MATH 2270-2 Final Exam Spring 2012

ANSWERS

Essay Questions

1. (10 points) Define an Elementary Matrix. Display the fundamental matrix multiply equation which summarizes a sequence of swap, combination and multiply operations, transforming a matrix A into a matrix B.

Answer:

An elementary matrix is a matrix E obtained from the identity matrix I by applying one combination, swap or multiply operation. The equation is

$$E_k \cdots E_2 E_1 A = B$$

where $E_1, E_2, \ldots, E_k$ are elementary matrices representing swap, multiply and combination operations that take A into B.

2. (20 points) State the Fundamental Theorem of Linear Algebra. Include **Part 1**: The dimensions of the four subspaces, and **Part 2**: The orthogonality equations for the four subspaces.

Answer:

Let A denote an $m \times n$ matrix of rank r. Part 1. The dimensions of the nullspace(A), colspace(A), rowspace(A), nullspace(A^T) are respectively $n-r$, r, r, $m-r$.
Part 2. rowspace(A) $\perp$ nullspace(A), colspace(A) $\perp$ nullspace(A^T). Both can be summarized by rowspace $\perp$ nullspace, applied to both A and A^T.

Problems 3 to 17

3. (30 points) Find a factorization $A = LU$ into lower and upper triangular matrices for the matrix $A = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{pmatrix}$.

Answer:

Let E_1 be the result of combo(1,2,-1/2) on I, and E_2 the result of combo(2,3,-2/3) on I. Then $E_2E_1A = U = \begin{pmatrix} 2 & 1 & 0 \\ 0 & \frac{3}{2} & 1 \\ 0 & 0 & \frac{1}{3} \end{pmatrix}$. Let $L = E_1^{-1}E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ 0 & \frac{2}{3} & 1 \end{pmatrix}$.

4. (30 points) Determine which values of k correspond to infinitely many solutions for the system $A\vec{x} = \vec{b}$ given by

$$A = \begin{pmatrix} 1 & 4 & k \\ 0 & k-3 & k-3 \\ 1 & 4 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ -1 \\ k \end{pmatrix}.$$

Answer:

There is a unique solution for $\det(A) \neq 0$, which implies $k \neq 1$ and $k \neq 3$. Elimination methods with swap, combo, multiply give $\begin{pmatrix} 1 & 4 & k & 1 \\ 0 & k-3 & k-3 & -1 \\ 0 & 0 & 1-k & k-1 \end{pmatrix}$. Then (1) Unique solution for three lead variables, equivalent to the determinant nonzero for the frame above, or $(k-3)(1-k) \neq 0$; (2) No solution for $k = 3$ [signal equation]; (3) Infinitely many solutions for $k = 1$ [one free variable].

5. (30 points) Find the complete solution $\vec{x} = \vec{x}_h + \vec{x}_p$ for the nonhomogeneous system

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}.$$

The homogeneous solution $\vec{x}_h$ is a linear combination of Strang's special solutions. Symbol $\vec{x}_p$ denotes a particular solution.

Answer:

The augmented matrix has reduced row echelon form (last frame) equal to the matrix $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. Then $x_1 = t_1, x_2 = 1, x_3 = 1, x_4 = t_2$ is the general solution in scalar form. The partial derivative on t_1 gives the homogeneous solution basis vector $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$.

The partial derivative on t_2 gives the homogeneous solution basis vector $\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Then

$\vec{x}_h = c_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Set $t_1 = t_2 = 0$ in the scalar solution to find a particular solution $\vec{x}_p = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$.

6. (20 points) Define S to be the set of all vectors $\vec{x}$ in $\mathcal{R}^3$ such that $x_1 + x_3 = 0$ and $x_3 + x_2 = x_1$. Apply a theorem which concludes that S is a subspace of $\mathcal{R}^3$. This implies stating the hypotheses and checking that they apply.

