Notes: You may use any resources to solve these problems. Make sure you understand your solution thoroughly if you submit for credit. Good luck!

Problem 1

Part a. Prove the Central Limit Theorem (for finite variance), which states that any random variable $X$ with a mean $\mu$ and finite variance $\sigma^2$ will have an IID sample mean distribution that converges to a normal distribution for large sample size $n$:

$$\lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} X_i \sim \mathcal{N} \left( \mu, \frac{\sigma^2}{n} \right)$$

Part b. What special properties of the normal distribution cause it to be the limiting distribution?

Problem 2

A particle is released at $(0,e) \in \mathbb{R}^2$ within the $xy$ plane. The particle undergoes decoupled temporal motion (with respect to time $t$) given by the equations

$$dx = \mathcal{N}(0,1)\sqrt{d}t \quad \text{(brownian motion)}$$

$$dy = -y \, dt$$

until it reaches the line $y = 0.1$. At this moment, the particle begins to obey the equations of motion

$$dx = \sin(ax) \, dt$$

$$dy = -y \, dt$$

for some $a \in \mathbb{R}$.

Let $1 \gg \epsilon > 0$, and determine the probability, with respect to $a$, of the particle reaching a stable state $S \in \mathbb{R}^2$ where $||S - (\pi,0)|| < \epsilon$ for sufficiently large $t$. You may express your answer in terms of the standard normal cumulative distribution function $\Phi$.

Problem 3

Suppose $\mu : \mathcal{B}(\mathbb{R}) \to \mathbb{Q} \cap [0,1]$ is a rational-valued probability measure on the Borel $\sigma$-algebra $\mathcal{B}$ over $\mathbb{R}$. Prove that $\mu$ takes finitely many values.