MATH 2270-2 Exam 1 Spring 2017

ANSWERS

1. (10 points)
Assume that matrices A and B are $n \times n$, matrix I is the $n \times n$ identity and C^T denotes the transpose of a matrix C.

(a) Give a counter example or explain why it is true. If matrix A is invertible, then $A^T(A^{-1}+B)^T = I + (BA)^T$.

(b) Give a counter example or explain why it is true. If $A^2B^2 = I$, then AB is the inverse of BA.

Answer:

(a) TRUE. In general $(CD)^{-1}$ is the product of the inverses in reverse order, $D^{-1}C^{-1}$. The same is true for transposes. And transpose and inverse commute: $(C^T)^{-1} = (C^{-1})^T$. Why it is true: $A^T(A^{-1}+B)^T = A^T((A^{-1})^T + B^T) = A^T(A^{-1})^T + A^TB^T = (A^{-1}A)^T + (BA)^T = I + (BA)^T$

(b) TRUE. It is a standard theorem that $CD = I$ implies $DC = I$ for square matrices C, D. The determinant product theorem applied to $A^2B^2 = I$ implies $|A| \neq 0$ and $|B| \neq 0$. To show AB is the inverse of BA we only have to prove $(AB)(BA) = I$ (use $C = AB, D = BA$ in the theorem cited). Here's how: $(AB)(BA) = A^{-1}AABBA = A^{-1}(A^2B^2)A = A^{-1}(I)A = I$.

2. (10 points) Definition: An **elementary matrix** is the answer after applying exactly one combo, swap or multiply to the identity matrix I. An **elimination matrix** is a product of elementary matrices.

Let A be a 3×4 matrix. Find the elimination matrix E which under left multiplication against matrix A performs both (1) and (2) below with one matrix multiply.

(1) Replace Row 3 of A with Row 3 minus Row 1.

(2) Replace Row 3 of A by Row 3 minus 5 times Row 2.

Answer:

Perform combo(1,3,-1) on I then combo(2,3,-5) on the result. The elimination matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -5 & 1 \end{pmatrix}$$

3. (20 points) Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & c & b \\ 2 & b+c & a \\ 1 & b & -a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -a \\ a \\ a \end{pmatrix}$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for $(c-b)(2a-b) \neq 0$.

(b). The system has no solution if $b = 2a$ and $a \neq 0$ (don't explain the other possibilities).

(c). The system has infinitely many solutions if $a = b = c = 0$ (don't explain the other possibilities).

Answer:

(a) Uniqueness requires zero free variables. Then the determinant of the coefficient matrix A must be nonzero. After the cofactor expansion the determinant is factored as $(b-2a)(b-c)$. The inverse of the coefficient matrix then exists for $(b-2a)(b-c) \neq 0$, which implies equation $A\vec{u} = \vec{b}$ has unique solution $\vec{u} = A^{-1}\vec{b}$.

(b) No solution: Combo, swap and mult are used in part (b) with $b = 2a$ substituted into the system of equations. After 3 combo steps the matrix is transformed into

$$A_3 = \begin{pmatrix} 1 & c & 2a & -a \\ 0 & 2a-c & -3a & 3a \\ 0 & 0 & 0 & a \end{pmatrix}$$

The last row of A_3 is a signal equation if $b = 2a = 0$ and $a \neq 0$.

(c) Infinitely many solutions: If $a = b = c = 0$, then from part (b)

$$A_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matrix A_3 has one lead variable and two free variables, because the last two rows of A_3 are zero. This homogeneous problem has no signal equation, therefore it has infinitely many solutions.

A full analysis of the three possibilities is fairly complex.

The sequence of steps are documented below for maple.

```
combo:=(A,s,t,m)->linalg[addrow](A,s,t,m);\newline
mult:=(A,t,m)->linalg[mulrow](A,t,m);\newline
swap:=(A,s,t)->linalg[swaprow](A,s,t);\newline
A:=(a,b,c)->Matrix([[1,c,b,-a],[2,b+c,a,a],[1,b,-a,a]]);\newline
A1:=combo(A(a,b,c),1,2,-2);\newline
A2:=combo(A1,1,3,-1);\newline
A3:=combo(A2,2,3,-1);
```

Definition. *Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called* **independent** *provided solving the equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ has the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called* **dependent**.

