1. (Chapter 1: 50 points) Consider the system \( A\vec{u} = \vec{b} \) with \( \vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \) defined by

\[
\begin{align*}
    x_1 + 3x_2 + 4x_3 + x_4 + x_5 &= 2 \\
    2x_1 + x_2 + 8x_3 + x_4 + 2x_5 &= 4 \\
    2x_1 + 2x_2 + 8x_3 + x_4 + x_5 &= 2
\end{align*}
\]

Solve the following parts (a) to (e):

(a) [10%] Find the reduced row echelon form of the augmented matrix.

(b) [10%] Write the scalar equations corresponding to the answer in (a). Then identify the free variables and the lead variables.

(c) [10%] Display a formula for the vector general solution \( \vec{u} \).

(d) [10%] Extract from the answer in (c) vector formulas for a particular solution \( \vec{u}_p \) and the homogeneous solution \( \vec{u}_h \).

(e) [10%] Extract from the answer in (d) a vector solution basis for \( A\vec{u} = \vec{0} \). These vectors are called \textit{Strang’s Special Solutions}.

2. (Chapter 2: 40 points)

Define \( A = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix} \) and \( B = A + A^T \), where \( A^T \) is the transpose of \( A \).
(a) [20%] Apply two different methods to find the inverse of the matrix $A$.

(b) [20%] Compute $(B^{-1})^T$.

3. (Chapter 3: 30 points) Let $P, Q, R$ be real numbers. Define matrix $A$ and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$  

Find the value of $x_2$ by Cramer’s Rule in the system $A\vec{x} = \vec{b}$.

4. (Chapters 1 to 4: 20 points) It is known that functions $x$, $\cos(x)$, $e^x$ are independent in the vector space $V$ of all functions on $(-\infty, \infty)$. Define functions in $V$ by the equations

$f_1(x) = x + e^x$, $f_2(x) = 2x - e^x$, $f_3(x) = 3\cos(x) + x + e^x$.

**Definition:** An Euler solution atom is a base atom multiplied by a factor $x^ne^{ax}$ where $n = 0, 1, 2, \ldots$ and $a$ is a real constant. A base atom is one of $1, \cos(bx), \sin(bx)$ where $b > 0$ is real.

Check the independence tests below which apply to prove that the functions $f_1, f_2, f_3$ are independent in the vector space $V$. Don’t check one which won’t work!

- **Wronskian test**
  Wronskian of $f_1, f_2, f_3$ nonzero at $x = x_0$ implies independence of $f_1, f_2, f_3$.

- **Euler Solution Atom test**
  Any finite set of distinct Euler atoms is independent.

- **Sample test**
  Functions $f_1, f_2, f_3$ are independent if a sampling matrix has nonzero determinant.

5. (Chapters 1 to 4: 30 points) It is known that functions $x$, $\cos(x)$, $e^x$ are independent in the vector space $V$ of all functions on $(-\infty, \infty)$. Define functions in $V$ by the equations

$f_1(x) = x + e^x$, $f_2(x) = 2x - e^x$, $f_3(x) = 3\cos(x) + x + e^x$.

(a) [10%] Independence of the functions $f_1, f_2, f_3$ in the vector space $V$ can be established
using the coordinate map

\[ c_1 x + c_2 e^x + c_3 \cos(x) \] maps into \[
\begin{pmatrix}
  c_1 \\
  c_2 \\
  c_3 
\end{pmatrix}.
\]

Reformulate the independence of functions \( f_1, f_2, f_3 \) into independence of column vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) in the vector space \( \mathbb{R}^3 \).

(b) [10%] Show details for one of the tests below applied to \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \), defined in part (a).

(c) [10%] Check all tests below that may be applied to \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \), as defined in part (a). Don’t check a test which won’t work!

- **Rank test**
  Vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are independent if their augmented matrix has rank 3.

- **Determinant test**
  Vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are independent if their square augmented matrix has nonzero determinant.

