MATH 2270-2 Exam 1 Spring 2012

ANSWERS

1. (10 points) Give a counter example or explain why it is true. If A and B are $n \times n$ invertible, and C^T denotes the transpose of a matrix C, then $(AB^{-1})^T = (B^T)^{-1}A^T$.

Answer:

In general $(CD)^{-1}$ is the product of the inverses in reverse order, $D^{-1}C^{-1}$. The same is true for transposes. And transpose and inverse commute: $(C^T)^{-1} = (C^{-1})^T$. Why it is true: $(AB^{-1})^T = (B^{-1})^T A^T = (B^T)^{-1}A^T$.

2. (10 points) Give a counter example or explain why it is true. If square matrices A and B satisfy $AB = I$, then the transposes satisfy $A^T B^T = I$.

Answer:

It is a standard theorem that $AB = I$ implies $BA = I$. Transpose this last equation of get $A^T B^T = (BA)^T = I^T = I$.

3. (10 points) Let A be a 3×4 matrix. Find the elimination matrix E which under left multiplication against A performs both (1) and (2) with one matrix multiply.

(1) Replace Row 2 of A with Row 2 minus Row 3.

(2) Replace Row 3 of A by Row 3 minus 4 times Row 1.

Answer:

Perform combo(3,2,-1) on I then combo(1,3,-4) on the result. The elimination matrix is

$$E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ -4 & 0 & 1 \end{pmatrix}$$

4. (30 points) Let a, b and c denote constants and consider the system of equations

$$\begin{pmatrix} 1 & b & c \\ 1 & c & -a \\ 2 & b+c & a \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -a \\ a \\ a \end{pmatrix}$$

Use techniques learned in this course to briefly explain the following facts. Only write what is needed to justify a statement.

(a). The system has a unique solution for $(c - b)(2a - c) \neq 0$.

(b). The system has no solution if $c = 2a$ and $a \neq 0$ (don't explain the other possibilities).

(c). The system has infinitely many solutions if $a = b = c = 0$ (don't explain the other possibilities).

Answer:
Combo, swap and mult are used to obtain in 3 combo steps the matrix

$$A_3 = \begin{pmatrix} 1 & b & c & -a \\ 0 & -b+c & -c-a & 2a \\ 0 & 0 & -c+2a & a \end{pmatrix}$$

(a) Uniqueness requires zeros free variables. Then the diagonal entries of the last frame must be nonzero, written simply as $(c-b)(2a-c) \neq 0$, which is equivalent to the determinant of A not equal to zero.
(b) No solution: The last row of A_3 is a signal equation if $-c+2a = 0$ and $a \neq 0$. There are other possibilities for no solution: see part (c).
(c) Infinitely many solutions: If $a = b = c = 0$, then A_3 has one lead variable and two free variables, because the last two rows of A_3 are zero. This homogeneous problem has infinitely many solutions. There are other possibilities for infinitely many solutions, because the last row of A_3 could have a zero row ($a = c = 0$), without the second row being zero ($a = b = c = 0$), or y could be a free variable ($c - b = 0$) with the last two equations consistent.
A full analysis of the three possibilities is fairly complex. For instance, $-b + c = 0$ causes one free variable y. The condition $-b + c = 0$ splits into two sub-cases: one for no solution and one for infinitely many solutions. Continued
Definition. *Vectors $\vec{v}_1, \ldots, \vec{v}_k$ are called* **independent** *provided solving the equation $c_1\vec{v}_1 + \cdots + c_k\vec{v}_k = \vec{0}$ for constants $c_1, \ldots, c_k$ has the unique solution $c_1 = \cdots = c_k = 0$. Otherwise the vectors are called* **dependent**.
5. (20 points) Classify the following sets of vectors as Independent or Dependent, using the Pivot Theorem or the definition of independence (above).

$$\text{Set 1:} \quad \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 0 \end{pmatrix}$$

$$\text{Set 2:} \quad \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$$

Answer:
The first set is independent. The two vectors are not scalar multiples of each other, so they are linearly independent.
The second set is dependent. The augmented matrix of the three vectors has pivot columns 1,2. Therefore, the first two vectors are independent. By the Pivot Theorem, the third vector is a linear combination of the pivot columns 1,2. Hence the set of three vectors is dependent.

6. (20 points) Find the vector general solution $\vec{x}$ to the equation $A\vec{x} = \vec{b}$ for

$$A = \begin{pmatrix} 1 & 0 & 0 & 4 \\ 3 & 0 & 1 & 0 \\ 4 & 0 & 0 & 1 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 0 \\ 4 \\ 0 \end{pmatrix}$$

Answer:
The augmented matrix for this system of equations is

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 3 & 0 & 1 & 0 & 4 \\ 4 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The reduced row echelon form is found as follows:

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 4 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{combo(1,2,-3)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 0 & -16 & 0 \end{pmatrix} \quad \text{combo(1,3,-4)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 4 & 0 \\ 0 & 0 & 1 & -12 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{mult(3,-1/16)}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{last frame}$$

The last frame, or RREF, implies the system

$$\begin{aligned} x_1 \qquad\qquad &= 0 \\ x_3 \qquad &= 4 \\ x_4 &= 0 \end{aligned}$$

The lead variables are x_1, x_3, x_4 and the free variables is x_2. The last frame algorithm introduces invented symbol t_1. The free variable is set to this symbol, then back-substitute into the lead variable equations of the last frame to obtain the general solution

$$\begin{aligned} x_1 &= 0, \\ x_2 &= t_1, \\ x_3 &= 4, \\ x_4 &= 0. \end{aligned}$$

Strang's *special solutions* $\vec{s}_2$ is the partial of $\vec{x}$ on the invented symbol t_1. A particular solution $\vec{x}_p$ is obtained by setting all invented symbols to zero. Then

$$\vec{x} = \vec{x}_p + t_1\vec{s}_2 = \begin{pmatrix} 0 \\ 0 \\ 4 \\ 0 \end{pmatrix} + t_1 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

End Exam 1.