Answer:

Let $A = \begin{pmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Then the restriction equations can be written as $A\vec{x} = \vec{0}$. Apply the nullspace theorem (also called the kernel theorem), which says that the nullspace of a matrix is a subspace.

Another solution: The given restriction equations are linear homogeneous algebraic equations. Therefore, S is the nullspace of some matrix B, hence a subspace of $\mathcal{R}^3$. This solution uses the fact that linear homogeneous algebraic equations can be written as a matrix equation $B\vec{x} = \vec{0}$, without actually finding the matrix.

7. (20 points) The 5×7 matrix A below has some independent columns. Report the independent columns of A, according to the Pivot Theorem.

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 & -2 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 6 & 0 & 0 & 0 & 6 & 0 & 3 \\ 2 & 0 & 0 & 0 & 2 & 0 & 1 \end{pmatrix}$$

Answer:

Compute $\mathbf{rref}(A) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$. The pivot columns are 1 and 5.

8. (40 points) Let S be the subspace of $\mathbb{R}^4$ spanned by the vectors $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$. Find the Gram-Schmidt orthonormal basis of S.

Answer:

Let $\vec{y}_1 = \vec{v}_1$ and $\vec{u}_1 = \frac{1}{\|\vec{y}_1\|}\vec{y}_1$. Then $\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}$. Let $\vec{y}_2 = \vec{v}_2$ minus the shadow projection of $\vec{v}_2$ onto the span of $\vec{v}_1$. Then

$$\vec{y}_2 = \vec{v}_2 - \frac{\vec{v}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 = \frac{1}{3} \begin{pmatrix} 2 \\ -1 \\ -1 \\ 3 \end{pmatrix}.$$

Finally, $\vec{u}_2 = \frac{1}{\|\vec{y}_2\|}\vec{y}_2$. We report the Gram-Schmidt basis:

$$\vec{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{u}_2 = \frac{1}{\sqrt{15}}\begin{pmatrix} 2 \\ -1 \\ -1 \\ 3 \end{pmatrix}.$$

9. (30 points) Define matrix A and vector $\vec{b}$ by the equations

$$A = \begin{pmatrix} -2 & 3 & 0 \\ 0 & -2 & 4 \\ 1 & 0 & -2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}.$$

Find the value of x_2 by Cramer's Rule in the system $A\vec{x} = \vec{b}$.

Answer:

$$x_2 = \Delta_2/\Delta,\ \Delta_2 = \det\begin{pmatrix} -2 & 1 & 0 \\ 0 & 2 & 4 \\ 1 & 3 & -2 \end{pmatrix} = 36,\ \Delta = \det(A) = 4,\ x_2 = 9.$$

10. (20 points) Display the entry in row 4, column 3 of the adjugate matrix of $A = \begin{pmatrix} 0 & 2 & -1 & 0 \\ 0 & 0 & 4 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & -2 & 0 \end{pmatrix}$.

The adjugate matrix is also called the adjoint matrix. It appears in the formula $C^{-1} = \dfrac{\mathbf{adj}(C)}{\det(C)}$.

Answer:

The answer is the cofactor of A in row 3, column 4 $= (-1)^7$ times the minor of A in row 3, column 4. The minor equals 8. The answer is -8.

11. (10 points) Consider a 3×3 real matrix A with eigenpairs

$$\left(-1, \begin{pmatrix} 5 \\ 6 \\ -4 \end{pmatrix}\right), \quad \left(1, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right), \quad \left(2, \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}\right).$$

Display an invertible matrix P and a diagonal matrix D such that $AP = PD$.

Answer:

The columns of P are the eigenvectors and the diagonal entries of D are the eigenvalues, taken in the same order.

12. (30 points) Find the eigenvalues of the matrix $A = \begin{pmatrix} 0 & -12 & 3 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 5 & 1 & 3 \end{pmatrix}$.

To save time, **do not** find eigenvectors!

