UPSC: Problem Set 2

Opens: 3 p.m. Friday March 3rd, 2023

Due: 12 p.m. Friday March 17th, 2023

- You must work independently.
- Write your solutions clearly and show all of your work.
- Include your name, student ID number, and email address.
- Email a pdf file of your solution to `ugrad_services@math.utah.edu` by the deadline.
- A winner will be decided on the basis of the best solution submitted. If no best solution can be determined (i.e. there exist relatively identical solutions), the winner will be the student who submitted their solution first.
- Each submission will be given 3 points for a fully correct solution and 1-2 points for a partially correct solution. The winner of each problem set will get a bonus of ϵ points.
- Please don't just search online for a solution - that isn't the point of this contest.
- Do not feel discouraged if you are unable to solve the problems! These are chosen to be difficult, and are meant to be struggled with. After the problem set due date, you can contact me (Emil) at u0539859@utah.edu if you would like to see the solutions, or want any guidance for math contest preparation.
- Enjoy!

Problem 1 [1 point]: Let $a_1, a_2, \ldots$ be a sequence of positive real numbers, and let $S_n = \sum_{j=1}^{n} a_j$ be the partial sums of $\{a_j\}$. Show that if

$$\sum_{j=1}^{\infty} \frac{a_j}{S_j}$$

converges, then

$$\sum_{j=1}^{\infty} a_j$$

converges.

Problem 2 [1 point]: Let n be an even integer. Let $f(n)$ be the number of ways to color an $n \times n$ chess board with three colors such that any two horizontally/vertically adjacent squares are different colors. Show that

$$\sqrt{2} \leq \lim_{n\to\infty} \left(f(n)\right)^{\frac{1}{n^2}} \leq \sqrt{3}$$

(Bonus challenge - can you determine $f(n)$?)

Problem 3 [1 point]: Let $a_0 = \frac{5}{2}$, and inductively define $a_{n+1} = a_n^2 - 2$ for $n \geq 0$. Determine a closed form expression for

$$\prod_{k=0}^{\infty} \left(1 - \frac{1}{a_k}\right)$$