_{\Bbbk}} \underset{\text{Ex 2.18}}{=} (\mathfrak{p}R_{\mathfrak{p}})^n = \mathfrak{p}^{(n)} R_{\mathfrak{p}},$$

as required. $\square$

**Remark 2.24.** The technical structure theorem Theorem 1.21 played a key role here, via Exercise 2.18. We can't just get by with the fact that differential operators behave well under localization, since we need to understand how differential operators act on elements that minimally generate the maximal ideal.

**Example 2.25.** To give a simple example with differential powers, let $\Bbbk$ be a field of characteristic zero, and $R = K[X_{3\times 3}]$, where $X = X_{3\times 3}$ is a $3 \times 3$ matrix of indeterminates. You have seen in Eloísa's exercises that $\det(X) \in I_2(X)^{\langle 2\rangle_\Bbbk}$. To see this with the Zariski-Nagata theorem, note that $D^1_{R|\Bbbk} = R \oplus \bigoplus_{ij} R\frac{\partial}{\partial x_{ij}}$, so it suffices to see that $\frac{\partial \det(X)}{\partial x_{ij}} \in I_2(X)$ for each pair $ij$. This is clear, since, by the Laplace expansion, $\frac{\partial \det(X)}{\partial x_{ij}}$ is equal to the complementary $2 \times 2$ minor.

**Remark 2.26.** We know from Example 2.19 that some sort of regularity hypothesis is necessary in the Zariski-Nagata theorem.

**Exercise 2.27** (Eisenbud-Hochster [EH79])**.** Let $R$ be essentially of finite type over a perfect field $\Bbbk$. Suppose that $\mathfrak{p}$ is a prime with $\mathfrak{p} = \bigcap_{\mathfrak{m}\in \mathrm{Max}(R)\,,\,\mathfrak{m}\supseteq\mathfrak{p}} \mathfrak{m}$, and $R_\mathfrak{p}$ is regular. Show that $\mathfrak{p}^{(n)} = \bigcap_{\mathfrak{m}\in \mathrm{Max}(R)\,,\,\mathfrak{m}\supseteq\mathfrak{p}} \mathfrak{m}^n$.

## 2.5 F-singularities and symbolic powers

We know that, under mild hypotheses, for a prime $\mathfrak{p} \subset R$, there is a constant $c$ such that $\mathfrak{p}^{(cn)} \subseteq \mathfrak{p}^n$ for all $n$. Since symbolic powers are contained in differential powers, we might hope for the stronger result that there is a constant $c$ such that $\mathfrak{p}^{\langle cn\rangle_\Bbbk} \subseteq \mathfrak{p}^n$ for all $n$. It turns out that we can do this if we have good (F-)singularities.

**Definition 2.28.** *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$, and dimension $d$. The splitting ideals of $R$ are defined as*

$$I_e(R) := \{r \in R \mid \varphi(r^{1/p^e}) \in \mathfrak{m} \textit{ for all } \varphi \in \mathrm{Hom}_R(R^{1/p^e}, R)\}.$$

This definition looks a lot like the definition of differential powers.

**Exercise 2.29.** Show that if $R$ is essentially of finite type over a perfect field $\Bbbk$, and $R$ is F-pure, then

$$I_e(R) = \{r \in R \mid \delta(r) \in \mathfrak{m} \text{ for all } \delta \in \mathrm{Hom}_{R^{p^e}}(R, R)\}.$$

The following proposition, originally due to Aberbach and Leuschke [AL03] and greatly simplified in [PT18], is a variation on the fact that positivity of F-signature is equivalent to strong F-regularity. You will have seen it in the end of Linquan and Thomas' lectures.

