Shimura varieties and the Mumford-Tate conjecture

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Abstract

Let A be an abelian variety over a number field E. Let p be a rational prime. Let $\bar{E}$ be the algebraic closure of E. Let

$$\rho\colon \mathrm{Gal}(\bar{E}/E) \to GL(H^1_{\text{ét}}(A_{\bar{E}}, \mathbb{Q}_p))$$

be the natural p-adic representation. Let $G_{\mathbb{Q}_p}$ be the connected component of the origin of the algebraic subgroup of $GL(H^1_{\text{ét}}(A_{\bar{E}}, \mathbb{Q}_p))$ obtained as the algebraic envelope of this representation. Let H_A be the reductive group over $\mathbb{Q}$ which is the Mumford-Tate group of A. It is known that $H_{A\mathbb{Q}_p}$ can be naturally viewed as a subgroup of $GL(H^1_{\text{ét}}(A_{\bar{E}}, \mathbb{Q}_p))$, containing $G_{\mathbb{Q}_p}$.

A major conjecture (of Mumford and Tate) predicts: for any A and p as above, as subgroups of $GL(H^1_{\text{ét}}(A_{\bar{E}}, \mathbb{Q}_p))$ we have $G_{\mathbb{Q}_p} = H_{A\mathbb{Q}_p}$.

We prove it. The proof is based on the following result:

The Split Criterion: There is $N(A) \in \mathbb{N}$ such that, if $p > N(A)$, and if $G_{\mathbb{Q}_p}$ is a split reductive group, then the Mumford-Tate conjecture is true for A and p.

The proof of the Split Criterion relies heavily on the crystalline machinery developed by Faltings and supplemented by the author, used in the context of Shimura varieties. The talk is divided into four parts. In the first part we review briefly the previously used tools in handling the conjecture, and show how the above Criterion implies the conjecture. In the second part we present the necessary arrangements needed to apply the mentioned machinery. In particular, a cocharacter of $G_{\mathbb{Q}_p}$ obtained via a result of Wintenberger in crystalline cohomology allows us to define the compact and the non-comapact part of $G^{der}_{\mathbb{Q}_p}$ and of $H^{der}_{A\mathbb{Q}_p}$. The machinery itself is presented in the third part. It is all that is needed to prove the conjecture for the non-compact factors, i.e. to get that thenon-compact part of G^{der} is isomorphic to the non-compact part of $H^{der}_{A\mathbb{Q}_p}$. In the last part we present a simple proof, based on Frobenius tori and a proposition on adjoint Shimura varieties of abelian type, of why what we know about the non-compact parts is enough to get a complete prof of the Criterion, and so to handle the compact parts and the toric parts.