1. **(10 points)**

(a) Give a counter example or explain why it is true. If $A$ and $B$ are $n \times n$ invertible, and $C^T$ denotes the transpose of a matrix $C$, then $((A + B)^{-1})^T = (B^T)^{-1} + (A^T)^{-1}$.

(b) Give a counter example or explain why it is true. If $A$ is a square matrix and $A^T A = I$, then both $A$ and $A^T$ are invertible.
2. **(10 points)** Let $A$ be a $3 \times 4$ matrix. Find the elimination matrix $E$ which under left multiplication against $A$ performs both (1) and (2) with one matrix multiply.

(1) Replace Row 2 of $A$ with Row 2 minus Row 1.

(2) Replace Row 3 of $A$ by Row 3 minus 5 times Row 2.
3. (30 points) Let $a$, $b$ and $c$ denote constants and consider the system of equations

\[
\begin{pmatrix}
1 & -b & c \\
1 & c & a \\
2 & -b + c & -a
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
=
\begin{pmatrix}
a \\
-a \\
-a
\end{pmatrix}
\]

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for $(b + c)(2a + c) \neq 0$.

(b). The system has no solution if $2a + c = 0$ and $a \neq 0$ (don’t explain the other possibilities).

(c). The system has infinitely many solutions if $a = c = 0$ (don’t explain the other possibilities).
4. **(20 points) Definition.** Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called **independent** provided solving the equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ has the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called **dependent**.

Find a largest set of independent vectors from the following set of vectors, using the definition of independence (above). You may use the Pivot Theorem without explanation. Any independence test from a reference textbook may be used, provided you state the test.

\[
\begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
2 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
0 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix},
\begin{pmatrix}
3 \\
2 \\
0 \\
1
\end{pmatrix},
\begin{pmatrix}
2 \\
0 \\
0 \\
1
\end{pmatrix}
\]
5. (20 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 4 \\ 4 \\ 0 \end{pmatrix}$$
(a) [10%] True or False? The value of a determinant is multiplied by $-1$ when two columns are swapped.
(b) [10%] True or False? The determinant of two times the $n \times n$ identity matrix is 2.
(c) [30%] Assume given $3 \times 3$ matrices $A, B$. Suppose $E_3E_2E_1A = BA^2$ and $E_1, E_2, E_3$ are elementary matrices representing respectively a multiply by 3, a swap and a combination. Assume $\det(B) = 3$. Find all possible values of $\det(-2A)$.
(d) [20%] Determine all values of $x$ for which $(I + 2C)^{-1}$ fails to exist, where $I$ is the $3 \times 3$ identity and $C = \begin{pmatrix} 2 & x & 1 \\ x & 0 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.
(e) [30%] Let symbols $a, b, c$ denote constants and define
\[
A = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & \frac{1}{2} \\ 1 & c & 1 & \frac{1}{2} \end{pmatrix}
\]
Apply the adjugate [adjoint] formula for the inverse
\[
A^{-1} = \frac{\text{adj}(A)}{|A|}
\]
to find the value of the entry in row 4, column 1 of $A^{-1}$.

End Exam 1.