**Proposition 2.30.** *Let $(R, \mathfrak{m})$ be a local ring of characteristic $p > 0$. If $R$ is strongly F-regular, then there is an integer $b > 0$ such that $I_{e+b}(R) \subseteq \mathfrak{m}^{[p^e]}$ for all $e > 0$.*

**Theorem 2.31** (Linear Zariski-Nagata theorem [BJNnB])**.** *Let $R$ be a strongly F-regular ring essentially of finite type over a perfect field $\Bbbk$, and $\mathfrak{p} \subset R$ be prime. Then, there is a constant $c$ such that for all $n$, $\mathfrak{p}^{\langle cn\rangle_\Bbbk} \subseteq \mathfrak{p}^{(n)}$.*

*Proof.* Since the ideals $\mathfrak{p}^{(n)}$ and $\mathfrak{p}^{\langle n\rangle_K}$ are $\mathfrak{p}$-primary for all $n$ by Proposition 2.16 (3), it suffices to check that for some $c$, $\mathfrak{p}^{\langle cn\rangle_\Bbbk} R_\mathfrak{p} \subseteq \mathfrak{p}^{(n)} R_\mathfrak{p} (= \mathfrak{p}^n R_\mathfrak{p})$ for all $n > 0$. Since $\mathfrak{p}^{\langle cn\rangle_\Bbbk} R_\mathfrak{p} \subseteq (\mathfrak{p}R_\mathfrak{p})^{\langle cn\rangle_\Bbbk}$, it suffices to show that for some $c$, $(\mathfrak{p}R_\mathfrak{p})^{\langle cn\rangle_\Bbbk} \subseteq (\mathfrak{p}R_\mathfrak{p})^n$ for all $n$. Thus, it suffices to show the statement after localizing at $\mathfrak{p}$, so we may assume that $(R, \mathfrak{m})$ is local, and $\mathfrak{p} = \mathfrak{m}$.

For an integer $n$, set $\ell(n) = \lceil \log_p(n) \rceil$: this is the smallest integer $e$ such that $p^e \geq n$.

1. Note that $n \leq p^{\ell(n)} \leq pn$.

2. By Proposition 2.1 and Remark 2.2, there is an $a$ such that $\mathrm{Hom}_{R^{p^e}}(R,R) \subseteq D^{ap^e}_{R|\Bbbk}$ for all $e$. So, if every operator in $D^{ap^e}_{R|\Bbbk}$ sends $f$ into $\mathfrak{m}$, every map in $\mathrm{Hom}_{R^{p^e}}(R,R)$ sends $f$ into $\mathfrak{m}$. Thus, by Exercise 2.29, we find that $\mathfrak{m}^{\langle ap^e \rangle_\Bbbk} \subseteq I_e(R)$ for all $e$.

3. By Proposition 2.30, there is an integer $b$ such that $I_{e+b}(R) \subseteq \mathfrak{m}^{[p^e]}$ for all $e > 0$.

Together, we see

$$\mathfrak{m}^{\langle ap^{b+1}n \rangle_\Bbbk} \underset{(1)}{\subseteq} \mathfrak{m}^{\langle ap^{\ell(n)+b} \rangle_\Bbbk} \underset{(2)}{\subseteq} I_{\ell(n)+b}(R) \underset{(3)}{\subseteq} \mathfrak{m}^{[p^{\ell(n)}]} \underset{(1)}{\subseteq} \mathfrak{m}^n .$$

That is, setting $c = ap^{b+1}$, one has $\mathfrak{m}^{\langle cn \rangle_\Bbbk} \subseteq \mathfrak{m}^n$ for all $n \in \mathbb{N}$, as required. □

**Exercise 2.32.** Show that if $R$ is F-pure and essentially of finite type over a perfect field $\Bbbk$, and $\mathfrak{p}$ is prime, if there exists a constant $c$ such that for all $n$, $\mathfrak{p}^{\langle cn \rangle_\Bbbk} \subseteq \mathfrak{p}^{(n)}$, then $R_\mathfrak{p}$ is strongly F-regular.

