NAME: 

Please, no books, notes or electronic devices.

The last four (4) questions are proofs. Please divide your time accordingly.

Extra details can be on the back side or on extra pages. Please supply a road map for details not on the front side.

Details count 75% and answers count 25%.
Problem 1. (100 points) Define matrix $A$, vector $\vec{b}$ and vector variable $\vec{x}$ by the equations

\[
A = \begin{pmatrix}
z_1 & z_2 & 0 \\
0 & z_3 & 0 \\
1 & z_4 & 1
\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}
-3 \\
5 \\
1
\end{pmatrix}, \quad \vec{x} = \begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix}.
\]

For the system $A\vec{x} = \vec{b}$, display the formula for $x_2$ according to Cramer’s Rule. To save time, do not compute determinants!
Problem 2. (100 points) Define matrix \( A = \begin{pmatrix} 2 & 1 & 0 \\ 6 & 8 & 1 \\ 8 & 14 & 1 \end{pmatrix} \). Find a lower triangular matrix \( L \) and an upper triangular matrix \( U \) such that \( A = LU \).
Problem 3. (100 points) Find the complete vector solution $\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p$ for the nonhomogeneous system

$$
\begin{pmatrix}
0 & 3 & 1 & 0 & 0 \\
0 & 3 & 3 & 1 & 1 \\
0 & 0 & 0 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
x_5
\end{pmatrix}
=
\begin{pmatrix}
3 \\
1 \\
0
\end{pmatrix}.
$$

Please display vector answers for both $\mathbf{x}_h$ and $\mathbf{x}_p$. The homogeneous solution $\mathbf{x}_h$ is a linear combination of Strang’s special solutions. Symbol $\mathbf{x}_p$ denotes a particular solution.
Problem 4. (100 points) Let $V$ be the vector space of all functions on $(-\infty, \infty)$. Define subspace $S = \text{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$ where $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent vectors defined respectively by the equations $y = x - 1, y = 1 + x^2, y = 2x + x^2$. If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, then the uniquely determined constants $c_1, c_2, c_3$ are called the coordinates of $\vec{v}$ relative to the basis $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

Compute $c_1, c_2, c_3$ for $\vec{v}$ defined by $y = 1 + 2x + 3x^2$. 
Problem 5. (100 points) The functions $1, x^2, x^5$ represent independent vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ in the vector space $V$ of all functions on $0 < x < \infty$. The set $S = \text{span}(\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a subspace of $V$. Let vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ in $V$ be defined by the functions $1 + x^2, x^5 - x^2, 5 + 2x^5$, respectively. The **coordinate map** defined by

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + c_3 \vec{u}_3 \rightarrow c_1 c_2 c_3$$

maps the vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ into the following images in $\mathbb{R}^3$, respectively:

$$110, \quad 0 - 11, \quad 502.$$ 

The independence tests below can decide independence of vectors $\vec{v}_1, \vec{v}_2, \vec{v}_3$ by formulating the independence question in vector space $V$ or in vector space $\mathbb{R}^3$, because the coordinate map takes independent sets to independent sets.

**Check below** all independence tests which apply to decide independence of $\vec{v}_1, \vec{v}_2, \vec{v}_3$.

- **Wronskian test**: Wronskian determinant of $f_1, f_2, f_3$ nonzero at $x = x_0$ implies independence of $f_1, f_2, f_3$.
- **Sampling test**: Sampling determinant for samples $x = x_1, x_2, x_3$ nonzero implies independence of $f_1, f_2, f_3$.
- **Rank test**: Three vectors are independent if their augmented matrix has rank 3.
- **Determinant test**: Three vectors are independent if their augmented matrix is square and has nonzero determinant.
- **Orthogonality test**: Three vectors are independent if they are all nonzero and pairwise orthogonal.
- **Pivot test**: Three vectors are independent if their augmented matrix $A$ has 3 pivot columns.
Problem 6. (100 points) The matrix \( A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 3 & 1 \\ -1 & -1 & 5 \end{pmatrix} \) has eigenvalues 3, 4, 4.

(a) [80%] Find all eigenvectors for \( \lambda = 4 \). To save time, don’t find \( \lambda = 3 \) eigenvectors.

(b) [20%] Report whether or not matrix \( A \) is diagonalizable. Explain.
Problem 7. (100 points) Define $S$ to be the set of all vectors $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in $\mathbb{R}^3$ such that $x_1 + x_3 = x_2$ and $x_3 x_2 = x_1 x_2$. Show that $S$ is NOT a subspace of $\mathbb{R}^3$, that is, exhibit a counterexample to one of the items in the Subspace Criterion.
Problem 8. (100 points) Let $A$ be a $4 \times 3$ matrix. Assume the columns of $A^T A$ are dependent. Prove or disprove that $A$ has dependent columns.
Problem 9. (100 points) Let $3 \times 3$ matrices $A$, $B$ and $C$ be related by $AP = PB$ and $BQ = QC$ for some invertible matrices $P$ and $Q$. Assume $B$ has eigenvalues 2, 3, 7. Prove that matrices $A$ and $C$ also have eigenvalues 2, 3, 7.
Problem 10. (100 points) The Fundamental Theorem of Linear Algebra says that the null space of a matrix is orthogonal to the row space of the matrix.

Let $A$ be an $m \times n$ matrix. Define subspaces $S_1 = \text{column space of } A$, $S_2 = \text{null space of } A^T$. Prove that the only vector $\vec{v}$ in both $S_1$ and $S_2$ is the zero vector.