1. (10 points)
Assume that matrices $A$ and $B$ are $n \times n$, matrix $I$ is the $n \times n$ identity and $C^T$ denotes the transpose of a matrix $C$.

(a) Give a counter example or explain why it is true. If matrix $A$ is invertible, then $A^T(A^{-1} + B)^T = I + (BA)^T$.

(b) Give a counter example or explain why it is true. If $A^2B^2 = I$, then $AB$ is the inverse of $BA$.

Answer:

(a) TRUE. In general $(CD)^{-1}$ is the product of the inverses in reverse order, $D^{-1}C^{-1}$. The same is true for transposes. And transpose and inverse commute: $(C^T)^{-1} = (C^{-1})^T$. Why it is true: $A^T(A^{-1} + B)^T = A^T((A^{-1})^T + B^T) = A^T(A^{-1})^T + A^T B^T = (A^{-1}A)^T + (BA)^T = I + (BA)^T$

(b) TRUE. It is a standard theorem that $CD = I$ implies $DC = I$ for square matrices $C$, $D$. The determinant product theorem applied to $A^2B^2 = I$ implies $|A| \neq 0$ and $|B| \neq 0$. To show $AB$ is the inverse of $BA$ we only have to prove $(AB)(BA) = I$ (use $C = AB, D = BA$ in the theorem cited). Here’s how: $(AB)(BA) = A^{-1}AABBBA = A^{-1}(A^2B^2)A = A^{-1}(I)A = I$.

2. (10 points) Definition: An elementary matrix is the answer after applying exactly one combo, swap or multiply to the identity matrix $I$. An elimination matrix is a product of elementary matrices.

Let $A$ be a $3 \times 4$ matrix. Find the elimination matrix $E$ which under left multiplication against matrix $A$ performs both (1) and (2) below with one matrix multiply.

(1) Replace Row 3 of $A$ with Row 3 minus Row 1.

(2) Replace Row 3 of $A$ by Row 3 minus 5 times Row 2.

Answer:

Perform combo(1,3,-1) on $I$ then combo(2,3,-5) on the result. The elimination matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -5 & 1 \end{pmatrix}$$
3. (20 points) Let \( a, b \) and \( c \) denote constants and consider the system of equations

\[
\begin{pmatrix}
1 & c & b \\
2 & b + c & a \\
1 & b & -a
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
-a \\
a \\
a
\end{pmatrix}
\]

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for \((c - b)(2a - b) \neq 0\).

(b). The system has no solution if \( b = 2a \) and \( a \neq 0 \) (don’t explain the other possibilities).

(c). The system has infinitely many solutions if \( a = b = c = 0 \) (don’t explain the other possibilities).

Answer:

(a) Uniqueness requires zero free variables. Then the determinant of the coefficient matrix \( A \) must be nonzero. After the cofactor expansion the determinant is factored as \((b - 2a)(b - c)\). The inverse of the coefficient matrix then exists for \((b - 2a)(b - c) \neq 0\), which implies equation \( A\vec{u} = \vec{b} \) has unique solution \( \vec{u} = A^{-1}\vec{b} \).

(b) No solution: Combo, swap and mult are used in part (b) with \( b = 2a \) substituted into the system of equations. After 3 combo steps the matrix is transformed into

\[
A_3 = 
\begin{pmatrix}
1 & c & 2a & -a \\
0 & 2a - c & -3a & 3a \\
0 & 0 & 0 & a
\end{pmatrix}
\]

The last row of \( A_3 \) is a signal equation if \( b = 2a = 0 \) and \( a \neq 0 \).

(c) Infinitely many solutions: If \( a = b = c = 0 \), then from part (b)

\[
A_3 = 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

Matrix \( A_3 \) has one lead variable and two free variables, because the last two rows of \( A_3 \) are zero. This homogeneous problem has no signal equation, therefore it has infinitely many solutions.

A full analysis of the three possibilities is fairly complex.

The sequence of steps are documented below for maple.

2
combo := (A, s, t, m) -> linalg[addrow](A, s, t, m); newline
mult := (A, t, m) -> linalg[mulrow](A, t, m); newline
swap := (A, s, t) -> linalg[swaprow](A, s, t); newline
A := (a, b, c) -> Matrix([[1, c, b, -a], [2, b+c, a, a], [1, b, -a, a]]); newline
A1 := combo(A(a, b, c), 1, 2, -2); newline
A2 := combo(A1, 1, 3, -1); newline
A3 := combo(A2, 2, 3, -1);

Definition. Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called independent provided solving the equation $c_1 \vec{v}_1 + \cdots + c_k \vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ has the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called dependent.

