Symbolic powers exercises

1) Show that a finite intersection of P-primary ideals is a P-primary ideal.

2) Let R be a noetherian ring and I an ideal in R. Show that a prime ideal P is associated to I if and only if $\operatorname{depth}(R_P/I_P) = 0$.

3) Given ideals I and J in a noetherian ring R, the following are equivalent:

   (a) $I \subseteq J$;

   (b) $I_P \subseteq J_P$ for all primes $P \in \operatorname{Supp}(R/J)$;

   (c) $I_P \subseteq J_P$ for all primes $P \in \operatorname{Ass}(R/J)$.

4) Let I be an ideal with no embedded primes in a noetherian ring R.

   (a) $I^{(1)} = I$;

   (b) For all $n \geqslant 1$, $I^n \subseteq I^{(n)}$;

   (c) $I^a \subseteq I^{(b)}$ if and only if $a \geqslant b$.

   (d) If $a \geqslant b$, then $I^{(a)} \subseteq I^{(b)}$;

   (e) For all $a, b \geqslant 1$, $I^{(a)}I^{(b)} \subseteq I^{(a+b)}$.

   (f) $I^n = I^{(n)}$ if and only if I^n has no embedded primes.

5) Show that if P is prime, $P^{(n)}$ is the smallest P-primary ideal containing P^n.

6) Use `Macaulay2` to find primary decompositions of I^2, I^3 and I^{10}, where I is each of the following ideals, and then use these decompositions to determine $I^{(2)}$, $I^{(3)}$ and $I^{(10)}$. Consider the fields $k = \mathbb{Q}, \mathbb{Z}/2$ and $\mathbb{Z}/101$.

   (a) I the defining ideal of the curve (t^3, t^4, t^5) in $k[x, y, z]$.

   (b) $I = (xy, yz, xz)$, in $k[x, y, z]$ and $k[x, y, z, u, v]$.

   (c) $I = (x(y^3 - z^3), y(z^3 - x^3), z(x^3 - y^3))$ in $k[x, y, z]$.

   (d) The ideal generated by all the degree 2 squarefree monomials in $k[x_1, \dots, x_5]$.

   Are there better methods you can use to determine the same symbolic powers using `Macaulay2`? If so, try asking Macaulay2 to compute the symbolic powers of the previous ideals using different methods. Did your answers change with the field?

7) Let I be an ideal in a noetherian ring. Show that $\left(I^d : I^{(d)}\right)$ always contains an element that is not in any minimal prime of I.

8) Show that if $\mathfrak{m}$ is a maximal ideal, $\mathfrak{m}^n = \mathfrak{m}^{(n)}$ for all n.

9) The monomial $I = (xy, xz, yz) \subseteq k[x, y, z]$ does not coincide with its square. However, show that the containment $I^{(3)} \subseteq I^2$ does hold.

10) Consider the ideal $I = I_2(X)$ of 2×2 minors of a generic 3×3 matrix X in the polynomial ring $R = k[X]$ generated by the variables in X over a field k. Show that $g = \det X \in P^{(2)}$, while $g \notin P^2$.

11) Let $I = I_2(X)$, where X is a generic 3×3 matrix. Find generators for $I^{(2)}$.

12) Show that if I is the ideal in $k[X]$ generated by the maximal minors of a generic matrix X over a field k of characteristic 0, then $I^n = I^{(n)}$ for all $n \geqslant 1$.

13) Let k be a field, $R = k[x, y, z]$, and consider $P = (x^2y - z^2, xz - y^2, yz - x^3)$ such that $R/P \cong k[t^3, t^4, t^5]$. Show that $P^{(n)} \neq P^n$ for all $n \geqslant 2$.

14) Show that if I is generated by a regular sequence, then $I^n = I^{(n)}$ for all $n \geqslant 1$.

15) Let R be a Cohen-Macaulay local ring and P be a prime ideal such that $\dim(R/P) = 1$. Show that $P^{(n)} = P^n$ for all $n \geqslant 1$ if and only if P is generated by a regular sequence.

16) Give an example of a prime P in a regular local ring R such that P is not generated by a regular sequence but $P^{(n)} = P^n$ for all $n \geqslant 1$.

17) If I is a squarefree monomial ideal in $k[x_1, \ldots, x_n]$, then I is a radical ideal whose minimal primes are generated by variables. Writing an irredundant decomposition $I = \bigcap_i Q_i$, where each Q_i is an ideal generated by variables, show that $I^{(n)} = \bigcap_i Q_i^n$.

18) Give examples of squarefree monomial ideals that are not könig.

19) Give an example of an ideal that is packed and of one that is not packed.