# 3 History and references

The notion of differential operators on polynomial rings, has of course, been around for a long time, playing a significant role in 19th and 20th century algebra, e.g., invariant theory [Wey16]. The general notion of differential operators here is (to my knowledge) due to Grothendieck, and dates to the 1960's. The main original source is [Gro67, Chapter 16], which contains most of the material in Subsection 1.4; one traces a long path backwards through EGA for the proof of Theorem 1.21 there without the help of Vasconcelos' theorem. Differential operators also show up around the same time in work of Heynemann and Sweedler [HS69] and Nakai [Nak70]. The study of the structure of rings of differential operators (as noncommutative rings) had a flurry of activity in the 1980's and 1990's: a couple of good places to get a first glimpse are in the survey of P. Smith [Smi86] and the intro of [LS89].

The theory of differential operators over polynomial rings and power series rings comes with a rich and rigid module theory (D-modules) that builds on Bernstein's inequality, Kashiwara's equivalence, and the Riemann-Hilbert correspondence. A friendly introduction to the first two of these topics can be found in [Cou95], and a tougher and more thorough treatment of all of these in [HT07].

The application of differential operators to local cohomology is due to Lyubeznik [Lyu93] in 1993. The approach is largely based on the aforementioned theory of D-modules over polynomial rings and power series rings that is outside the scope of these notes. Apropos the theme of these notes is the work of Àlvarez-Montaner, Huneke, and Núñez-Betancourt [MHNB17], which jacks up the idea of Exercise 1.15 to extend results on the structure of local cohomology of polynomial rings to direct summands of polynomial rings.

Proposition 2.1 in various levels of generality has been known for a while; the most general version (moreso than appearing here) is the work of Yekutieli [Yek92]. The connections between differential operators and F-singularities begin with the work of K. Smith [Smi95] (1995) and Smith and Van den Bergh [SVdB97] (1997). In particular, Theorem 2.9 and

Exercises 2.11 and 2.12 are from [Smi95]. Joint work in progress with Brenner and Núñez-Betancourt [BJNnB] pushes this theme, and has motivated a lot of the presentation here.

The connections between differential operators and symbolic powers go back a bit further, in some sense. The Zariski-Nagata theorem [Zar49, Nag62] characterizing symbolic powers on smooth varieties in terms of order of vanishing was established in the 1960's, and the interpretation in terms of differential operators over fields of characteristic zero has been known probably since around then; see e.g., [Eis95, Chapter 3]. This connection has been applied to compute symbolic powers by Simis [Sim96], also in the 1990's. The point of view pursued here, via differential powers, follows the survey [DDSG+17], with some simplifications from [DSGJ] and [BJNnB]. Theorem 2.31 is from [BJNnB]. Also worth noting is a generalization of Theorem 2.23 to mixed characteristic in [DSGJ].

Exercise 1.31 is a special case of a result of Kantor [Kan77] that shows that, for a polynomial ring $S$ and a "small" group action $G$, $D_{S^G} \cong (D_S)^G$. The exercises 1.33,1.34,1.35, and 2.13 are from [Jef].

# 4 Curation of exercises

**Warmup exercises:**

- Exercise 1.2
- Exercise 1.8
- Exercise 1.10
- Exercise 1.12
- Exercise 1.19
- Exercise 1.22
- Exercise 1.23
- Exercise 1.28
- Exercise 2.3
- Exercise 2.7
- Exercise 2.10
- Exercise 2.15
- Exercise 2.18
- Exercise 2.20
- Exercise 2.29

**Somewhat tougher exercises:**

- Exercise 1.13
- Exercise 1.15
- Exercise 1.24
- Exercise 1.29
- Exercise 1.30
- Exercise 1.31
- Exercise 1.33
- Exercise 1.34
- Exercise 1.35
- Exercise 2.6
- Exercise 2.12
- Exercise 2.13
- Exercise 2.27
- Exercise 2.32

**Exercises used later:**

- Exercise 1.8
- Exercise 1.10
- Exercise 1.15
- Exercise 1.23
- Exercise 1.24
- Exercise 2.18
- Exercise 2.29

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