# MATH 2270-2 Final Exam Spring 2017

**1. (Chapter 1: 50 points)** Consider the system $A\vec{u} = \vec{b}$ with $\vec{u} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix}$ defined by

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + x_4 + x_5 &= 2 \\ 2x_1 + x_2 + 8x_3 + x_4 + 2x_5 &= 4 \\ 2x_1 + 2x_2 + 8x_3 + x_4 + x_5 &= 2 \end{aligned}$$

Solve the following parts (a) to (e):

**(a)** [10%] Find the reduced row echelon form of the augmented matrix.

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**(b)** [10%] Write the scalar equations corresponding to the answer in (a). Then identify the **free** variables and the **lead** variables.

**(c)** [10%] Display a formula for the **vector** general solution $\vec{u}$.

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**(d)** [10%] Extract from the answer in (c) vector formulas for a particular solution $\vec{u}_p$ and the homogeneous solution $\vec{u}_h$.

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**(e)** [10%] Extract from the answer in (d) a vector solution basis for $A\vec{u} = \vec{0}$. These vectors are called **Strang's Special Solutions**.

**2. (Chapter 2: 40 points)**

Define $A = \begin{pmatrix} 1 & 4 \\ 1 & 5 \end{pmatrix}$ and $B = A + A^T$, where $A^T$ is the transpose of $A$.

**(a)** [20%] Apply two different methods to find the inverse of the matrix $A$.

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**(b)** [20%] Compute $(B^{-1})^T$.

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**3. (Chapter 3: 30 points)** Let $P, Q, R$ be real numbers. Define matrix $A$ and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 2 & 0 \\ 0 & -1 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}.$$

Find the value of $x_2$ by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

**4. (Chapters 1 to 4: 20 points)** It is known that functions $x$, $\cos(x)$, $e^x$ are independent in the vector space $V$ of all functions on $(-\infty, \infty)$. Define functions in $V$ by the equations $f_1(x) = x + e^x$, $f_2(x) = 2x - e^x$, $f_3(x) = 3\cos(x) + x + e^x$.

**Definition**: *An Euler solution atom is a base atom multiplied by a factor* $x^n e^{ax}$ *where* $n = 0, 1, 2, \ldots$ *and* $a$ *is a real constant. A base atom is one of* $1$, $\cos(bx)$, $\sin(bx)$ *where* $b > 0$ *is real.*

Check the independence tests below which apply to prove that the functions $f_1, f_2, f_3$ are independent in the vector space $V$. Don't check one which won't work!

| | | |
|---|---|---|
| ☐ | **Wronskian test** | Wronskian of $\vec{f}_1, \vec{f}_2, \vec{f}_3$ nonzero at $x = x_0$ implies independence of $\vec{f}_1, \vec{f}_2, \vec{f}_3$. |
| ☐ | **Euler Solution Atom test** | Any finite set of distinct Euler atoms is independent. |
| ☐ | **Sample test** | Functions $\vec{f}_1, \vec{f}_2, \vec{f}_3$ are independent if a sampling matrix has nonzero determinant. |

**5. (Chapters 1 to 4: 30 points)** It is known that functions $x$, $\cos(x)$, $e^x$ are independent in the vector space $V$ of all functions on $(-\infty, \infty)$. Define functions in $V$ by the equations $f_1(x) = x + e^x$, $f_2(x) = 2x - e^x$, $f_3(x) = 3\cos(x) + x + e^x$.

**(a)** [10%] Independence of the functions $f_1, f_2, f_3$ in the vector space $V$ can be established using the coordinate map

$$c_1 x + c_2 e^x + c_3 \cos(x) \quad \text{maps into} \quad \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Reformulate the independence of functions $f_1, f_2, f_3$ into independence of column vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in the vector space $\mathcal{R}^3$.

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**(b)** [10%] Show details for **one** of the tests below applied to $\vec{v}_1, \vec{v}_2, \vec{v}_3$, defined in part (a).

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**(c)** [10%] Check all tests below that may be applied to $\vec{v}_1, \vec{v}_2, \vec{v}_3$, as defined in part (a). Don't check a test which won't work!