20) Let I be a squarefree monomial ideal. Show that if $I^{(n)} = I^n$ for all $n \geqslant 1$ then I must be packed.

21) Show the Eisenbud–Mazur conjecture for squarefree monomial ideals.

22) Show that the symbolic Rees algebra of an ideal I in a ring R is a finitely generated R-algebra if and only if it is a noetherian ring.

23) If the symbolic Rees algebra of an ideal I in a ring R is finitely generated, show that there exists k such that $I^{(kn)} = \left(I^{(k)}\right)^n$ for all $n \geqslant 1$. The converse also holds as long as R is excellent.

24) Give an example of an ideal I that is generated by the maximal minors of a matrix in a polynomial ring but such that $I^n \neq I^{(n)}$ for all $n \geqslant 1$.

25) Let I be an ideal in a noetherian ring R with no embedded primes. Show that there exists an ideal J such that for all $n \geqslant 1$,

$$I^{(n)} = (I^n : J^\infty).$$

26) Let $(R, \mathfrak{m})$ be a local ring and P a prime ideal of height $\dim R - 1$. Show that $P^{(n)} = (P^n : \mathfrak{m}^\infty)$ for all $n \geqslant 1$. Can you generalize this statement for a larger class of ideals?

27) Solve the containment problem for generic determinantal ideals.

28) Let $(R, \mathfrak{m})$ be a Gorenstein local ring and P a prime ideal of height $\dim R - 1$. Given $a \geqslant b$, show that $P^{(a)} \subseteq P^b$ if and only if the map $\operatorname{Ext}^d_R(R/P^b, R) \to \operatorname{Ext}^d_R(R/P^a, R)$ induced by the canonical projection vanishes.

29) Let $R = k[x_1, \ldots, x_d]$ and consider the squarefree monomial ideal

$$I = \bigcap_{i<j} (x_i, x_j)\,.$$

Show that while $I^{(2n-1)} \nsubseteq I^n$ holds for all $n \geqslant 1$, $I^{(2n-2)} \nsubseteq I^n$ for $n < d$. What happens when $n = d$? How does this example generalize to higher height?

30) Given integers $c < h$, construct an ideal I with height c and big height h in a polynomial ring such that $I^{(cn)} \nsubseteq I^n$ for some n.

31) Let I be a squarefree monomial ideal. Show that I verifies Harbourne's Conjecture.

32) Let R be a regular ring, essentially of finite type over a perfect field, and $P \subseteq Q$ prime ideals. Show that $P^{(n)} \subseteq Q^{(n)}$ for all $n \geqslant 1$.

33) Consider the ring $R = k[u, v, w, x, y, z]/(ux+vy+wz)$. This is a Cohen-Macaulay, normal ring, with an isolated singularity, and even a UFD. However, we can prime ideals $P \subseteq Q$ that fail $P^{(n)} \subseteq Q^{(n)}$. Show that this is the case when Q is the maximal ideal generated by all the variables, and P the prime ideal generated by all the variables but one.

34) What questions could you ask Macaulay2 in an attempt to determine if $I^n = I^{(n)}$ for a given value of n *without* computing $I^{(n)}$?

Characteristic p exercises

35) Let R be a regular ring of characteristic p. For all ideals I and J in R and all $q = p^e$,

$$(J : I)^{[q]} = \left(J^{[q]} : I^{[q]}\right).$$

36) Prove that if R is a regular ring of characteristic p, the Frobenius map preserves associated primes, that is, $\operatorname{Ass}(R/I) = \operatorname{Ass}\left(R/I^{[q]}\right)$ for all $q = p^e$.

37) Suppose that I is a radical ideal of big height h in a regular ring R containing a field of characteristic $p > 0$. Show that for all $q = p^e$,

$$I^{(hq-h+1)} \subseteq I^{[q]} \subseteq I^q.$$

38) Let I be a radical ideal in a regular ring R of characteristic $p > 0$ and h the big height of I. For all $q = p^e$,

$$I^{(hq+kq-h+1)} \subseteq \left(I^{(k+1)}\right)^{[q]}.$$

39) Show that if R is a regular ring of characteristic p and R/I is F-pure, then I verifies Harbourne's Conjecture.

40) Let R be a regular ring of characteristic $p > 0$, and consider an ideal I in R such that R/I is F-pure. Show that given any integer $c \geqslant 1$, if $I^{(hn-c)} \subseteq I^n$ for some n, then $I^{(hn-c)} \subseteq I^n$ for all $n \gg 0$.

41) Find an example of a prime ideal P in a polynomial ring $R = k[x_1, \ldots, x_d]$ of characteristic p such that P is not a complete intersection but $P^{(n)} = P^n$ for all $n \geqslant 1$.