Essay Questions

1. **(10 points)** Define an Elementary Matrix. Display the fundamental matrix multiply equation which summarizes a sequence of swap, combination and multiply operations, transforming a matrix $A$ into a matrix $B$.

**Answer:**

An elementary matrix is a matrix $E$ obtained from the identity matrix $I$ by applying one combination, swap or multiply operation. The equation is

$$E_k \cdots E_2 E_1 A = B$$

where $E_1, E_2, \ldots, E_k$ are elementary matrices representing swap, multiply and combination operations that take $A$ into $B$.

2. **(20 points)** State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

**Answer:**

Let $A$ denote an $m \times n$ matrix of rank $r$. Part 1. The dimensions of the nullspace($A$), colspace($A$), rowspace($A$), nullspace($A^T$) are respectively $n - r$, $r$, $r$, $m - r$.

Part 2. rowspace($A$) $\perp$ nullspace($A$), colspace($A$) $\perp$ nullspace($A^T$). Both can be summarized by rowspace $\perp$ nullspace, applied to both $A$ and $A^T$.

Problems 3 to 17

3. **(30 points)** Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$. 

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Answer:

Let $E_1$ be the result of combo(1,2,-1/2) on $I$, and $E_2$ the result of combo(2,3,-2/3) on $I$.

Then $E_2E_1A = U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 3/2 & 1 \\ 0 & 0 & 1/3 \end{pmatrix}$. Let $L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1/2 & 1 & 0 \\ 0 & 2/3 & 1 \end{pmatrix}$.

4. (30 points) Determine which values of $k$ correspond to infinitely many solutions for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-3 & k-3 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer:

There is a unique solution for $\det(A) \neq 0$, which implies $k \neq 1$ and $k \neq 3$. Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-3 & k-3 & -1 \\ 0 & 0 & 1-k & k-1 \end{pmatrix}$. Then (1) Unique solution for three lead variables, equivalent to the determinant nonzero for the frame above, or $(k-3)(1-k) \neq 0$; (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 1$ [one free variable].

5. (30 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang’s special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:

The augmented matrix has reduced row echelon form (last frame) equal to the matrix

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Then $x_1 = t_1, x_2 = 1, x_3 = 1, x_4 = t_2$ is the general solution in scalar
The partial derivative on $t_1$ gives the homogeneous solution basis vector \[
\begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix}
\].

The partial derivative on $t_2$ gives the homogeneous solution basis vector \[
\begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}
\]. Then

\[
x_h = c_1 \begin{pmatrix}
1 \\
0 \\
0 \\
0
\end{pmatrix} + c_2 \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix}.
\]

Set $t_1 = t_2 = 0$ in the scalar solution to find a particular solution \[
\bar{x}_p = \begin{pmatrix}
0 \\
1 \\
1 \\
0
\end{pmatrix}.
\]

6. (20 points) Define $S$ to be the set of all vectors $\bar{x}$ in $\mathbb{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Apply a theorem which concludes that $S$ is a subspace of $\mathbb{R}^3$. This implies stating the hypotheses and checking that they apply.

Answer:

Let $A = \begin{pmatrix}
1 & 0 & 1 \\
-1 & 1 & 1 \\
0 & 0 & 0
\end{pmatrix}$. Then the restriction equations can be written as $A\bar{x} = \vec{0}$. Apply the nullspace theorem (also called the kernel theorem), which says that the nullspace of a matrix is a subspace.

Another solution: The given restriction equations are linear homogeneous algebraic equations. Therefore, $S$ is the nullspace of some matrix $B$, hence a subspace of $\mathbb{R}^3$. This solution uses the fact that linear homogeneous algebraic equations can be written as a matrix equation $B\bar{x} = \vec{0}$, without actually finding the matrix.

7. (20 points) The $5 \times 7$ matrix $A$ below has some independent columns. Report the
independent columns of $A$, according to the Pivot Theorem.

\[
A = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-3 & 0 & 0 & 0 & -2 & 1 & -1 \\
-1 & 0 & 0 & 0 & 0 & 1 & 0 \\
6 & 0 & 0 & 0 & 6 & 0 & 3 \\
2 & 0 & 0 & 0 & 2 & 0 & 1
\end{pmatrix}
\]

**Answer:**

\[
\text{Compute } \text{rref}(A) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1 & 1 & 1/2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

The pivot columns are 1 and 5.

8. **(40 points)** Let $S$ be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Find the Gram-Schmidt orthonormal basis of $S$.

**Answer:**

Let $\vec{y}_1 = \vec{v}_1$ and $\vec{u}_1 = \frac{1}{\|\vec{y}_1\|} \vec{y}_1$. Then $\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$. Let $\vec{y}_2 = \vec{v}_2$ minus the shadow projection of $\vec{v}_2$ onto the span of $\vec{v}_1$. Then

\[
\vec{y}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1} \vec{v}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 3 \end{pmatrix}.
\]
Finally, \( \vec{u}_2 = \frac{1}{\|y_2\|} \vec{y}_2 \). We report the Gram-Schmidt basis:

\[
\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{15}} \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}.
\]

9. **(30 points)** Define matrix \( A \) and vector \( \vec{b} \) by the equations

\[
A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.
\]

Find the value of \( x_2 \) by Cramer’s Rule in the system \( A \vec{x} = \vec{b} \).