4. (20 points) Classify the following set of vectors as Independent or Dependent, using the Pivot Theorem, the Rank Theorem or the definition of independence (above). Details are 75%, answer 25%.

$$\begin{pmatrix}1\\0\\0\\1\\1\end{pmatrix}, \begin{pmatrix}0\\2\\0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\0\\2\\2\end{pmatrix}, \begin{pmatrix}1\\0\\0\\2\\2\end{pmatrix}$$

Answer:

The vectors are dependent. The augmented matrix of the vectors has pivot columns 1,2,4. Therefore, vectors 1, 2, 4 are independent. By the Pivot Theorem, the third vector is a linear combination of the pivot columns 1,2,4.

$$\begin{pmatrix}1&0&1&1\\0&2&2&0\\0&0&0&0\\1&1&2&2\\1&1&2&2\end{pmatrix} \quad => \quad \begin{pmatrix}1&0&1&1\\0&1&1&0\\0&0&0&1\\0&0&0&0\\0&0&0&0\end{pmatrix}$$

5. (20 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix}1&0&0&4\\4&0&0&1\\6&0&2&0\end{pmatrix}, \quad \vec{b} = \begin{pmatrix}0\\0\\4\end{pmatrix}$$

Answer:

The augmented matrix for this system of equations is

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 4 & 0 & 0 & 1 & 0 \\ 6 & 0 & 2 & 0 & 4 \end{pmatrix}$$

The reduced row echelon form is found as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -15 & 0 \\ 6 & 0 & 2 & 0 & 4 \end{pmatrix} \quad \text{combo(1,2,-4)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & -15 & 0 \\ 0 & 0 & 2 & -24 & 4 \end{pmatrix} \quad \text{combo(1,3,-6)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & -24 & 4 \end{pmatrix} \quad \text{mult(2,-1/15)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 0 & 4 \end{pmatrix} \quad \text{combo(2,3,24)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix} \quad \text{mult(3,1/2)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{swap(2,3); last frame}$$

The last frame, or RREF, implies the system

$$\begin{aligned} x_1 \qquad\qquad\qquad &= 0 \\ x_3 \qquad &= 2 \\ x_4 &= 0 \end{aligned}$$

The lead variables are x_1, x_3, x_4 and the free variable is x_2. The last frame algorithm introduces invented symbol t_1. The free variable is set to this symbol, then back-substitute into the lead variable equations of the last frame to obtain the general solution

$$\begin{aligned} x_1 &= 0, \\ x_2 &= t_1, \\ x_3 &= 2, \\ x_4 &= 0. \end{aligned}$$

Strang's *special solution* $\vec{s}_1$ is the partial of $\vec{x}$ on the invented symbol t_1. A particular solution $\vec{x}_p$ is obtained by setting all invented symbols to zero. Then

$$\vec{x} = \vec{x}_p + t_1\vec{s}_1 = \begin{pmatrix} 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

6. (20 points) Determinant problem, chapter 3.
(a) [30%] Assume given 3×3 matrices A, B. Suppose $E_3E_2E_1A = A^2B^2$ and E_1, E_2, E_3 are elementary matrices representing respectively a combination, a swap and a multiply by 3. Assume $\det(B) = -5$. Let $C = 2A$. Find all possible values of $\det(C)$.
(b) [20%] Determine all values of x for which $(I+C)^{-1}$ exists, where I is the 3×3 identity and $C = \begin{pmatrix} 2 & x & -1 \\ x & 1 & 1 \\ 1 & 0 & -1 \end{pmatrix}$.
c) [30%] Let symbols a, b, c denote constants and define

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -2 & 0 & 0 \\ a & b & 0 & 1 \\ 1 & c & 1 & 2 \end{pmatrix}$$

Apply the adjugate [adjoint] formula for the inverse

$$A^{-1} = \frac{\mathbf{adj}(A)}{|A|}$$

to find the value of the entry in row 3, column 2 of A^{-1}.
Answer:

(a) Start with the determinant product theorem $|FG| = |F||G|$. Apply it to obtain $|E_3||E_2||E_1||A| = |A|^2|B|^2$. Let $x = |A|$ in this equation and solve for x. You will need to know that $|B| = -5$, $|E_1| = 1$, $|E_2| = -1$ and $|E_3| = 3$. Then $|C| = |(2I)A| = |2I||A| = 8x$. The answer is $|C| = 0$ or $|C| = 24/5$. (b) Find $C + I = \begin{pmatrix} 3 & x & -1 \\ x & 2 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, then evaluate its determinant, to eventually solve for $x = -2$. Used here is F^{-1} exists if and only if $|F| \neq 0$. The answer is $I + C$ has an inverse for all $x \neq -2$.
(c) Find the cross-out determinant in row 2, column 3 (no mistake, the transpose swaps

rows and columns). Form the fraction, top=checkboard sign times cross-out determinant, bottom=$|A|$. The value is $b - \frac{c}{2}$. A maple check:

```
C4:=Matrix([[1,0,0,0],[1,-2,0,0],[a,b,0,1],[1,c,1,2]]);
1/C4; # The inverse matrix
C5:=linalg[minor](C4,2,3); linalg[det](C5)*(-1)^(2+3);linalg[det](C4);
# ans =-c+2b divided by 2
```

End Exam 1.