1. (10 points)
Assume that matrices $A$ and $B$ are $n \times n$, matrix $I$ is the $n \times n$ identity and $C^T$ denotes the transpose of a matrix $C$.

(a) Give a counter example or explain why it is true. If matrix $A$ is invertible, then $A^T(A^{-1} + B)^T = I + (BA)^T$.

(b) Give a counter example or explain why it is true. If $A^2B^2 = I$, then $AB$ is the inverse of $BA$.

2. (10 points) Definition: An **elementary matrix** is the answer after applying exactly one combo, swap or multiply to the identity matrix $I$. An **elimination matrix** is a product of elementary matrices.

Let $A$ be a $3 \times 4$ matrix. Find the elimination matrix $E$ which under left multiplication against matrix $A$ performs both (1) and (2) below with one matrix multiply.

(1) Replace Row 3 of $A$ with Row 3 minus Row 1.

(2) Replace Row 3 of $A$ by Row 3 minus 5 times Row 2.

3. (20 points) Let $a, b$ and $c$ denote constants and consider the system of equations

$$
\begin{pmatrix}
1 & c & b \\
2 & b+c & a \\
1 & b & -a
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
-a \\
a \\
a
\end{pmatrix}
$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for $(c - b)(2a - b) \neq 0$.

(b). The system has no solution if $b = 2a$ and $a \neq 0$ (don’t explain the other possibilities).

(c). The system has infinitely many solutions if $a = b = c = 0$ (don’t explain the other possibilities).
Definition. Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called independent provided solving the equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ has the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called dependent.

4. (20 points) Classify the following set of vectors as Independent or Dependent, using the Pivot Theorem, the Rank Theorem or the definition of independence (above). Details are 75%, answer 25%.

$$
\begin{pmatrix}
1 & 0 & 0 & 4 \\
0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 2 & 2 \\
1 & 1 & 2 & 2
\end{pmatrix}
$$

5. (20 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$
A = \begin{pmatrix}
1 & 0 & 0 & 4 \\
0 & 2 & 2 & 0 \\
4 & 0 & 0 & 1 \\
6 & 0 & 2 & 0
\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}
0 \\
0 \\
4
\end{pmatrix}
$$

6. (20 points) Determinant problem, chapter 3.
(a) [30%] Assume given 3 × 3 matrices $A$, $B$. Suppose $E_3E_2E_1A = A^2B^2$ and $E_1$, $E_2$, $E_3$ are elementary matrices representing respectively a combination, a swap and a multiply by 3. Assume $\det(B) = -5$. Let $C = 2A$. Find all possible values of $\det(C)$.
(b) [20%] Determine all values of $x$ for which $(I + C)^{-1}$ exists, where $I$ is the 3 × 3 identity and $C = \begin{pmatrix}
2 & x & -1 \\
x & 1 & 1 \\
1 & 0 & -1
\end{pmatrix}$.
(c) [30%] Let symbols $a, b, c$ denote constants and define

$$
A = \begin{pmatrix}
1 & 0 & 0 & 0 \\
1 & -2 & 0 & 0 \\
a & b & 0 & 1 \\
1 & c & 1 & 2
\end{pmatrix}
$$

Apply the adjugate [adjoint] formula for the inverse

$$
A^{-1} = \frac{\text{adj}(A)}{|A|}
$$

to find the value of the entry in row 3, column 2 of $A^{-1}$.

End Exam 1.