Answer:

\[
x_2 = \frac{\Delta_2}{\Delta}, \quad \Delta_2 = \det \begin{pmatrix} -2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 3 & -2 \end{pmatrix} = 36, \quad \Delta = \det(A) = 4, \quad x_2 = 9.
\]

10. **(20 points)** Display the entry in row 4, column 3 of the adjugate matrix of

\[
A = \begin{pmatrix} 0 & 2 & -1 \\ 0 & 0 & 4 \\ 0 & 1 & 1 \\ 1 & 3 & -2 \end{pmatrix}.
\]

The adjugate matrix is also called the adjoint matrix. It appears in the formula \( C^{-1} = \frac{\text{adj}(C)}{\det(C)} \).

Answer:

The answer is the cofactor of \( A \) in row 3, column 4 = \((-1)^7\) times the minor of \( A \) in row 3, column 4. The minor equals 8. The answer is \(-8\).

11. **(10 points)** Consider a \( 3 \times 3 \) real matrix \( A \) with eigenpairs

\[
\left( -1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix} \right), \quad \left( 1, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right), \quad \left( 2, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} \right).
\]

Display an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( AP = PD \).

Answer:
The columns of $P$ are the eigenvectors and the diagonal entries of $D$ are the eigenvalues, taken in the same order.

12. (30 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, do not find eigenvectors!

Answer:

The characteristic polynomial is $\det(A - rI) = (-r)(3-r)(r-2)(r+2)$. The eigenvalues are 0, 2, -2, 3. Determinant expansion of $\det(A - \lambda I)$ is by the cofactor method along column 1. This reduces it to a $3 \times 3$ determinant, which can be expanded by the cofactor method along column 3.

13. (20 points) Let $I$ denote the $3 \times 3$ identity matrix. Assume given two $3 \times 3$ matrices $B, C$, which satisfy $CP = PB$ for some invertible matrix $P$. Let $C$ have eigenvalues $-1, 1, 0$. Let $A = 2I + 3B$. Find the eigenvalues of $A^3$.

Answer:

Both $B$ and $C$ have the same eigenvalues, because $\det(B - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(PCP^{-1} - \lambda PP^{-1}) = \det(C - \lambda I)$. Further, both $B$ and $C$ are diagonalizable. The answer is the same for all such matrices, so the computation can be done for a diagonal matrix $B = \text{diag}(-1, 1, 0)$. In this case, $A = 2I + 3B = \text{diag}(2, 2, 2) + \text{diag}(-3, 3, 0) = \text{diag}(-1, 5, 2)$ and the eigenvalues of $A$ are $-1, 5, 2$. To finish, observe that the eigenvalues of $A^3$ and $A$ are related, because $A\vec{v} = \lambda \vec{v}$ implies $A^3\vec{v} = \lambda^3 \vec{v}$. Therefore, the eigenvalues of $A^3$ are $-1, 125, 8$.

14. (10 points) The Cayley-Hamilton theorem suggests that there are real $2 \times 2$ matrices $A$ such that $A^2 = A$. Give an example of one such matrix $A$.

Answer:

Choose any matrix whose characteristic equation is $\lambda^2 - \lambda = 0$. Then $A^2 - A = 0$ by the Cayley-Hamilton theorem.

15. (40 points) The spectral theorem says that a symmetric matrix $A$ can be factored into $A = QDQ^T$ where $Q$ is orthogonal and $D$ is diagonal. Find $Q$ and $D$ for the symmetric
matrix \( A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \).

**Answer:**

Start with the equation \( r^2 - 4r + 3 = 0 \) having roots \( r = 1, 3 \). Compute the eigenpairs \((1, \vec{v}_1), (3, \vec{v}_2)\) where \( \vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) and \( \vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \). The two vectors are orthogonal but not of unit length. Unitize them to get \( \vec{u}_1 = \frac{1}{\sqrt{2}} \vec{v}_1 \), \( \vec{u}_2 = \frac{1}{\sqrt{2}} \vec{v}_2 \). Then \( Q = \langle \vec{u}_1 | \vec{u}_2 \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \), \( D = \text{diag}(1, 3) \).

16. (30 points) Let the linear transformation \( T \) from \( \mathbb{R}^2 \) to \( \mathbb{R}^2 \) be defined by its action on two independent vectors:

\[
T \left( \begin{pmatrix} 3 \\ 0 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \quad T \left( \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.
\]

Find the unique \( 2 \times 2 \) matrix \( A \) such that \( T \) is defined by the matrix multiply equation \( T(\vec{x}) = A\vec{x} \).

**Answer:**

Let \( A \) be the matrix for \( T \), such that \( T(\vec{v}) = A\vec{v} \) (matrix multiply). Then \( A \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix} \). Solving for \( A \) gives \( A = \begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 4/3 & 4/3 \\ 4/3 & -2/3 \end{pmatrix} \).

17. (10 points) Assume singular value decomposition \( A = U\Sigma V^T \) given by

\[
\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ \sqrt{2} & 1 \end{pmatrix}^T.
\]

Find a formula for the pseudo-inverse, but don’t bother to multiply out matrices.

**Answer:**

The definition is \( A^+ = V\Sigma^+U^T \). Identifying the matrices from the formula gives \( A^+ = \begin{pmatrix} 1 & -1 \\ \sqrt{2} & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{8} & 0 \\ 0 & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T \).