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|---|---|---|
| ☐ | **Rank test** | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix has rank 3. |
| ☐ | **Determinant test** | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their square augmented matrix has nonzero determinant. |
| ☐ | **Pivot test** | Vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent if their augmented matrix $A$ has 3 pivot columns. |
| ☐ | **Orthogonality test** | A set of nonzero pairwise orthogonal vectors is independent. |
| ☐ | **Combination test** | A list of vectors is independent if each vector is not a linear combination of the preceding vectors. |

**6. (Chapters 2, 4: 20 points)** Define $S$ to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + 2x_3 - x_2 = 0$, $x_3 = 0$ and $x_3 + x_2 = x_1$. Supply the proof details which verify that $S$ is a subspace of $\mathcal{R}^3$.

**7. (Chapter 6: 40 points)** Let $S$ be the subspace of $\mathbb{R}^4$ spanned by the vectors

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

**(a)** [10%] Explain, by citing a theorem, why $S$ is a subspace.

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**(b)** [30%] Find a Gram-Schmidt orthonormal basis $\vec{u}_1, \vec{u}_2, \vec{v}_3$ for subspace $S$.

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**8. (Chapters 1 to 6: 30 points)** Let $A$ be an $m \times n$ matrix and assume that $A^T A$ has rank $n-1$. Prove that the rank of $A$ cannot equal $n$.

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**9. (Chapter 5: 40 points)** The matrix $A$ below has eigenvalues 2, 3 and 3. Compute all eigenpairs of $A$. Is $A$ diagonalizable?

$$A = \begin{pmatrix} 4 & 1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

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**10. (Chapter 6: 30 points)** Define $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix}$ and $\vec{\mathbf{b}} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$. Let $W$ be the column space of $A$. Write the normal equations for the inconsistent problem $A\vec{x} = \vec{b}$ and solve for the least squares solution $\hat{\vec{x}}$.

Remark. *Vector* $\hat{\vec{\mathbf{b}}} = \mathbf{A}\hat{\vec{\mathbf{x}}}$ *is the near point to* $\vec{\mathbf{b}}$ *in the subspace* $W$*.*

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**11. (Chapter 6: 30 points)** Given vectors $\vec{q}_1, \vec{q}_2, \vec{q}_3$ in $\mathcal{R}^3$, define

$$A = 2\vec{q}_1\vec{q}_1^{\,T} + 5\vec{q}_2\vec{q}_2^{\,T} + 7\vec{q}_3\vec{q}_3^{\,T}.$$

**(a)** [10%] Prove that $A$ is symmetric.

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**(b)** [20%] The Spectral Theorem for symmetric matrices produces a similar formula where

$2, 5, 7$ are replaced by the eigenvalues of $A$. Write the formula for $3 \times 3$ matrices $A$ and explain all the symbols used in the formula.

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**12. (Chapter 7: 30 points)** The spectral theorem says that a symmetric matrix $A$ satisfies $AQ = QD$ where $Q$ is orthogonal and $D$ is diagonal. Find $Q$ and $D$ for the symmetric matrix $A = \begin{pmatrix} 4 & -1 \\ -1 & 4 \end{pmatrix}$.

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**13. (Chapter 7: 40 points)** Write out the singular value decomposition for the matrix $A = \begin{pmatrix} 5 & 1 \\ 1 & 5 \end{pmatrix}$.

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**14. (Chapter 4: 30 points)** Let the linear transformation $T$ from $\mathcal{R}^2$ to $\mathcal{R}^2$ be defined by its action on two independent vectors:

$$T\left(\begin{pmatrix} 3 \\ 2 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, \quad T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Find the unique $2 \times 2$ matrix $A$ such that $T$ is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$

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**15. (Chapter 4, 7: 40 points)** Let $A$ be an $m \times n$ matrix. Denote by $S_1$ the row space of of $A$ and $S_2$ the column space of $A$. Using only the Pivot Theorem and the Toolkit of swap, combo, multiply, prove that $S_1$ and $S_2$ have the same dimension.

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**16. (Chapter 4: 20 points)** Least squares can be used to find the best fit line $y = ax + b$ for the points $(1, 2)$, $(2, 2)$, $(3, 0)$. Find the line equation by the method of least squares.

**17. (Chapters 1 to 7: 20 points)** State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

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