- **Pivot test**
  Vectors \( \vec{v}_1, \vec{v}_2, \vec{v}_3 \) are independent if their augmented matrix \( A \) has 3 pivot columns.

- **Orthogonality test**
  A set of nonzero pairwise orthogonal vectors is independent.

- **Combination test**
  A list of vectors is independent if each vector is not a linear combination of the preceding vectors.

6. (Chapters 2, 4: 20 points) Define \( S \) to be the set of all vectors \( \vec{x} \) in \( \mathbb{R}^3 \) such that \( x_1 + 2x_3 - x_2 = 0 \), \( x_3 = 0 \) and \( x_3 + x_2 = x_1 \). Supply the proof details which verify that \( S \) is a subspace of \( \mathbb{R}^3 \).

7. (Chapter 6: 40 points) Let \( S \) be the subspace of \( \mathbb{R}^4 \) spanned by the vectors

\[
\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.
\]
(a) [10%] Explain, by citing a theorem, why $S$ is a subspace.

(b) [30%] Find a Gram-Schmidt orthonormal basis $\mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_3$ for subspace $S$.

8. (Chapters 1 to 6: 30 points) Let $A$ be an $m \times n$ matrix and assume that $A^T A$ has rank $n - 1$. Prove that the rank of $A$ cannot equal $n$.

9. (Chapter 5: 40 points) The matrix $A$ below has eigenvalues 2, 3 and 3. Compute all eigenpairs of $A$. Is $A$ diagonalizable?

\[
A = \begin{pmatrix}
4 & 1 & 1 \\
-1 & 2 & 1 \\
0 & 0 & 2
\end{pmatrix}
\]

10. (Chapter 6: 30 points) Define $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Let $W$ be the column space of $A$. Write the normal equations for the inconsistent problem $A\mathbf{x} = \mathbf{b}$ and solve for the least squares solution $\mathbf{x}$.

Remark. Vector $\tilde{\mathbf{b}} = A\tilde{\mathbf{x}}$ is the near point to $\mathbf{b}$ in the subspace $W$.

11. (Chapter 6: 30 points) Given vectors $\vec{q}_1, \vec{q}_2, \vec{q}_3$ in $\mathbb{R}^3$, define

\[
A = 2\vec{q}_1\vec{q}_1^T + 5\vec{q}_2\vec{q}_2^T + 7\vec{q}_3\vec{q}_3^T.
\]

(a) [10%] Prove that $A$ is symmetric.

(b) [20%] The Spectral Theorem for symmetric matrices produces a similar formula where
2, 5, 7 are replaced by the eigenvalues of \( A \). Write the formula for \( 3 \times 3 \) matrices \( A \) and explain all the symbols used in the formula.

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12. **(Chapter 7: 30 points)** The spectral theorem says that a symmetric matrix \( A \) satisfies \( AQ = QD \) where \( Q \) is orthogonal and \( D \) is diagonal. Find \( Q \) and \( D \) for the symmetric matrix \( A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix} \).

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13. **(Chapter 7: 40 points)** Write out the singular value decomposition for the matrix
\[ A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}. \]

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14. **(Chapter 4: 30 points)** Let the linear transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) be defined by its action on two independent vectors:
\[ T \left( \begin{pmatrix} 3 \\ 2 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad T \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}. \]
Find the unique \( 2 \times 2 \) matrix \( A \) such that \( T \) is defined by the matrix multiply equation \( T(\vec{x}) = A\vec{x} \).

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15. **(Chapter 4, 7: 40 points)** Let \( A \) be an \( m \times n \) matrix. Denote by \( S_1 \) the row space of \( A \) and \( S_2 \) the column space of \( A \). Using only the Pivot Theorem and the Toolkit of swap, combo, multiply, prove that \( S_1 \) and \( S_2 \) have the same dimension.

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16. **(Chapter 4: 20 points)** Least squares can be used to find the best fit line \( y = ax + b \) for the points \( (1, 2), (2, 2), (3, 0) \). Find the line equation by the method of least squares.

17. **(Chapters 1 to 7: 20 points)** State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.