Answer:

The characteristic polynomial is $\det(A-rI) = (-r)(3-r)(r-2)(r+2)$. The eigenvalues are $0, 2, -2, 3$. Determinant expansion of $\det(A - \lambda I)$ is by the cofactor method along column 1. This reduces it to a 3×3 determinant, which can be expanded by the cofactor method along column 3.

13. (20 points) Let I denote the 3×3 identity matrix. Assume given two 3×3 matrices B, C, which satisfy $CP = PB$ for some invertible matrix P. Let C have eigenvalues -1, 1, 0. Let $A = 2I + 3B$. Find the eigenvalues of A^3.

Answer:

Both B and C have the same eigenvalues, because $\det(B - \lambda I) = \det(P(B - \lambda I)P^{-1}) = \det(PCP^{-1} - \lambda PP^{-1}) = \det(C - \lambda I)$. Further, both B and C are diagonalizable. The answer is the same for all such matrices, so the computation can be done for a diagonal matrix $B = \mathbf{diag}(-1, 1, 0)$. In this case, $A = 2I + 3B = \mathbf{diag}(2, 2, 2) + \mathbf{diag}(-3, 3, 0) = \mathbf{diag}(-1, 5, 2)$ and the eigenvalues of A are $-1, 5, 2$. To finish, observe that the eigenvalues of A^3 and A are related, because $A\vec{v} = \lambda\vec{v}$ implies $A^3\vec{v} = \lambda^3\vec{v}$. Therefore, the eigenvalues of A^3 are $-1, 125, 8$.

14. (10 points) The Cayley-Hamilton theorem suggests that there are real 2×2 matrices A such that $A^2 = A$. Give an example of one such matrix A.

Answer:

Choose any matrix whose characteristic equation is $\lambda^2 - \lambda = 0$. Then $A^2 - A = 0$ by the Cayley-Hamilton theorem.

15. (40 points) The spectral theorem says that a symmetric matrix A can be factored into $A = QDQ^T$ where Q is orthogonal and D is diagonal. Find Q and D for the symmetric matrix $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$.

Answer:

Start with the equation $r^2 - 4r + 3 = 0$ having roots $r = 1, 3$. Compute the eigenpairs $(1, \vec{v}_1)$, $(3, \vec{v}_2)$ where $\vec{v}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\vec{v}_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$. The two vectors are orthogonal but not of unit length. Unitize them to get $\vec{u}_1 = \frac{1}{\sqrt{2}}\vec{v}_1$, $\vec{u}_2 = \frac{1}{\sqrt{2}}\vec{v}_2$. Then $Q = < \vec{u}_1 | \vec{u}_2 > = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $D = \mathbf{diag}(1, 3)$.

16. (30 points) Let the linear transformation T from $\mathcal{R}^2$ to $\mathcal{R}^2$ be defined by its action on two independent vectors:

$$T\left(\begin{pmatrix} 3 \\ 0 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 4 \end{pmatrix}, T\left(\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right) = \begin{pmatrix} 4 \\ 2 \end{pmatrix}.$$

Find the unique 2×2 matrix A such that T is defined by the matrix multiply equation $T(\vec{x}) = A\vec{x}$.

Answer:

Let A be the matrix for T, such that $T(\vec{v}) = A\vec{v}$ (matrix multiply). Then $A \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix}$. Solving for A gives $A = \begin{pmatrix} 4 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 4/3 & 4/3 \\ 4/3 & -2/3 \end{pmatrix}$.

17. (10 points) Assume singular value decomposition $A = U\Sigma V^T$ given by

$$\begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{8} & 0 \\ 0 & \sqrt{2} \end{pmatrix} \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right)^T.$$

Find a formula for the pseudo-inverse, but don't bother to multiply out matrices.

Answer:

The definition is $A^+ = V\Sigma^+ U^T$. Identifying the matrices from the formula gives $A^+ = \left(\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \right) \begin{pmatrix} \frac{1}{\sqrt{8}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^T$.