4. (20 points) Classify the following set of vectors as Independent or Dependent, using the Pivot Theorem, the Rank Theorem or the definition of independence (above). Details are 75%, answer 25%.

\[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2
\end{pmatrix}
\]

Answer:
The vectors are dependent. The augmented matrix of the vectors has pivot columns 1, 2, 4. Therefore, vectors 1, 2, 4 are independent. By the Pivot Theorem, the third vector is a linear combination of the pivot columns 1, 2, 4.

\[
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2
\end{pmatrix}
\Rightarrow
\begin{pmatrix}
1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

5. (20 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

\[
A = \begin{pmatrix}
1 & 0 & 0 & 4 \\
4 & 0 & 0 & 1 \\
6 & 0 & 2 & 0
\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}
0 \\
0 \\
4
\end{pmatrix}
\]

Answer:
The augmented matrix for this system of equations is

\[
\begin{pmatrix}
1 & 0 & 0 & 4 & 0 \\
4 & 0 & 0 & 1 & 0 \\
6 & 0 & 2 & 0 & 4 \\
\end{pmatrix}
\]

The reduced row echelon form is found as follows:

\[
\begin{pmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & -15 & 0 \\
6 & 0 & 2 & 0 & 4 \\
\end{pmatrix}
\]

combo(1,2,-4)

\[
\begin{pmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & -15 & 0 \\
0 & 0 & 2 & -24 & 4 \\
\end{pmatrix}
\]

combo(1,3,-6)

\[
\begin{pmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & -24 & 4 \\
\end{pmatrix}
\]

mult(2,-1/15)

\[
\begin{pmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & 0 & 4 \\
\end{pmatrix}
\]

combo(2,3,24)

\[
\begin{pmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 2 \\
\end{pmatrix}
\]

mult(3,1/2)

\[
\begin{pmatrix}
1 & 0 & 0 & 4 & 0 \\
0 & 0 & 1 & 0 & 2 \\
0 & 0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

swap(2,3); last frame

The last frame, or RREF, implies the system

\[
\begin{align*}
x_1 &= 0 \\
x_3 &= 2 \\
x_4 &= 0
\end{align*}
\]

The lead variables are \(x_1, x_3, x_4\) and the free variable is \(x_2\). The last frame algorithm introduces invented symbol \(t_1\). The free variable is set to this symbol, then back-substitute into the lead variable equations of the last frame to obtain the general solution

\[
\begin{align*}
x_1 &= 0, \\
x_2 &= t_1, \\
x_3 &= 2, \\
x_4 &= 0.
\end{align*}
\]
Strang’s special solution \( \vec{s}_1 \) is the partial of \( \vec{x} \) on the invented symbol \( t_1 \). A particular solution \( \vec{x}_p \) is obtained by setting all invented symbols to zero. Then

\[
\vec{x} = \vec{x}_p + t_1 \vec{s}_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}
\]

6. (20 points) Determinant problem, chapter 3.
(a) [30%] Assume given 3 \( \times \) 3 matrices \( A, B \). Suppose \( E_3 E_2 E_1 A = A^2 B^2 \) and \( E_1, E_2, E_3 \) are elementary matrices representing respectively a combination, a swap and a multiply by 3. Assume \( \det(B) = -5 \). Let \( C = 2A \). Find all possible values of \( \det(C) \).
(b) [20%] Determine all values of \( x \) for which \((I + C)^{-1}\) exists, where \( I \) is the 3 \( \times \) 3 identity and \( C = \begin{pmatrix} 2 & x & -1 \\ x & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix} \).
(c) [30%] Let symbols \( a, b, c \) denote constants and define

\[
A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}
\]

Apply the adjugate [adjoint] formula for the inverse

\[
A^{-1} = \frac{\text{adj}(A)}{|A|}
\]

to find the value of the entry in row 3, column 2 of \( A^{-1} \).

Answer:

(a) Start with the determinant product theorem \( |FG| = |F||G| \). Apply it to obtain \( |E_3||E_2||E_1||A| = |A|^2|B|^2 \). Let \( x = |A| \) in this equation and solve for \( x \). You will need to know that \( |B| = -5, \) \( |E_1| = 1, |E_2| = -1 \) and \( |E_3| = 3 \). Then \( |C| = |(2I)A| = |2I||A| = 8x \). The answer is \( |C| = 0 \) or \( |C| = 24/5 \). (b) Find \( C + I = \begin{pmatrix} 3 & x & -1 \\ x & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix} \), then evaluate its determinant, to eventually solve for \( x = -2 \). Used here is \( F^{-1} \) exists if and only if \( |F| \neq 0 \). The answer is \( I + C \) has an inverse for all \( x \neq -2 \).
(c) Find the cross-out determinant in row 2, column 3 (no mistake, the transpose swaps
rows and columns). Form the fraction, top=checkboard sign times cross-out determinant, bottom=|A|. The value is $b - \frac{c}{2}$. A maple check:

```maple
C4:=Matrix([[1,0,0,0],[1,-2,0,0],[a,b,0,1],[1,c,1,2]]);
1/C4; # The inverse matrix
C5:=linalg[minor](C4,2,3); linalg[det](C5)*(-1)^(2+3);linalg[det](C4);
# ans =-c+2b divided by 2
```

End Exam 1.