1. (5 points) State the Three Possibilities for a linear system $A\mathbf{x} = \mathbf{b}$.

2. (5 points) Completely describe each operation in the basic Toolkit for solving a linear system (combo, swap, mult).

3. (5 points) Can the following system have no solution for some choice of $b_1, b_2, b_3$?

$$
\begin{align*}
    x_1 + x_2 + x_3 + x_4 &= b_1 \\
    4x_1 + 2x_2 + 3x_3 &= b_2 \\
    x_3 + x_4 &= b_3
\end{align*}
$$

4. (5 points) True or false? Explain. If $A$ and $B$ are $n \times n$ invertible, then $(AB)^{-1} = A^{-1}B^{-1}$.

5. (5 points) True or false? Explain. If square matrices $A$ and $B$ satisfy $AB = I$, then $A\mathbf{x} = \mathbf{b}$ has a unique solution $\mathbf{x}$ for each vector $\mathbf{b}$.

6. (5 points) Give an example of a $3 \times 2$ matrix $A$ and a frame sequence with three or more frames, starting at $A$, which proves that the invented system $A\mathbf{x} = \mathbf{0}$ has a unique solution.

7. (5 points) Give an example of a $4 \times 3$ matrix $A$ and a frame sequence with three or more frames, starting at $A$, which proves that the invented system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

8. (5 points) Let $A$ be a $3 \times 4$ matrix. What is the elimination matrix that replaces Row 2 of $A$ with Row 2 minus Row 1 and replaces Row 3 of $A$ by Row 3 minus 2 times Row 1?

9. (5 points) Let $A$ be a $3 \times 3$ matrix. Let

$$
F = \begin{pmatrix}
1 & 2 & 3 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{pmatrix}.
$$

Assume $F$ is obtained from $A$ by the following sequential row operations: (1) Swap rows 2 and 3; (2) Add $-2$ times row 2 to row 3; (3) Add 3 times row 1 to row 2; (4) Multiply row 2 by $-3$. Find $A$.

10. (5 points) What is the inverse of the following matrix?

$$
E = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & -5 & 1 & 0 \\
0 & 0 & 1 & 0 & 1
\end{pmatrix}
$$
11. (5 points) Describe in words the effect of multiplying
\[ E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -3 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix} \]
on the left of a 4 × 5 matrix A to get EA.

12. (20 points) Let a, b and c denote constants and consider the system of equations
\[
\begin{pmatrix} 1 & b-c & a \\ 1 & c & -a \\ 2 & b & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -a \\ a \\ a \end{pmatrix}
\]
(a). Determine those values of a, b and c such that the system has a unique solution.
(b). Determine those values of a, b and c such that the system has no solution.
(c). Determine those values of a, b and c such that the system has infinitely many solutions.

To save time, don’t attempt to solve the equations or to produce a complete frame sequence.

```
macro(combo=linalg[addrow]);macro(mult=linalg[mulrow]);
macro(swap=linalg[swaprow]);
A:=(a,b,c)->Matrix([[1,b-c,a,-a],[1,c,-a,a],[2,b,a,a]]);
A1:=combo(A(a,b,c),1,2,-1);
A2:=combo(A1,1,3,-2);
A3:=combo(A2,2,3,-1);
A4:=combo(A3,3,2,2);
A5:=combo(A4,3,1,-1);
A5 := Matrix([[1,b-c,0,-2*a],[0,-b+2*c,0,4*a],[0,0,a,a]]);
```

13. (15 points) Let \( A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \). Let \( B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \). Let \( C = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \). Calculate the following:
\[ A(BC), \quad (AB)C, \quad C^2, \quad 2A + B - C, \quad A(B - C) \]

14. (20 points) Classify the following sets of vectors for (1) Independence or (2) Dependence. For each set of vectors, report whether or not they form a basis for the indicated vector space.
15. **(20 points)** Find all solutions to the equation $A\bar{x} = \bar{b}$ for

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{pmatrix}$$

$$\bar{b} = \begin{pmatrix} 0 \\ 4 \end{pmatrix}$$

16. **(20 points)** Prove that a system of 4 linear equations in 5 unknowns has either no solution or infinitely many solutions.

17. **(20 points)** Let matrix $A$ have 201 rows and 201 columns. All entries of $A$ are zero off the diagonal, except for the number $-7$ in row 107, column 35. The diagonal entries of $A$ are all one. Describe the inverse of $A$, in words.

End of the sample exam questions.