1. (10 points) Let \( A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \). Find a basis of vectors for each of the four fundamental subspaces.

2. (10 points) Assume \( V = \text{span}(\vec{v}_1, \vec{v}_2) \) with \( \vec{v}_1 = \begin{pmatrix} 3 \\ 0 \\ 4 \\ 0 \end{pmatrix} \), \( \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \). Find the Gram-Schmidt orthonormal vectors \( \vec{q}_1, \vec{q}_2 \) whose span equals \( V \).

3. (10 points) Let \( Q \) be an orthonormal matrix. The normal equations for the system \( Q\vec{x} = \vec{b} \) finds the least squares solution \( \vec{v} = QQ^T\vec{b} \). The equations imply that \( P = QQ^T \) projects \( \vec{b} \) onto the span of the columns of \( Q \). For the subspace \( V = \text{span}(\vec{v}_1, \vec{v}_2) \) in the previous problem, find matrix \( P \). This matrix projects \( \mathbb{R}^4 \) onto \( V \), while \( I - P \) projects \( \mathbb{R}^4 \) onto \( V^\perp \).

4. (10 points) Find the least squares best fit line \( y = v_1 x + v_2 \) for the points \( (0, 1), (2, 3), (4, 4) \).

5. (5 points) Find the determinant of the matrix

\[
\begin{pmatrix}
1 & 2 & 3 & 0 \\
1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 \\
5 & 6 & 7 & 8
\end{pmatrix}
\]

6. (10 points) Let \( A = \begin{pmatrix} 1 & 3 \\ 2 & 2 \end{pmatrix} \). Find all eigenpairs of \( A \).

7. (10 points) Let \( A = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix} \). Find all eigenpairs.
8. **(15 points)** Find an equation for the plane in \( \mathbb{R}^3 \) that contains the three points \((1, 0, 0), (1, 1, 1), (1, 2, 0)\).

9. **(10 points)** Suppose an \( n \times n \) matrix \( A \) has all eigenvalues equal to 0. Show from the Cayley-Hamilton Theorem that \( A^n \) has all entries equal to 0.

10. **(15 points)** Prove the Cayley-Hamilton Theorem for \( 2 \times 2 \) matrices with real eigenvalues. Write the characteristic equation as \( \lambda^2 + c_1 \lambda = -c_2 \), then substitute as in the Cayley-Hamilton theorem, arriving at the proposed equation \( A^2 + c_1 A = -c_2 I \). Expand the left side:

\[
A^2 + c_1 A = A(A + c_1 I) = A(A - (a + d)I) = -A \text{adj}(A), \quad \text{adj}(A) = \begin{pmatrix}
d & -b \\
- c & a
\end{pmatrix}.
\]

Because \( A \text{adj}(A) = |A| I \) (the adjugate identity), then the right side of the preceding display simplifies to \(- \det(A) I = -c_2 I\). This proves the Cayley-Hamilton theorem for \( 2 \times 2 \) matrices: \( A^2 + c_1 A = -c_2 I \).

11. **(5 points)** Suppose a \( 3 \times 3 \) matrix \( A \) has eigenpairs

\[
\left( 3, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right), \quad \left( 3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right), \quad \left( 0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).
\]

Display an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( AP = PD \).

12. **(10 points)** Suppose a \( 3 \times 3 \) matrix \( A \) has eigenpairs

\[
\left( 3, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \right), \quad \left( 3, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right), \quad \left( 0, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right).
\]

Find \( A \).

13. **(10 points)** Assume \( A \) is \( 2 \times 2 \) and Fourier's model holds:

\[
A \begin{pmatrix} c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{pmatrix} = 2c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.
\]

Find \( A \).

14. **(10 points)** How many eigenpairs? (a) \( \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \), (b) \( \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \).
15. (5 points) True or False? A Jordan block has one and only one eigenpair.

16. (5 points) True or False? A diagonal block matrix $A = \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix}$ where $B_1, B_2$ are Jordan blocks has exactly two eigenpairs.

No new questions beyond this point.